Unexpected quadratic behaviors for the small-time local null controllability of scalar-input parabolic equations

Karine Beauchard*, Frédéric Marbach*

December 29, 2017

Abstract

We consider scalar-input control systems in the vicinity of an equilibrium, at which the linearized systems are not controllable. For finite dimensional control systems, the authors recently classified the possible quadratic behaviors. Quadratic terms introduce coercive drifts in the dynamics, quantified by integer negative Sobolev norms, which are linked to Lie brackets and which prevent smooth small-time local controllability for the full nonlinear system.

In the context of nonlinear parabolic equations, we prove that the same obstructions persist. More importantly, we prove that two new behaviors occur, which are impossible in finite dimension. First, there exists a continuous family of quadratic obstructions quantified by fractional negative Sobolev norms. Second, and more strikingly, small-time local null controllability can sometimes be recovered from the quadratic expansion.

As in the finite dimensional case, the relation between the regularity of the controls and the strength of the possible quadratic obstructions plays a key role in our analysis.

Contents

*Univ Rennes, CNRS, IRMAR - UMR 6625, F - 35000 Rennes, France

1 Introduction and main results

The goal of this work is to illustrate possible behaviors for parabolic scalar-input control systems, stemming from the analysis of their second-order expansions. Some of these quadratic behaviors are already present in finite dimension (see [6], where the authors classified the possible quadratic behaviors for scalar-input control systems in finite dimension, or [28] for a short survey in French by the second author). Others are new and specific to control systems in infinite dimension.

1.1 Description of the control system

We present our results in the simple setting of a scalar-input control system governed by a nonlinear heat equation set on the line segment $x \in (0, \pi)$. We consider the following nonlinear control system:

$$\begin{cases} \partial_t z(t,x) - \partial_{xx} z(t,x) = u(t)\Gamma[z(t)](x), \\ \partial_x z(t,0) = \partial_x z(t,\pi) = 0, \\ z(0,x) = z_0(x), \end{cases} \tag{1.1}$$

where z is the state, u is a scalar control, and Γ is an appropriate nonlinearity. We will be interested in the notion of small-time local null controllability: given a small time $T > 0$ and an initial data z_0 sufficiently small, does there exist a small control u such that $z(T) = 0$?

Remark 1.1. *The abstract system* (1.1) *is not intended to model the behavior of a real-world physical system. Even so, we expect that most of the techniques and methods we introduce in the sequel could be applied or extended to other more complex or realistic classes of systems. We chose this abstract system as it lightens the computations and reduces the amount of technicalities in the proof (see also Section 4.6 for an example of difficulties to be expected for other systems).*

1.2 Notations and functional settings

1.2.1 Functional setting in space

We consider the Lebesgue space $L^2(0,\pi)$, equipped with its usual scalar product $\langle \cdot, \cdot \rangle$. Let Δ_N be the Neumann-Laplacian operator on $(0, \pi)$

$$D(\Delta_N) := \{f \in H^2(0,\pi);\ \ f'(0) = f'(\pi) = 0\}, \quad \Delta_N f = -\partial_{xx} f \tag{1.2}$$

and $(\varphi_k)_{k\in\mathbb{N}}$, the orthonormal basis of $L^2(0,\pi)$ of its eigenfunctions

$$\varphi_k(x) := \frac{1}{\sqrt{\pi}} \begin{cases} 1 & \text{if } k = 0, \\ \sqrt{2}\cos(k\pi x) & \text{if } k \in \mathbb{N}^*. \end{cases} \tag{1.3}$$

We introduce the Sobolev space $H^1_N(0,\pi)$

$$H^1_N(0,\pi) := \left\{ f \in L^2(0,\pi);\ \ \|f\|^2_{H^1_N(0,\pi)} = \sum_{k=0}^{\infty} |(1+k)\langle f, \varphi_k\rangle|^2 < \infty \right\} \tag{1.4}$$

and its dual space $H^{-1}_N(0,\pi)$, which is equipped with the norm

$$\|f\|^2_{H^{-1}_N(0,\pi)} = \left(\sum_{k=0}^{\infty} \left| (1+k)^{-1} (f, \varphi_k)_{H^{-1}_N, H^1_N} \right|^2 \right)^{1/2}. \tag{1.5}$$

1.2.2 Assumptions on the nonlinearity and regularity of solutions

Throughout this work, we will assume that there exists $C_\Gamma > 0$ such that the nonlinearity Γ appearing in (1.1) satisfies

$$\begin{gathered}\Gamma \in C^2\left(H^1_N(0,\pi); H^{-1}_N(0,\pi)\right) \quad \text{and} \\ \forall z \in H^1_N(0,\pi), \quad \|\Gamma'[z]\|_{\mathcal{L}(H^1_N;H^{-1}_N)} + \|\Gamma''[z]\|_{\mathcal{L}(H^1_N\times H^1_N;H^{-1}_N)} \leq C_\Gamma,\end{gathered} \tag{1.6}$$

so that, for every control $u \in L^\infty(0,T)$, system (1.1) is locally well-posed (see Lemma 2.2 below) in the space

$$Z := C^0\left([0,T]; L^2(0,\pi)\right) \cap L^2\left((0,T); H^1_N(0,\pi)\right), \tag{1.7}$$

which we endow with the norm

$$\|z\|^2_Z := \|z\|^2_{C^0([0,T];L^2(0,\pi))} + \|z\|^2_{L^2((0,T);H^1_N(0,\pi))} \tag{1.8}$$

and its solution z has a C^2-dependence with respect to control the u. Eventually, we will use as a shorthand notation

$$\mu := \Gamma[0] \in H^{-1}_N(0,\pi). \tag{1.9}$$

Remark 1.2. *The regularity assumption* (1.6) *includes bilinear systems as a particular case. Indeed, for any $\lambda, \mu \in H^{-1}_N(0,\pi)$, the map $\Gamma : z \mapsto \mu + \lambda z$ satisfies* (1.6)*. Moreover, even when Γ is such an affine map, the control system is already fully nonlinear since the source is $u\Gamma[z]$.*

1.2.3 Functional setting in time

For $T > 0$ and $f \in L^\infty(0,T)$, we consider the iterated primitives of f defined by induction as

$$f_0 := f, \quad f_{n+1}(t) := \int_0^t f_n(\tau)\,\mathrm{d}\tau. \tag{1.10}$$

Implicitly, when required, we identify a function $f \in L^\infty(0,T)$ with its extension by zero to the real line, which allows us to consider

- its non-unitary Fourier transform $\widehat{f}$, defined for $\xi \in \mathbb{R}$ as

$$\widehat{f}(\xi) := \int_{\mathbb{R}} f(t)e^{-i\xi t}\,\mathrm{d}t = \int_0^T f(t)e^{-i\xi t}\,\mathrm{d}t, \tag{1.11}$$

- its (negative) fractional Sobolev norm in $H^{-s}(\mathbb{R})$, for any $s \in (0,1]$,

$$\|f\|_{H^{-s}(\mathbb{R})} := \left(\int_{\mathbb{R}} \frac{|\widehat{f}(\xi)|^2}{(1+\xi^2)^s}\,\mathrm{d}\xi\right)^{\frac{1}{2}}. \tag{1.12}$$

We will also consider, for any integer $m \in \mathbb{N}$ the usual integer-order Sobolev space

$$H^m(0,T) := \left\{f \in L^2(0,T);\ f^{(k)} \in L^2(0,T) \text{ for } k = 0,...,m\right\}, \tag{1.13}$$

equipped with the norm

$$\|f\|_{H^m(0,T)} := \left(\sum_{k=0}^m \int_0^T |f^{(k)}(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}, \tag{1.14}$$

and, for $a \in [0,\infty)$ the (positive) fractionnal Sobolev space $H^a(0,T)$ defined by interpolation and equipped with the norm $\|\cdot\|_{H^a(0,T)}$.

1.3 Controllability stemming from the linear order

We start by studying the linearized system of (1.1) around the null equilibrium $(z,u)=(0,0)$:

$$\begin{cases} \partial_t z(t,x) - \partial_{xx} z(t,x) = u(t)\mu(x), \\ \partial_x z(t,0) = \partial_x z(t,\pi) = 0, \\ z(0,x) = z_0(x). \end{cases} \tag{1.15}$$

The controllability of systems such as (1.15) has been extensively studied. We refer in particular to [19] and [20] for the introduction of the *moment method*. The assumption that $\langle \mu, \varphi_k \rangle \neq 0$ for all $k \in \mathbb{N}$ is obviously necessary for the linearized system to be null controllable (otherwise the component $\langle z(t), \varphi_k \rangle$ of the state would evolve freely). Moreover, in order for the linearized system to be small-time null controllable, one must add the assumption that the sequence $\langle \mu, \varphi_k \rangle$ does not decay too fast (see Section 2.3).

Theorem 1. *Let $\mu \in H_N^{-1}(0,\pi)$ such that $\langle \mu, \varphi_k \rangle \neq 0$ for all $k \in \mathbb{N}$ and satisfying the decay assumption* (2.25)*. The linear system* (1.15) *is small-time null controllable with L^2 controls.*

When dealing with nonlinear behaviors, especially in infinite dimension, the regularity of the controls plays a crucial role. In fact, and quite surprisingly, the regularity of the controls already plays an important role for control systems in finite dimension (see [6]). We define the following more precise notions, stressing the regularity imposed on the controls.

Definition 1.3 (Small-time local null controllability)**.** *Let Γ be such that* (1.6) *holds. For $m \in \mathbb{N}^*$, we say that system* (1.1) *is H^m-STLNC (respectively H_0^m-STLNC) when, for every $T, \eta > 0$, there exists $\delta > 0$ such that, for every $z_0 \in L^2(0,\pi)$ with $\|z_0\|_{L^2} \leq \delta$, there exists $u \in H^m(0,T)$ (resp. $u \in H_0^m(0,T)$) with $\|u\|_{H^m} \leq \eta$ such that the solution $z \in Z$ to* (1.1) *satisfies $z(T) = 0$.*

Definition 1.4 (Smooth small-time local null controllability)**.** *Let Γ be such that* (1.6) *holds. We say that system* (1.1) *is Smoothly-STLNC, when it is H_0^m-STLNC for every $m \in \mathbb{N}^*$.*

Under appropriate assumptions, the smooth small-time null controllability of the linearized system around the null equilibrium implies that the nonlinear system is Smoothly-STLNC. This was also the case in finite dimension (see [6, Theorem 1]). Although the following theorem is quite classical, we include a proof in Section 2 to highlight that we can indeed construct regular controls.

Theorem 2. *Let Γ satisfying* (1.6)*, $\langle \mu, \varphi_k \rangle \neq 0$ for all $k \in \mathbb{N}$ and the decay assumption* (2.25)*. Then, the nonlinear system* (1.1) *is Smoothly-STLNC with a linear cost.*

More precisely, for every $m \in \mathbb{N}^*$, there exist constants $C_{\mathfrak{L}}, \delta_{\mathfrak{L}} > 0$ and a continuous map $\mathfrak{L}$: $\{z_0 \in L^2(0,\pi);\ \|z_0\|_{L^2} \leq \delta_{\mathfrak{L}}\} \to H_0^m(0,T)$ such that, for every $z_0 \in L^2(0,\pi)$ with $\|z_0\|_{L^2} \leq \delta_{\mathfrak{L}}$, the solution $z \in Z$ to (1.1) *with a control $u := \mathfrak{L}(z_0)$ satisfies $z(T) = 0$. Moreover, the control and the trajectory satisfy the estimate*

$$\|z\|_Z + \|u\|_{H^m(0,T)} \leq C_{\mathfrak{L}} \|z_0\|_{L^2(0,\pi)}. \tag{1.16}$$

Remark 1.5. *In fact, the small-time null-controllability of the linearized system is a necessary condition for the Smooth-STLNC of the full nonlinear system* with linear cost *(see Section 2.6).*

1.4 Obstructions caused by quadratic integer drifts

When the linearized system misses some directions, a natural question is whether a quadratic expansion can help to recover controllability along the lost directions. As a representative situation, we consider the case when the linearized system misses one direction: the first one. This corresponds to the assumption that

$$\langle \mu, \varphi_0 \rangle = 0. \tag{1.17}$$

To study the quadratic behavior of the system along this lost direction, we introduce, for $j \in \mathbb{N}^*$, the sequence $(c_j)_{j\in\mathbb{N}^*} \in \mathbb{R}^{\mathbb{N}^*}$ defined by

$$c_j := \langle \mu, \varphi_j \rangle \langle \Gamma'[0]\varphi_j, \varphi_0 \rangle. \tag{1.18}$$

Similarly as in finite dimension (see [6, Theorems 2 and 3]), Lie bracket considerations lead to obstructions related to quadratic coercive drifts, quantified by integer-order negative Sobolev norms. The particular case of the first obstruction was encountered by Morancey and the first author in [7], in the context of a bilinear Schrödinger equation. The following theorem, proved in Section 3, shows that any integer-order quadratic obstruction is possible and has consequences for the controllability of the full nonlinear system.

Theorem 3. *Let $n \in \mathbb{N}^*$, Γ satisfying* (1.6) *and* (1.17)*. Assume that there exists n different directions φ_j such that $\langle \mu, \varphi_j \rangle \neq 0$ and that the sequence c_j defined in* (1.18) *satisfies the assumptions*

$$\sum_{j=1}^{\infty} j^{4n} |c_j| < +\infty, \tag{1.19}$$

$$\sum_{j=1}^{\infty} j^{2(2\ell-1)} c_j = 0, \qquad \text{for every } \ell \in \{1, ..., n-1\}, \tag{1.20}$$

$$a := \sum_{j=1}^{\infty} j^{2(2n-1)} c_j \neq 0. \tag{1.21}$$

Then the system (1.1) is not H^{2n+2}-STLNC.

More precisely, for every $\varepsilon > 0$, there exists $T^* > 0$ such that, for every $T \in (0, T^*)$, there exists $\eta > 0$ such that, for every $\delta \in [-1, 1]$, for every $u \in H^{2n+2}(0,T)$ with $\|u\|_{H^{2n+2}(0,T)} \leq \eta$, if the solution of (1.1) *with initial data $z_0 = \delta\varphi_0$ satisfies*

$$\forall j \in \mathbb{N}^*, \quad \langle z(T), \varphi_j \rangle = 0, \tag{1.22}$$

then

$$\left| \langle z(T), \varphi_0 \rangle - \delta + a(-1)^n \|u_n\|^2_{L^2(0,T)} \right| \leq \varepsilon \left(|\delta| + \|u_n\|^2_{L^2(0,T)} \right). \tag{1.23}$$

Remark 1.6. *At an heuristic level, the estimate* (1.23) *corresponds to the fact that, in the asymptotic of small controls in $H^{2n+2}(0,T)$, one has*

$$\langle z(T), \varphi_0 \rangle \approx \langle z_0, \varphi_0 \rangle - a(-1)^n \|u_n\|^2_{L^2(0,T)}. \tag{1.24}$$

The approximate equality (1.24) *indicates that the quadratic terms induce a drift in the dynamics of the system, which is quantified by the $L^2(0,T)$ norm of u_n. In particular, initial states for which $\langle z_0, \varphi_0 \rangle$ has the same sign as $-a(-1)^n$ cannot be driven to zero.*

Remark 1.7. *Under assumption (1.19), the series considered in (1.20) and (1.21) converge. Under appropriate regularity assumptions on Γ, these two relations may be rewritten in term of the Lie brackets that appear in finite dimension as*

$$\langle [ad_{f_0}^{\ell-1}(f_1), ad_{f_0}^{\ell}(f_1)](0), \varphi_0 \rangle = 0, \qquad \text{for every } \ell \in \{1, ..., n-1\}, \tag{1.25}$$

$$\langle [ad_{f_0}^{n-1}(f_1), ad_{f_0}^{n}(f_1)](0), \varphi_0 \rangle \neq 0, \tag{1.26}$$

where $f_0 := \Delta_N$ and $f_1 := \Gamma$.

Remark 1.8. *For control-affine systems in finite dimension, we proved in [6, Theorem 3] that the optimal norm for the smallness assumption on u is the $W^{2n-3,\infty}$ one. The H^{2n+2}-norm used above, for the nonlinear heat equation, is not the optimal one, but allows a lighter exposition.*

1.5 Obstructions caused by quadratic fractional drifts

In the context of parabolic equations, which are control systems in infinite dimension, a whole new continuous family of drifts can occur, quantified by fractional-order negative Sobolev norms. A fractional drift quantified by the $H^{-5/4}$ norm of the control had already been observed by the second author for a Burgers equation (see [27]). The following theorem, proved in Section 4, is the first main result of our work and proves that any negative fractional drift is possible.

Theorem 4. *Let $n \in \mathbb{N}$, $s \in (0,1)$, $a \in \mathbb{R}^*$, $\alpha > 4s-1$ and Γ be such that* (1.6) *and* (1.17) *hold. Assume that the coefficients c_j defined in* (1.18) *satisfy*

$$c_j = \frac{a}{j^{4n-1+4s}} + O\left(\frac{1}{j^{4n+\alpha}}\right), \tag{1.27}$$

$$\sum_{j=1}^{\infty} j^{2(2\ell-1)} c_j = 0, \qquad \text{for every } \ell \in \{1, ..., n\}. \tag{1.28}$$

Then the system (1.1) is not $H^{2n+2s+\frac{3}{2}+}$-STLNC.

More precisely, there exists a constant $\gamma(s) > 0$ (see (4.15)*) such that, for every $\varepsilon, \nu > 0$, there exists $T^* > 0$ such that, for every $T \in (0, T^*)$, there exists $\eta > 0$ such that, for every $\delta \in [-1, 1]$, for every $u \in H^{2n+2s+\frac{3}{2}+\nu}(0,T)$ with $\|u\|_{H^{2n+2s+\frac{3}{2}+\nu}(0,T)} \leq \eta$, if the solution of* (1.1) *with initial data $z_0 = \delta\varphi_0$ satisfies the condition* (1.22)*, then*

$$\left|\langle z(T), \varphi_0\rangle - \delta - a\gamma(s)(-1)^n \|u_n\|^2_{H^{-s}(0,T)}\right| \leq \varepsilon\left(|\delta| + \|u_n\|^2_{H^{-s}(0,T)}\right). \tag{1.29}$$

Remark 1.9. *As in the integer-order obstructions, the following comments can be made.*

- *At an heuristic level, the estimate* (1.29) *corresponds to the fact that, in the asymptotic of small controls in $H^{2n+2s+\frac{3}{2}+}(0,T)$, one has*

$$\langle z(T), \varphi_0\rangle \approx \langle z_0, \varphi_0\rangle + a\gamma(s)(-1)^n \|u_n\|^2_{H^{-s}(0,T)}. \tag{1.30}$$

- *The conditions* (1.28) *can be interpreted in terms of Lie brackets between Δ_N and Γ.*
- *The smallness in $H^{2n+2s+\frac{3}{2}+}$ of the controls is not the optimal assumption.*

Remark 1.10. *Despite the resemblance between the integer-order and the fractional-order statements, we stress that the nature of the underlying cause might be different. Indeed, the integer-order obstructions occur when the weighted sum* (1.21) *of the coefficients c_j is non-zero, whereas the fractional-order obstructions depend only on the asymptotic behavior* (1.27) *of the coefficients.*

1.6 Controllability stemming from the quadratic order

Finally, and even more strikingly, for parabolic equations, we can sometimes recover small-time local null controllability from the quadratic expansion. This fact is most surprising, as it is never possible in finite dimension. Indeed, for finite dimensional control systems, if the linearized system misses one direction, then small-time local controllability can only be recovered thanks to cubic terms (thus, with a cubic control cost).

Up to our knowledge, the following result is the first example in which small-time controllability is restored at the quadratic order for a scalar-input system. In all previously known situations where the linearized system misses at least one direction and controllability is restored, either:

- controllability is restored in small time but by means of a *cubic expansion* with a vanishing quadratic term, (see e.g. [16] where the authors prove small-time local null controllability for a Korteweg-de-Vries system with a critical length thanks to cubic terms),

- controllability is restored by means of a quadratic expansion but only *in large time* (see e.g. [11, 12] where the authors obtain controllability in large time for Korteweg-de-Vries systems with critical lengths and [4, 5, 7] where the authors obtain controllability of bilinear Schrödinger equation),
- controllability is restored in *large time* by means of the return method and possibly other technics (see [14] about the Saint-Venant equation, and [3, 5] about bilinear Schrödinger equations where the large time is due to quasi-static transformations, and [29] where a large time is needed to construct the reference trajectory of the return method),
- controllability is restored *with non-scalar controls* (see e.g. [13] where the author prove small-time controllability for the Euler equation using boundary controls, which corresponds to an infinite number of scalar controls).

The following result, proved in Section 5, is the second main result of our work.

Theorem 5. *There exists a nonlinearity* Γ *satisfying* (1.6) *such that* $\langle \mu, \varphi_0 \rangle = 0$ *and the nonlinear system* (1.1) *is small-time locally null controllable with* quadratic cost. *More precisely, for any* $T > 0$*, there exists* $C, \delta > 0$ *such that, for any* $z_0 \in L^2(0, \pi)$ *with* $\|z_0\|_{L^2} \leq \delta$*, there exists* $u \in L^2(0, T)$ *driving* z_0 *to* 0 *with a size* $\|u\|_{L^2}^2 \leq C\|z_0\|_{L^2}$*.*

Remark 1.11. *The size estimate for the control is reminiscent of the fact that the controllability stems from the quadratic order. In finite dimension, there is no system for which the linearized system misses a direction and for which small-time local controllability is recovered with quadratic cost (we refer to [6] for more precise statements).*

2 Smooth controllability stemming from the linear order

The goal of this section is to prove Theorem 2. The idea that small-time controllability of the linearized system implies controllability of the nonlinear system is quite classical. However, there are two difficulties here. First, we are seeking a null controllability result and we only assumed the controllability of the linearized system at the null equilibrium (and not around any state near the equilibrium). This prevents us from using classical fixed-point methods and requires a specific powerful method. We will use the *source term method* introduced by Liu, Takahashi and Tucsnak in [26] in the context of a fluid-structure system. Second, we wish to build regular controls, even for the nonlinear system. This will require that we adapt accordingly the source term method.

2.1 Classical well-posedness results

We recall, for the sake of completeness, usual well-posedness results for the class of parabolic equations we are looking at, under the regularity assumption (1.6) on Γ.

Lemma 2.1. *Let* $T > 0$*. There exists* $C_T > 0$*, which is a non-decreasing function of* T*, such that, for any* $f \in L^2((0, T); H_N^{-1}(0, \pi))$ *and any* $z_0 \in L^2(0, \pi)$*, there exists a unique solution* $z \in Z$ *to*

$$\begin{cases} \partial_t z(t,x) - \partial_{xx} z(t,x) = f(t,x), \\ z_x(t,0) = z_x(t,\pi) = 0, \\ z(0,x) = z_0(x). \end{cases} \tag{2.1}$$

Moreover, it satisfies the estimate

$$\|z\|_Z \leq C_T \left(\|z_0\|_{L^2} + \|f\|_{L^2((0,T);H_N^{-1}(0,\pi))} \right). \tag{2.2}$$

Proof. The proof of this statement is classical. We refer for example to [33]. The fact that C_T depends on T may seem unusual at first glance. However, for the Neumann-Laplacian (1.2), the eigenvector φ_0 is associated with the eigenvalue 0. Thus, the contribution of this first mode to the $L^2((0,T);H^1_N(0,\pi))$ norm of z is not bounded as $T \to +\infty$. □

Lemma 2.2. *Let Γ satisfying (1.6) and $T > 0$. There exist constants $C, \eta > 0$ such that, for every $z_0 \in L^2(0,\pi)$, $f \in L^2((0,T);H^{-1}_N(0,\pi))$ and $u \in L^\infty(0,T)$ with $\|u\|_{L^\infty} \leq \eta$, there exists a unique solution $z \in Z$ to*

$$\begin{cases} \partial_t z(t,x) - \partial_{xx} z(t,x) = u(t)\Gamma[z(t)](x) + f(t,x), \\ \partial_x z(t,0) = \partial_x z(t,\pi) = 0, \\ z(0,x) = z_0(x). \end{cases} \tag{2.3}$$

Moreover, this solution satisfies

$$\|z\|_Z \leq C \left(\|z_0\|_{L^2} + \|u\|_{L^\infty(0,T)} + \|f\|_{L^2((0,T);H^{-1}_N(0,\pi))} \right). \tag{2.4}$$

Proof. Let $T > 0$, Γ satisfying (1.6), $z_0 \in L^2(0,\pi)$, $f \in L^2((0,T),H^{-1}_N(0,\pi))$ and $u \in L^\infty(0,T)$. We construct a map $\mathcal{F} : Z \to Z$ by associating, to any $z \in Z$, the value $\mathcal{F}(z) := h$, where h is the solution to

$$\begin{cases} \partial_t h(t,x) - \partial_{xx} h(t,x) = u(t)\Gamma[z(t)](x) + f(t,x), \\ \partial_x h(t,0) = \partial_x h(t,\pi) = 0, \\ h(0,x) = z_0(x). \end{cases} \tag{2.5}$$

From the regularity assumption (1.6) on Γ, for every $z_1, z_2 \in H^1_N(0,\pi)$,

$$\|\Gamma[z_1] - \Gamma[z_2]\|_{H^{-1}_N(0,\pi)} \leq C_\Gamma \|z_1 - z_2\|_{H^1_N(0,\pi)}. \tag{2.6}$$

Then, by Lemma 2.1, the map $\mathcal{F}$ is well-defined and moreover, for every $z_1, z_2 \in Z$,

$$\|\mathcal{F}(z_1) - \mathcal{F}(z_2)\|_Z \leq C_T C_\Gamma \|u\|_{L^\infty(0,T)} \|z_1 - z_2\|_Z. \tag{2.7}$$

For $\|u\|_{L^\infty(0,T)} \leq \eta := (2C_T C_\Gamma)^{-1}$, this proves that $\mathcal{F}$ is a contraction mapping on Z. Thanks to the Banach fixed point theorem, it admits a unique fixed point. Using once more Lemma 2.1, we obtain estimate (2.4) with $C := 2C_T(\|\mu\|_{H^{-1}_N(0,\pi)} + 1)$. □

2.2 Smooth resolution of moment problems

To study the linear problem and obtain estimates useful for the nonlinear problem, we will use the well-known *moment method* introduced by Fattorini and Russel (see e.g. the seminal works [19, 20]). We start with the following results, concerning the solvability of moment problems with smooth controls.

Lemma 2.3 (Existence of biorthogonal families)**.** *Let $m \in \mathbb{N}$. There exist $C_1, T_1 > 0$, such that, for any $T \in (0, T_1]$, there exists a family $(\psi^T_{k,j})_{k\in\mathbb{N}, 0\leq j\leq m}$ of functions in $L^2(-\frac{T}{2}, \frac{T}{2})$, such that, for any $k, k' \in \mathbb{N}$ and $0 \leq j, j' \leq m$,*

$$\int_{-T/2}^{T/2} t^j \, e^{-(1+k^2)t} \, \psi^T_{k',j'}(t) \, \mathrm{d}t = \delta_{j,j'} \delta_{k,k'}, \tag{2.8}$$

which moreover satisfies, for $k \in \mathbb{N}$ and $0 \leq j \leq m$,

$$\|\psi^T_{k,j}\|_{L^2(-\frac{T}{2},\frac{T}{2})} \leq C_1 e^{kC_1 + C_1/T}. \tag{2.9}$$

Proof. This result is stated more generally in [8, Theorem 1.5] for any sequence of eigenvalues satisfying a set of appropriate assumptions. Here, the sequence of eigenvalues given by $\Lambda_k := 1 + k^2$ for $k \in \mathbb{N}$ satisfies all the required assumptions. □

Lemma 2.3 was introduced to control systems of parabolic equations, which required being able to solve moment problems with polynomial terms. Here, we consider a single scalar parabolic equation, but we wish to build regular controls and estimate their size.

Proposition 2.4 (Solvability of moment problems)**.** *Let $m \in \mathbb{N}$ and $\eta_1 > \frac{1}{2}$. There exist $M_1, T_1 > 0$, such that the following property holds. Let $T \in (0, T_1]$ and define the normed vector space*

$$D_T := \left\{ d = (d_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}, \;\; \|d\|_{D_T} := \sup_{k\in\mathbb{N}} |d_k| e^{\eta_1 T k^2} < +\infty \right\}. \tag{2.10}$$

There exists a continuous linear map $\mathfrak{L}_1^T : D_T \to H_0^m(0,T)$ such that, for any sequence $d = (d_k)_{k\in\mathbb{N}} \in D_T$, the control $u := \mathfrak{L}_1^T(d)$ satisfies, for all $k \in \mathbb{N}$, the moment condition

$$\int_0^T u(t) e^{-k^2(T-t)} \,\mathrm{d}t = d_k \tag{2.11}$$

and the size estimate

$$\|u\|_{H^m(0,T)} \leq M_1 e^{M_1/T} \|d\|_{D_T}. \tag{2.12}$$

Proof. Let $C_1, T_1 > 0$ be given by Lemma 2.3. Let $m \in \mathbb{N}$. First, there exists a constant $S_1 > 0$ such that, for any $T \in (0, T_1]$ and any $u \in H_0^m(0,T)$, one has

$$\|u\|_{H^m(0,T)} \leq S_1 \|u^{(m)}\|_{L^2(0,T)}. \tag{2.13}$$

Let $\eta_1 > \frac{1}{2}$. We define $\alpha_1 := \frac{1}{2}(\eta_1 - \frac{1}{2})$ and

$$M_1 := \max\left\{ 1 + C_1 + \frac{(C_1 + 2m)^2}{4\alpha_1}, \;\; C_1 S_1 \left(1 + m! + \frac{T_1}{1 - e^{-\alpha_1 T_1}} \right) \right\}. \tag{2.14}$$

Let $T \in (0, T_1]$ and $d \in D_T$. To build a control $u := \mathfrak{L}_1^T(d) \in H_0^m(0,T)$, we look for its m-th derivative $u^{(m)}$ under the form

$$u^{(m)}(t) = v\left(\frac{T}{2} - t\right) e^{-\left(\frac{T}{2} - t\right)}, \tag{2.15}$$

where $v \in L^2(-\frac{T}{2}, \frac{T}{2})$. Using (2.15) and iterated integration by parts, we obtain that $u \in H_0^m(0,T)$ if and only if, for $0 \leq j \leq m-1$,

$$\int_{-T/2}^{T/2} \tau^j v(\tau) e^{-\tau} \,\mathrm{d}\tau = 0. \tag{2.16}$$

Integrating by parts and using (2.16), we obtain that (2.11) for $k = 0$ is equivalent to

$$\int_{-T/2}^{T/2} \tau^m v(\tau) e^{-\tau} \,\mathrm{d}\tau = m!\, d_0 \tag{2.17}$$

and that (2.11) for $k \geq 1$ is equivalent to

$$\int_{-T/2}^{T/2} v(\tau) e^{-(1+k^2)\tau} \,\mathrm{d}\tau = (-1)^m k^{2m} e^{k^2 T/2} d_k. \tag{2.18}$$

We set

$$v(t) := m!\, d_0 \psi_{0,m}^T(t) + (-1)^m \sum_{k\geq 1} k^{2m} e^{k^2 T/2} d_k \psi_{k,0}^T(t). \tag{2.19}$$

Thanks to the size estimate of the biorthogonal family (2.9) and the decay of d_k (2.10), the above serie converges in $L^2(-T/2,T/2)$ and

$$\|v\|_{L^2} \leq m!\,|d_0|C_1 e^{C_1/T} + C_1 \sum_{k\geq 1} |d_k| k^{2m} e^{k^2T/2} e^{kC_1 + C_1/T}. \tag{2.20}$$

Thanks to the biorthogonality condition (2.8), the relations (2.16), (2.17) and (2.18) are satisfied. Thus $u \in H_0^m(0,T)$ and solves the moment problem (2.11) for $k \in \mathbb{N}$. Since $d \in D_T$, using the definition of $\|d\|_{D_T}$ in (2.10) and the relations $k^{2m} \leq e^{2mk}$ and $2\alpha_1 = \eta_1 - \frac{1}{2}$, we obtain from (2.20) that

$$\|v\|_{L^2} \leq C_1 e^{C_1/T} \left(m! + \sum_{k\geq 1} e^{(C_1+2m)k} e^{-2\alpha_1 k^2 T} \right) \|d\|_{D_T}. \tag{2.21}$$

For any $k \in \mathbb{N}$,

$$\exp\left((C_1+2m)k\right) \leq \exp\left(\alpha_1 k^2 T + \frac{(C_1+2m)^2}{4T\alpha_1} \right). \tag{2.22}$$

Moreover, for $T \in (0,T_1]$,

$$\sum_{k\geq 1} e^{-\alpha_1 T k^2} \leq \frac{1}{T} \frac{T_1}{1-e^{-\alpha_1 T_1}} \leq e^{1/T} \frac{T_1}{1-e^{-\alpha_1 T_1}}. \tag{2.23}$$

Gathering (2.13) (2.21), (2.22), and (2.23) proves the size estimate (2.12) with the claimed constant M_1 defined in (2.14). □

2.3 Cost of controllability for the linearized system

The first step of the source-term method is to compute an estimate of the cost of controllability for the linearized system (1.15). Roughly speaking, the cost of controllability is the minimal size of the controls one must use to drive an initial state to zero in a given time. This topic has received much attention. In the particular case of (1.15), we refer to the recent work [25] and the references therein.

In order for the linear system (1.15) to be small-time null controllable, it is necessary to assume that the coefficients $|\langle \mu, \varphi_k \rangle| \neq 0$ do not decay too fast. More precisely, that

$$T_\mu := \limsup_{k\to+\infty} \frac{-\log|\langle \mu, \varphi_k\rangle|}{k^2} = 0. \tag{2.24}$$

When T_μ is positive, then it is the minimal time of null controllability for the linearized system (see [1, 25] for more details). In the sequel, we always assume that (2.24) holds as we are interested in obstructions to controllability caused by the quadratic properties of the system.

In order for the linear system (1.15) to be controllable with the usual control cost for the heat equation of the form $e^{1/T}$, we must add a stronger assumption than (2.24). We assume

$$b_\mu := \limsup_{k\to+\infty} \frac{-\log|\langle \mu, \varphi_k\rangle|}{k} < +\infty. \tag{2.25}$$

Assumption (2.25) implies (2.24) and is satisfied for a very wide class of $\mu \in H_N^{-1}(0,\pi)$.

The following result is quite classical. We include a proof for the sake of completeness and because it is not so frequent to build regular controls in the dissipative case. For time-reversible systems, some authors studied the behavior of the HUM operator (see e.g. [17, 24]) or developed methods to obtain regular controls from the HUM method (see e.g. [18]).

Proposition 2.5. *Let $\mu \in H_N^{-1}(0,\pi)$. Assume that, for every $k \in \mathbb{N}$, $\langle \mu, \varphi_k \rangle \neq 0$ and that μ satisfies* (2.25)*. Let $m \in \mathbb{N}$. There exists $M_2 > 0$ such that, for any $T > 0$, there exists a continuous linear map $\mathfrak{L}_2^T : L^2(0,\pi) \to H_0^m(0,T)$ such that, for any $z_0 \in L^2(0,\pi)$, the solution $z \in Z$ to the linear system* (1.15) *with control $u := \mathfrak{L}_2^T(z_0)$ satisfies $z(T) = 0$ and*

$$\|u\|_{H^m(0,T)} \leq M_2 e^{M_2/T} \|z_0\|_{L^2(0,\pi)}. \tag{2.26}$$

Proof. From the decay assumption (2.25), there exists $k_b \in \mathbb{N}$ such that, for $k \geq k_b$, one has, for any $\tau > 0$,

$$\frac{1}{|\langle \mu, \varphi_k \rangle|} \leq e^{(b_\mu+1)k} \leq e^{(b_\mu+1)^2/\tau} e^{k^2\tau/4}. \tag{2.27}$$

Moreover, for $k \leq k_b$, one has

$$\frac{1}{|\langle \mu, \varphi_k \rangle|} \leq B_\mu := \sup_{k' \leq k_b} \frac{1}{|\langle \mu, \varphi_{k'} \rangle|} < +\infty. \tag{2.28}$$

Let $m \in \mathbb{N}$ and $M_1, T_1 > 0$ be given by Proposition 2.4 for the exponent $\eta_1 = 3/4$. We set $M := \max\{M_1, M_1 B_\mu, M_1 + (1+b_\mu)^2\}$ and $M_2 := \max\{M, M e^{M/T_1}\}$.

Let $T > 0$ and $z_0 \in L^2(0,\pi)$. Let $\tau := \min\{T, T_1\}$. Each component of the state $z(\tau)$ can be computed explicitly as

$$\langle z(\tau), \varphi_k \rangle = \langle z_0, \varphi_k \rangle e^{-k^2\tau} + \langle \mu, \varphi_k \rangle \int_0^\tau u(t) e^{-k^2(\tau - t)} \,\mathrm{d}t. \tag{2.29}$$

We construct a sequence $(d_k)_{k\in\mathbb{N}}$ as

$$d_k := -\frac{\langle z_0, \varphi_k \rangle}{\langle \mu, \varphi_k \rangle} e^{-k^2\tau}. \tag{2.30}$$

Since $|\langle z_0, \varphi_k \rangle| \leq \|z_0\|_{L^2}$, we obtain thanks to (2.27) and (2.28), for any $k \in \mathbb{N}$,

$$|d_k| \leq \|z_0\|_{L^2} \max\{B_\mu, e^{(1+b_\mu)^2/\tau}\} e^{-3k^2\tau/4}. \tag{2.31}$$

From (2.31), $d \in D_\tau$ with $\eta_1 = 3/4$. We set $u = \mathfrak{L}_2^T(z_0) := \mathfrak{L}_1^\tau(d)$, which we extend by zero on $[\tau, T]$ if $T > \tau$. From the size estimate for the resolution of the moment problem (2.12) and (2.31), we have

$$\begin{aligned}
\|u\|_{H^m(0,T)} &\leq M_1 e^{M_1/\tau} \|d\|_{D_\tau} \\
&\leq M_1 e^{M_1/\tau} \|z_0\|_{L^2} \max\{B_\mu, e^{(1+b_\mu)^2/\tau}\} \\
&\leq M e^{M/\tau} \|z_0\|_{L^2} \\
&\leq M_2 e^{M_2/T} \|z_0\|_{L^2}.
\end{aligned} \tag{2.32}$$

This concludes the proof of the cost estimate (2.26). □

2.4 Controllability despite a source term

The key point of the source term method is to prove that, if a linear system is null controllable (i.e. if one can use a control to drive a non-zero initial state back to zero), then one can also use a control to drive the state to zero despite a source term, provided that it vanishes quick enough near the final time, compared to the control cost in small time. We consider the forced version of the linear control system (1.15):

$$\begin{cases}
\partial_t z(t,x) - \partial_{xx} z(t,x) = u(t)\mu(x) + f(t,x), \\
\partial_x z(t,0) = \partial_x z(t,\pi) = 0, \\
z(0,x) = z_0(x).
\end{cases} \tag{2.33}$$

Let $\mu \in H_N^{-1}(0,\pi)$ satisfying (2.25). Let M_2 be given by Proposition 2.5. Let $T > 0$, $q \in (1, \sqrt{2})$ and $p > q^2/(2-q^2)$. We define the weights

$$\rho_0(t) := M_2^{-p} \exp\left(-\frac{pM_2}{(q-1)(T-t)}\right), \tag{2.34}$$

$$\rho_{\mathcal{F}}(t) := M_2^{-1-p} \exp\left(-\frac{(1+p)q^2 M_2}{(q-1)(T-t)}\right). \tag{2.35}$$

Then we define associated spaces for the source term, the state and the control

$$\mathcal{F} := \left\{ f \in L^2((0,T); H_N^{-1}(0,\pi)), \ \int_0^T \|f(t)\|^2_{H_N^{-1}(0,\pi)}/\rho_{\mathcal{F}}^2(t)\,\mathrm{d}t < +\infty \right\}, \tag{2.36}$$

$$\mathcal{Z} := \left\{ z \in L^2((0,T); H_N^1(0,\pi)), \ \int_0^T \|z(t)\|^2_{H_N^1(0,\pi)}/\rho_0^2(t)\,\mathrm{d}t < +\infty \right\}, \tag{2.37}$$

$$\mathcal{U} := \left\{ u \in L^\infty(0,T), \ \sup_{t\in[0,T]} |u(t)|^2/\rho_0^2(t) < +\infty \right\}. \tag{2.38}$$

Proposition 2.6. *Let $\mu \in H_N^{-1}(0,\pi)$. Assume that, for every $k \in \mathbb{N}$, $\langle \mu, \varphi_k \rangle \neq 0$ and that μ satisfies (2.25). Let $m \in \mathbb{N}^*$ and $T > 0$. There exists $C_3 > 0$ and a continuous linear map $\mathfrak{L}_3 : L^2(0,\pi) \times \mathcal{F} \to \mathcal{U} \cap H_0^m(0,T)$ such that, for any $z_0 \in L^2(0,\pi)$ and any $f \in \mathcal{F}$, the solution $z \in Z$ to (2.33) with a control $u := \mathfrak{L}_3(z_0, f)$ satisfies $z(T) = 0$ and*

$$\|z\|_{\mathcal{Z}} + \|z/\rho_0\|_{C^0([0,T];L^2(0,\pi))} + \|u\|_{\mathcal{U}} + \|u\|_{H^m(0,T)} \le C_3 \left(\|z_0\|_{L^2(0,\pi)} + \|f\|_{\mathcal{F}}\right). \tag{2.39}$$

Remark 2.7. *Proposition 2.6 is inspired by [26, Proposition 2.3]. However, we give a proof below because we introduced changes in the definitions of the functional spaces. Namely, we use forces with weaker regularity ($H_N^{-1}(0,\pi)$ instead of $H_N^1(0,\pi)$), we ask for stronger regularity on the state ($H_N^1(0,\pi)$ instead of $L^2(0,\pi)$) and stronger regularity on the constructed controls. Our proof follows the time decomposition scheme introduced in the original paper.*

Proof. For $k \ge 0$, we define $T_k := T(1-q^{-k})$. On the one hand, we let $a_0 := z_0$ and, for $k \ge 0$, we define $a_{k+1} := z_f(T_{k+1}^-)$ where z_f is the solution to

$$\begin{cases} \partial_t z_f - \Delta_N z_f = f & \text{on } (T_k, T_{k+1}), \\ z_f(T_k^+, .) = 0 \end{cases} \tag{2.40}$$

From Lemma 2.1, using the energy estimate (2.2), we have

$$\|a_{k+1}\|_{L^2(0,\pi)} \le \|z_f\|_{C^0([T_k,T_{k+1}],L^2(0,\pi))} \le C_T \|f\|_{L^2((T_k,T_{k+1}),H_N^{-1}(0,\pi))}. \tag{2.41}$$

On the other hand, for $k \ge 0$, we also consider the control systems

$$\begin{cases} \partial_t z_u - \Delta_N z_u = u_k \mu & \text{on } (T_k, T_{k+1}), \\ z_u(T_k^+) = a_k. \end{cases} \tag{2.42}$$

Using Proposition 2.5, we define $u_k \in H_0^m(T_k, T_{k+1})$ as $u_k(t) := (\mathfrak{L}_2^{T_{k+1}-T_k}(a_k))(t - T_k)$. Hence, $z_u(T_{k+1}^-) = 0$ and, thanks to the cost estimate (2.26),

$$\|u_k\|^2_{H^m(T_k,T_{k+1})} \le M_2^2 e^{2M_2/(T_{k+1}-T_k)} \|a_k\|^2_{L^2(0,\pi)}. \tag{2.43}$$

In particular, for $k = 0$,

$$\|u_0\|^2_{H^m(T_0,T_1)} \le M_2^2 e^{2qM_2/(T(q-1))} \|z_0\|^2_{L^2(0,\pi)}. \tag{2.44}$$

And, since ρ_0 is decreasing

$$\|u_0^{(m)}/\rho_0\|^2_{L^2(T_0,T_1)} + \|u_0/\rho_0\|^2_{L^\infty(T_0,T_1)} \leq (1+T)\rho_0^{-2}(T_1)M_2^2 e^{2qM_2/(T(q-1))}\|z_0\|^2_{L^2(0,\pi)}. \quad (2.45)$$

For $k \geq 0$, since $\rho_{\mathcal{F}}$ is decreasing, combining (2.41) and (2.43) yields

$$\|u_{k+1}\|^2_{H^m(T_{k+1},T_{k+2})} \leq C_T^2 M_2^2 e^{2M_2/(T_{k+2}-T_{k+1})}\rho_{\mathcal{F}}^2(T_k)\|f/\rho_{\mathcal{F}}\|^2_{L^2((T_k,T_{k+1}),H_N^{-1}(0,\pi))}. \quad (2.46)$$

In particular, since ρ_0 is decreasing and $m \geq 1$,

$$\begin{aligned}&\|u_{k+1}^{(m)}/\rho_0\|^2_{L^2(T_{k+1},T_{k+2})} + \|u_{k+1}/\rho_0\|^2_{L^\infty(T_{k+1},T_{k+2})}\\ &\leq C_T^2(1+T)M_2^2 e^{2M_2/(T_{k+2}-T_{k+1})}\rho_0^{-2}(T_{k+2})\rho_{\mathcal{F}}^2(T_k)\|f/\rho_{\mathcal{F}}\|^2_{L^2((T_k,T_{k+1}),H_N^{-1}(0,\pi))}.\end{aligned} \quad (2.47)$$

Using the definitions of the weights (2.34) and (2.35), we obtain

$$\begin{aligned}\|u_{k+1}^{(m)}/\rho_0\|^2_{L^2(T_{k+1},T_{k+2})} &+ \|u_{k+1}/\rho_0\|^2_{L^\infty(T_{k+1},T_{k+2})}\\ &\leq C_T^2(1+T)\|f/\rho_{\mathcal{F}}\|^2_{L^2((T_k,T_{k+1}),H_N^{-1}(0,\pi))}.\end{aligned} \quad (2.48)$$

As in the original proof, we can paste the controls u_k for $k \geq 0$ together by defining

$$\mathfrak{L}_3(z_0,f) := u_0(z_0) + \sum_{k\geq 1} u_k(f). \quad (2.49)$$

The concatenated control $u := \mathfrak{L}_3(z_0,f)$ remains H_0^m even across the junctions because its derivatives vanish at each T_k. And we have the estimate

$$\begin{aligned}\|u^{(m)}/\rho_0\|^2_{L^2(0,T)} + \|u/\rho_0\|^2_{L^\infty(0,T)} &\leq C_T^2(1+T)\|f\|^2_{\mathcal{F}}\\ &+ (1+T)\rho_0^{-2}(T_1)M_2^2 e^{2qM_2/(T(q-1))}\|z_0\|^2_{L^2(0,\pi)}.\end{aligned} \quad (2.50)$$

The state z can also be reconstructed by concatenation of $z_f + z_u$, which are continuous at each junction thanks to the construction. Then we estimate the state. We use the energy estimate (2.2) from Lemma 2.1 on each time interval. Hence

$$\|z_f\|^2_{C^0([T_k,T_{k+1}],L^2(0,\pi))} + \|z_f\|^2_{L^2((T_k,T_{k+1}),H_N^1(0,\pi))} \leq C_T^2\|f\|^2_{L^2((T_k,T_{k+1}),H_N^{-1}(0,\pi))}. \quad (2.51)$$

and

$$\begin{aligned}\|z_u\|^2_{C^0([T_k,T_{k+1}],L^2(0,\pi))} &+ \|z_u\|^2_{L^2((T_k,T_{k+1}),H_N^1(0,\pi))}\\ &\leq C_T^2\|a_k\|^2_{L^2} + C_T^2\|\mu\|^2_{H^{-1}(0,\pi)}\|u_k\|^2_{L^2(T_k,T_{k+1})}.\end{aligned} \quad (2.52)$$

Proceeding similarly as for the estimate on the control, we obtain respectively

$$\|z_f/\rho_0\|^2_{C^0([0;T],L^2(0,\pi))} + \|z_f/\rho_0\|^2_{L^2((0,T),H_N^1(0,\pi))} \leq C_T^2 M_2^{-2}\|f\|^2_{\mathcal{F}} \quad (2.53)$$

and

$$\begin{aligned}&\|z_u/\rho_0\|^2_{C^0([0,T],L^2(0,\pi))} + \|z_u/\rho_0\|^2_{L^2((0,T),H_N^1(0,\pi))}\\ &\quad\leq C_T^2(M_2^{-2} + \|\mu\|^2_{H^{-1}(0,\pi)})\|f\|^2_{\mathcal{F}}\\ &\quad+ C_T^2\rho_0^{-2}(T_1)\left(1+\|\mu\|^2_{H^{-1}(0,\pi)}M_2^2 e^{2qM_2/(T(q-1))}\right)\|z_0\|^2_{L^2(0,\pi)}.\end{aligned} \quad (2.54)$$

This concludes the proof of estimate (2.39) for an appropriate choice of constant C_3. The estimate of z/ρ_0 in $C^0([0,T],L^2(0,\pi))$ implies that $z(T,\cdot)=0$ because $\rho_0(T)=0$. □

2.5 Fixed-point argument for the nonlinear system

We conclude the proof of Theorem 2 thanks to a fixed point argument.

Let Γ satisfying (1.6). Let $T > 0$ and $m \in \mathbb{N}^*$. Let $C_3 > 0$ be given by Proposition 2.6. We define a small radius

$$\delta_{\mathfrak{L}} := M_2^{p-1} \left(8C_\Gamma C_3^2\right)^{-1} \tag{2.55}$$

and the associated ball of $L^2(0, \pi)$:

$$B_{\delta_{\mathfrak{L}}} := \left\{ z_0 \in L^2(0, \pi); \ \ \|z_0\|_{L^2(0,\pi)} \leq \delta_{\mathfrak{L}} \right\}. \tag{2.56}$$

Moreover, for any $r > 0$, we set

$$\mathcal{F}_r := \{ f \in \mathcal{F}; \ \ \|f\|_{\mathcal{F}} \leq r \}. \tag{2.57}$$

We construct a map $\mathcal{N} : B_{\delta_{\mathfrak{L}}} \times \mathcal{F}_{\delta_{\mathfrak{L}}} \to \mathcal{F}_{\delta_{\mathfrak{L}}}$ by setting, for $z_0 \in B_{\delta_{\mathfrak{L}}}$ and $f \in \mathcal{F}_{\delta_{\mathfrak{L}}}$,

$$\mathcal{N}(z_0, f) := u(\Gamma[z] - \Gamma[0]), \tag{2.58}$$

where $u := \mathfrak{L}_3(z_0, f)$ is given by Proposition 2.6 and z is the associated trajectory to (2.33) with initial data z_0, control u and source f.

- *First step.* For each $z_0 \in B_{\delta_{\mathfrak{L}}}$, the application $\mathcal{N}(z_0, \cdot)$ maps $\mathcal{F}_{\delta_{\mathfrak{L}}}$ to itself. Indeed, thanks to Lemma 2.8 (see below), the source-cost estimate (2.39) and the definition of $\delta_{\mathfrak{L}}$ in (2.55), one has

$$\begin{aligned} \|\mathcal{N}(z_0, f)\|_{\mathcal{F}} &\leq C_\Gamma M_2^{1-p} \|u\|_{\mathcal{U}} \|z\|_{\mathcal{Z}} \\ &\leq C_\Gamma M_2^{1-p} C_3^2 \left(\|z_0\|_{L^2(0,\pi)} + \|f\|_{\mathcal{F}} \right)^2 \\ &\leq C_\Gamma M_2^{1-p} C_3^2 \cdot 4\delta_{\mathfrak{L}}^2 \leq \frac{1}{2} \delta_{\mathfrak{L}}. \end{aligned} \tag{2.59}$$

- *Second step.* For each $z_0 \in B_{\delta_{\mathfrak{L}}}$, the application $\mathcal{N}(z_0, \cdot)$ is a contraction on $\mathcal{F}_{\delta_{\mathfrak{L}}}$ with a uniform constant $\frac{1}{2}$. Indeed, using Lemma 2.8 and Proposition 2.6 once more, for $f_1, f_2 \in \mathcal{F}_{\delta_{\mathfrak{L}}}$,

$$\begin{aligned} &\|\mathcal{N}(z_0, f_1) - \mathcal{N}(z_0, f_2)\|_{\mathcal{F}} \\ &= \|(u_1 - u_2)(\Gamma[z_1] - \Gamma[0]) + \ u_2(\Gamma[z_2] - \Gamma[z_1])\|_{\mathcal{F}} \\ &\leq C_\Gamma M_2^{1-p} \Big(\|u_1 - u_2\|_{\mathcal{U}} \|z_1\|_{\mathcal{Z}} + \|u_2\|_{\mathcal{U}} \|z_1 - z_2\|_{\mathcal{Z}} \Big) \\ &\leq C_\Gamma C_3^2 M_2^{1-p} \left(2\|z_0\|_{L^2(0,\pi)} + \|f_1\|_{\mathcal{F}} + \|f_2\|_{\mathcal{F}} \right) \|f_1 - f_2\|_{\mathcal{F}} \\ &\leq C_\Gamma C_3^2 M_2^{1-p} \cdot 4\delta_{\mathfrak{L}} \|f_1 - f_2\|_{\mathcal{F}} \leq \frac{1}{2} \|f_1 - f_2\|_{\mathcal{F}}. \end{aligned} \tag{2.60}$$

- *Third step.* Thanks to the Banach fixed point theorem, for any $z_0 \in B_{\delta_{\mathfrak{L}}}$, the application $\mathcal{N}(z_0, \cdot)$ admits a unique fixed point $f_{z_0} \in \mathcal{F}_{\delta_{\mathfrak{L}}}$. We define

$$\mathfrak{L}(z_0) := \mathfrak{L}_3(z_0, f_{z_0}) \in H_0^m(0, T). \tag{2.61}$$

  Since the application $\mathcal{N}(z_0, \cdot)$ is continuous and has a uniform contraction constant, this defines a continuous map $\mathfrak{L}$ on $B_{\delta_{\mathfrak{L}}}$.

- *Fourth step.* We estimate the size of $\mathfrak{L}(z_0)$. Let $z_0 \in B_{\delta_{\mathfrak{L}}}$ and $r := \|z_0\|_{L^2(0,\pi)} < \delta_{\mathfrak{L}}$. For $f \in \mathcal{F}_r$, repeating the same estimates as in the first step yields

$$\|\mathcal{N}(z_0, f)\|_{\mathcal{F}} \leq 4C_\Gamma C_3^2 M_2^{1-p} r^2 \leq r. \tag{2.62}$$

  Hence the application $\mathcal{N}(z_0, \cdot)$ actually leaves $\mathcal{F}_r$ invariant. This ensures that $f_{z_0} \in \mathcal{F}_r$. We conclude using (2.39) that

$$\|\mathfrak{L}(z_0)\|_{H^m(0,T)} \leq C_3 \left(\|z_0\|_{L^2(0,\pi)} + \|f_{z_0}\|_{\mathcal{F}} \right) \leq 2C_3 \|z_0\|_{L^2(0,\pi)}. \tag{2.63}$$

Therefore, the nonlinear system is smoothly small-time locally null controllable, with a control cost which depends linearly on the size of the initial data. This concludes the proof of Theorem 2.

Lemma 2.8. *Let* Γ *satisfying* (1.6)*,* $v \in \mathcal{U}$ *and* $\zeta_1, \zeta_2 \in \mathcal{Z}$*. Then*

$$\|v(\Gamma[\zeta_1] - \Gamma[\zeta_2])\|_{\mathcal{F}} \leq C_\Gamma M_2^{1-p} \|v\|_{\mathcal{U}} \|\zeta_1 - \zeta_2\|_{\mathcal{Z}}. \tag{2.64}$$

Proof. For almost every $t \in (0,T)$, using (2.38) and (1.6), one has

$$\|v(t)(\Gamma[\zeta_1(t)] - \Gamma[\zeta_2(t)])\|^2_{H_N^{-1}(0,\pi)} \leq C_\Gamma^2 \rho_0^2(t) \|v\|^2_{\mathcal{U}} \|(\zeta_1 - \zeta_2)(t)\|^2_{H_N^1(0,\pi)}. \tag{2.65}$$

Hence, using (2.36) and (2.37)

$$\|v(\Gamma[\zeta_1] - \Gamma[\zeta_2])\|^2_{\mathcal{F}} \leq C_\Gamma^2 \sup_{t\in[0,T]} \frac{\rho_0^4(t)}{\rho_{\mathcal{F}}^2(t)} \cdot \|v\|^2_{\mathcal{U}} \|\zeta_1 - \zeta_2\|^2_{\mathcal{Z}}. \tag{2.66}$$

The supremum is finite and bounded by M_2^{1-p}, provided that $p \geq q^2/(2-q^2)$, which we assumed. This concludes the proof of estimate (2.64). ☐

2.6 Controllability with linear cost and linear controllability

As stated in Remark 1.5, obtaining controllability with a linear cost for the nonlinear system implies controllability for the linear system. We provide a short proof below.

Let Γ satisfying (1.6) and $T > 0$. Let us assume that there exists $C, \delta > 0$ such that, for any $z_0 \in L^2(0,\pi)$ with $\|z_0\|_{L^2} \leq \delta$, there exists $u \in L^\infty(0,T)$ with $\|u\|_{L^\infty(0,T)} \leq C\|z_0\|_{L^2}$ such that the solution to the nonlinear system (1.1) satisfies $z(T) = 0$. We want to prove that this implies that the linear system is null-controllable in time T.

Let $z_0 \in L^2(0,\pi)$. We consider the family of initial data εz_0 for $\varepsilon > 0$. From our assumption, for ε small enough, there exists u^ε with $\|u^\varepsilon\|_{L^\infty(0,T)} \leq C\varepsilon\|z_0\|_{L^2}$ such that the associated solution $z^\varepsilon \in Z$ to the nonlinear system (1.1) satisfies $z^\varepsilon(T) = 0$. The sequence of controls $u^\varepsilon/\varepsilon$ is bounded in $L^2(0,T)$ and thus weakly converges in $L^2(0,T)$ towards some given control $u \in L^2(0,T)$. We will prove that the control u drives the initial state z_0 to 0 for the linear system (1.15).

On the one hand, let $y^\varepsilon \in Z$ be the solution to the linear system (1.15) with initial data z_0 and control $u^\varepsilon/\varepsilon$. Let y be the solution to (1.15) with initial data z_0 and control u. Then

$$\|y(T) - y^\varepsilon(T)\|^2_{L^2(0,\pi)} = \sum_{k=0}^{\infty} \left| \langle \mu, \varphi_k \rangle \left(\int_0^T \left(u(t) - \frac{u^\varepsilon(t)}{\varepsilon} \right) e^{-k^2(T-t)} \, \mathrm{d}t \right) \right|^2. \tag{2.67}$$

Since $u^\varepsilon/\varepsilon$ converges weakly to u, each term of the series (2.67) converges to 0. Moreover, using that $\mu \in H_N^{-1}(0,\pi)$, we can get a uniform bound to apply the dominated convergence theorem. Hence $y^\varepsilon(T)$ converges (strongly) towards $y(T)$ in $L^2(0,\pi)$.

On the other hand, one has

$$z^\varepsilon(T) = \varepsilon y^\varepsilon(T) + \underset{\varepsilon\to 0}{O}(\varepsilon^2). \tag{2.68}$$

Since $z^\varepsilon(T) = 0$ and $y^\varepsilon(T)$ converges to $y(T)$, this proves that $y(T) = 0$. Hence, for any $z_0 \in L^2(0,\pi)$ we build a control $u \in L^2(0,T)$ driving the solution of the linear system (1.15) to zero.

3 Obstructions caused by quadratic integer drifts

The goal of this section is to prove Theorem 3. We start by explaining the heuristic of the proof. Then, we justify the successive steps of the heuristic in the following paragraphs.

3.1 Heuristic

Let Γ satisfying (1.6) and (1.17). Let $T > 0$. Let $\eta > 0$ small enough be given by Lemma 2.2. Let $u \in L^\infty(0,T)$ with $\|u\|_{L^\infty} < \eta$. Let $\delta \in [-1,1]$. We consider the solution z to (1.1) with $z_0 = \delta\varphi_0$.

By Lemma 2.2, one may consider the solutions $z_1, z_2 \in Z$ of

- the linearized system of (1.1), i.e.,

$$\begin{cases} \partial_t z_1(t,x) - \partial_{xx} z_1(t,x) = u(t)\mu(x), \\ \partial_x z_1(t,0) = \partial_x z_1(t,\pi) = 0, \\ z_1(0,.) = 0, \end{cases} \tag{3.1}$$

  which can be explicitly computed as

$$z_1(t) = \sum_{j=0}^{\infty} \langle \mu, \varphi_j \rangle \left(\int_0^t u(\tau) e^{-j^2(t-\tau)} \,\mathrm{d}\tau \right) \varphi_j, \tag{3.2}$$

- the second-order system of (1.1), i.e.,

$$\begin{cases} \partial_t z_2(t,x) - \partial_{xx} z_2(t,x) = u(t)(\Gamma'[0] z_1(t))(x), \\ \partial_x z_2(t,0) = \partial_x z_2(t,\pi) = 0, \\ z_2(0,.) = 0. \end{cases} \tag{3.3}$$

Under appropriate assumptions, and in an appropriate sense, the nonlinear solution can be approximated by its second-order Taylor expansion with respect to u, so that one has

$$z(T) \approx \delta\varphi_0 + z_1(T) + z_2(T). \tag{3.4}$$

On the one hand, the assumption (1.17) leads to

$$\langle z_1(t), \varphi_0 \rangle = 0. \tag{3.5}$$

Thus, the component along φ_0 is not controlled on the linearized system. On the other hand, straightforward computations lead to

$$\langle z_2(T), \varphi_0 \rangle = \int_0^T u(t) \int_0^t u(\tau) K(t-\tau) \,\mathrm{d}\tau \,\mathrm{d}t, \tag{3.6}$$

where we introduce the quadratic kernel

$$K(\sigma) := \sum_{j=1}^{\infty} c_j e^{-j^2|\sigma|} \tag{3.7}$$

and the coefficients c_j are defined in (1.18). Using integration by parts, we will prove that, in an appropriate sense, there holds

$$\int_0^T u(t) \int_0^t u(\tau) K(t-\tau) \,\mathrm{d}\tau \,\mathrm{d}t \approx (-1)^n K^{(2n-1)}(0) \|u_n\|^2_{L^2(0,T)}. \tag{3.8}$$

Combining (3.5), (3.6) and (3.8) will prove the asymptotic behavior (1.24), under the assumption that the control is small in an appropriate Sobolev space. In the following paragraphs:

- we quantify (3.4) in Section 3.2, using the regularity assumption (1.6),
- we quantify (3.8) in Section 3.3, under the assumption that the final state satisfies (1.22),
- we combine these elements to conclude the proof of Theorem 3 in Section 3.4.

3.2 Approximation of the nonlinear solution

We start with a definition that lightens the notations in the sequel.

Definition 3.1. *Given two observable quantities $A(T,u,\delta)$ and $B(T,u,\delta)$ of interest, we will write $A(T,u,\delta)=\mathcal{O}(B(T,u,\delta))$ when there exist $C,T^*>0$ such that, for any $T\in(0,T^*]$, there exists $\eta>0$ such that, for any $u\in L^\infty(0,T)$ with $\|u\|_{L^\infty(0,T)}\leq\eta$ and any $\delta\in[-1,1]$, then one has the estimate $|A(T,u)|\leq C|B(T,u)|$. In particular, the following examples hold true and will be used in the sequel:*

$$\|u\|_{L^\infty(0,T)}=\mathcal{O}(T), \tag{3.9}$$

$$A=\mathcal{O}(TA+B) \quad\Rightarrow\quad A=\mathcal{O}(B). \tag{3.10}$$

Proposition 3.2. *Let Γ satisfying (1.6). For $\delta\in[-1,1]$, let z denote the solution of system (1.1) with $z_0=\delta\varphi_0$, and z_1,z_2 denote the solutions of systems (3.1), (3.3). There holds*

$$\|z-(\delta\varphi_0+z_1)\|_Z=\mathcal{O}\left(\|u\|^2_{L^\infty(0,T)}+|\delta|\|u\|_{L^\infty(0,T)}\right), \tag{3.11}$$

$$\|z-(\delta\varphi_0+z_1+z_2)\|_Z=\mathcal{O}\left(\|u\|^3_{L^\infty(0,T)}+|\delta|\|u\|_{L^\infty(0,T)}\right). \tag{3.12}$$

Proof. We denote by z^u the "pure-control" solution to

$$\begin{cases}\partial_t z^u(t,x)-\partial_{xx}z^u(t,x)=u(t)\Gamma[z^u(t)](x),\\ \partial_x z^u(t,0)=\partial_x z^u(t,\pi)=0,\\ z^u(0,x)=0,\end{cases} \tag{3.13}$$

The following estimates are direct consequences of the iterated application of Lemma 2.2 and the regularity assumption (1.6) on Γ. There holds

$$\|z^u\|_Z=\mathcal{O}\Big(\|u\|_{L^\infty(0,T)}\Big), \tag{3.14}$$

$$\|z^u-z_1\|_Z=\mathcal{O}\Big(\|u\|^2_{L^\infty(0,T)}\Big), \tag{3.15}$$

$$\|z^u-z_1-z_2\|_Z=\mathcal{O}\Big(\|u\|^3_{L^\infty(0,T)}\Big). \tag{3.16}$$

Moreover, we can write $z=\delta\varphi_0+z^u+\overline{z}$ where the function $\overline{z}$ solves

$$\begin{cases}\partial_t\overline{z}(t,x)-\partial_{xx}\overline{z}(t,x)=u(t)\left(\Gamma[z(t)]-\Gamma[z^u(t)]\right),\\ \partial_x\overline{z}(t,0)=\partial_x\overline{z}(t,\pi)=0,\\ \overline{z}(0,x)=0,\end{cases} \tag{3.17}$$

By Lemma 2.2,

$$\|\overline{z}\|_Z=\mathcal{O}\Big(|\delta|\|u\|_{L^\infty(0,T)}\Big). \tag{3.18}$$

Combining (3.15) and (3.18) proves (3.11). Combining (3.16) and (3.18) proves (3.12). □

3.3 Study of the quadratic form

Under assumption (1.19), the function K belongs to $C^{2n}([0,\infty),\mathbb{R})$ and we can integrate by parts $2n$ times in the quadratic form, which yields the following result.

Proposition 3.3. *Let $n \in \mathbb{N}^*$ and $K \in C^{2n}(0,+\infty) \cap C^{2n-1}([0,+\infty))$ with $K^{(2n)} \in L^1_{\text{loc}}([0,+\infty))$. There exists a quadratic form Q_n on $\mathbb{R}^{2n}$, such that, for $T > 0$ and $u \in L^\infty(0,T)$,*

$$\begin{aligned}\int_0^T u(t) \int_0^t u(\tau) K(t-\tau)\, \mathrm{d}\tau\, \mathrm{d}t &= (-1)^n \int_0^T u_n(t) \int_0^t u_n(\tau) K^{(2n)}(t-\tau)\, \mathrm{d}\tau\, \mathrm{d}t \\ &+ \sum_{\ell=1}^n (-1)^\ell K^{(2\ell-1)}(0) \int_0^T u_\ell(t)^2\, \mathrm{d}t + Q_n\Big(u_1(T), ..., u_n(T), \alpha_1, ..., \alpha_n\Big),\end{aligned} \tag{3.19}$$

where we use the shorthand notation, for $j \in \{1, ..., n\}$,

$$\alpha_j := \int_0^T u_n(t) K^{(n+j-1)}(T-t)\, \mathrm{d}t. \tag{3.20}$$

Proof. Let K be a function satisfying the above assumptions. We prove, by finite induction on $m \in \{0, ..., n\}$, that there exists a quadratic form Q_n^m on $\mathbb{R}^{2m}$ such that

$$\begin{aligned}\int_0^T u(t) \int_0^t u(\tau) K(t-\tau)\, \mathrm{d}\tau\, \mathrm{d}t &= (-1)^m \int_0^T u_m(t) \int_0^t u_m(\tau) K^{(2m)}(t-\tau)\, \mathrm{d}\tau\, \mathrm{d}t \\ &+ \sum_{\ell=1}^m (-1)^\ell K^{(2\ell-1)}(0) \int_0^T u_\ell(t)^2\, \mathrm{d}t + Q_n^m\Big(u_1(T), ..., u_m(T), \alpha_1, ..., \alpha_m\Big),\end{aligned} \tag{3.21}$$

with the convention that the sum is empty when $m = 0$ and $Q_n^0 := 0$, so that the equality clearly holds for $m = 0$. Let $m \in \{0, ..., n-1\}$ be such that (3.21) holds. We prove it at step $m+1$. Two integrations by part prove that

$$\begin{aligned}\int_0^T u_m(t) \int_0^t u_m(\tau) K^{(2m)}(t-\tau)\, \mathrm{d}\tau\, \mathrm{d}t &= -K^{(2m+1)}(0) \int_0^T u_{m+1}^2(t)\, \mathrm{d}t \\ &- \int_0^T u_{m+1}(t) \int_0^t u_{m+1}(\tau) K^{(2m+2)}(t-\tau)\, \mathrm{d}\tau\, \mathrm{d}t - \frac{K^{(2m)}(0)}{2} u_{m+1}(T)^2 \\ &+ u_{m+1}(T) \int_0^T u_m(\tau) K^{(2m)}(T-\tau)\, \mathrm{d}\tau.\end{aligned} \tag{3.22}$$

Moreover, the integral in the last term can be rewritten as

$$\int_0^T u_m(\tau) K^{(2m)}(T-\tau)\, \mathrm{d}\tau = \alpha_{m+1} + \sum_{\ell=1}^{n-m} u_{m+\ell}(T) K^{(2m+\ell-1)}(0). \tag{3.23}$$

This concludes the proof of (3.21) at step $m+1$. □

The following statement proves that, for particular motions of z, the boundary terms arising in Proposition 3.3 can be neglected.

Proposition 3.4. *Let Γ be such that (1.6) and (1.17) hold. Let $m \in \mathbb{N}^*$ and J be a finite subset of $\mathbb{N}^*$ with cardinal $|J| = m$ and such that $\langle \mu, \varphi_j \rangle \neq 0$ for every $j \in J$. Let z be a solution to (1.1) with $z_0 = \delta \varphi_0$ satisfying $\langle z(T), \varphi_j \rangle = 0$ for every $j \in J$. Then*

$$\sum_{\ell=1}^m |u_\ell(T)| = \mathcal{O}\Big(\sqrt{T} \|u_m\|_{L^2(0,T)} + \|u\|^2_{L^\infty(0,T)} + |\delta|\, \|u\|_{L^\infty(0,T)}\Big). \tag{3.24}$$

Proof. Let $j \in J$. Since $\langle z(T), \varphi_j \rangle = 0$, estimate (3.11) yields

$$\langle \mu, \varphi_j \rangle \int_0^T u(t) e^{-j^2(T-t)}\, \mathrm{d}t = \langle z_1(T), \varphi_j \rangle = \mathcal{O}\left(\|u\|^2_{L^\infty(0,T)} + |\delta| \|u\|_{L^\infty(0,T)}\right). \tag{3.25}$$

Then, m iterated integrations by part and the Cauchy-Schwarz inequality prove that

$$\begin{aligned} &u_1(T) - j^2 u_2(T) + ... + (-1)^{m-1} j^{2(m-1)} u_m(T) \\ =&(-1)^{m+1} j^{2m} \int_0^T u_m(t) e^{-j^2(T-t)}\,\mathrm{d}t + \int_0^T u(t) e^{-j^2(T-t)}\,\mathrm{d}t \\ =&\mathcal{O}\Big(\sqrt{T}\|u_m\|_{L^2(0,T)} + \|u\|^2_{L^\infty(0,T)} + |\delta|\|u\|_{L^\infty(0,T)}\Big). \end{aligned} \tag{3.26}$$

Let $U = (u_k(T))_{1\leq k\leq m} \in \mathbb{R}^m$ and V the Vandermonde matrix associated to the family $(-j^2)_{j\in J}$. The invertibility of V concludes the proof of (3.24). ☐

3.4 Proof of the integer drift theorem

We will use the following interpolation inequality. Such inequalities are referred to as Gagliardo-Nirenberg inequalities (see e.g. [32, Theorem p.125]).

Lemma 3.5. *Let $n \in \mathbb{N}$. There exists $C > 0$ such that, for every $T > 0$ and $v \in H^{3n+2}(0,T)$,*

$$\|v^{(n)}\|^3_{L^\infty(0,T)} \leq C\|v\|^2_{L^2(0,T)}\|v^{(3n+2)}\|_{L^1(0,T)} + CT^{-3n-\frac{3}{2}}\|v\|^3_{L^2(0,T)}. \tag{3.27}$$

We proceed as explained in the heuristic paragraph. Thanks to (3.12) of Proposition 3.2,

$$\langle z(T), \varphi_0\rangle = \epsilon\delta + \langle z_2(T), \varphi_0\rangle + \mathcal{O}\big(\|u\|^3_{L^\infty(0,T)} + |\delta|\|u\|_{L^\infty(0,T)}\big). \tag{3.28}$$

Moreover, thanks to the assumption (1.20), $K^{(2\ell-1)}(0) = 0$ for $\ell = 1, ...n-1$. Thus, applying Proposition 3.3 yields

$$\begin{aligned} \langle z_2(T), \varphi_0\rangle =& \int_0^T u(t) \int_0^t u(\tau) K(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}t \\ =&(-1)^n K^{(2n-1)}(0) \int_0^T |u_n|^2 + (-1)^n \int_0^T u_n(t) \int_0^t u_n(\tau) K^{(2n)}(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}t \\ &+ \mathcal{O}\left(\sum_{\ell=1}^n |u_\ell(T)|^2 + \left|\int_0^T u_n(t) K^{(n+\ell-1)}(t)\,\mathrm{d}t\right|^2\right). \end{aligned} \tag{3.29}$$

Thanks to assumption (1.19), $K^{(2n)} \in L^\infty(\mathbb{R})$. Thus, using the Cauchy-Schwarz inequality,

$$\langle z_2(T), \varphi_0\rangle = (-1)^n K^{(2n-1)}(0) \int_0^T |u_n|^2 + \mathcal{O}\Big(T\|u_n\|^2_{L^2(0,T)} + \sum_{\ell=1}^n |u_\ell(T)|^2\Big). \tag{3.30}$$

Applying Proposition 3.4 with $m = n$, we obtain

$$\sum_{\ell=1}^n |u_\ell(T)|^2 = \mathcal{O}\Big(T\|u_n\|^2_{L^2(0,T)} + \|u\|^3_{L^\infty(0,T)} + |\delta|\|u\|_{L^\infty}\Big). \tag{3.31}$$

Thus, we conclude that

$$\langle z(T), \varphi_0\rangle = \delta + (-1)^n K^{(2n-1)}(0) \int_0^T |u_n|^2 + \mathcal{O}\Big(T\|u_n\|^2_{L^2(0,T)} + \|u\|^3_{L^\infty(0,T)} + |\delta|\|u\|_{L^\infty}\Big). \tag{3.32}$$

Applying the Gagliardo-Nirenberg inequality (3.27) to $v = u_n$, we get

$$\begin{aligned} \|u\|^3_{L^\infty(0,T)} &\leq C\|u_n\|^2_{L^2(0,T)} \left(\|u^{(2n+2)}\|_{L^1(0,T)} + T^{-3n-\frac{3}{2}}\|u_n\|_{L^2(0,T)}\right) \\ &\leq C\|u_n\|^2_{L^2(0,T)} \left(\sqrt{T}\|u\|_{H^{2n+2}(0,T)} + T^{-2n-1}\|u\|_{L^\infty(0,T)}\right). \end{aligned} \tag{3.33}$$

Recalling (1.21), gathering (3.32) and (3.33) proves that

$$\begin{aligned}\left|\langle z(T),\varphi_0\rangle - \delta + (-1)^n a\|u_n\|^2_{L^2(0,T)}\right| &= \\ \mathcal{O}\Big((T + \|u\|_{H^{2n+2}(0,T)} + T^{-2n-1}\|u\|_{L^\infty(0,T)})\|u_n\|^2_{L^2(0,T)} &+ |\delta|\|u\|_{L^\infty}\Big).\end{aligned} \tag{3.34}$$

Expanding the definition of the notation $\mathcal{O}$, this means that, there exists $C_1, T_1 > 0$ such that, for any $T \in (0, T_1]$, there exists $\eta_1 > 0$ such that, for any $u \in L^\infty(0,T)$ with $\|u\|_{L^\infty(0,T)} \leq \eta_1$, the left-hand side is dominated by C_1 times the right-hand side.

Thus, let $\varepsilon > 0$. Let $T^* := \min\{1, T_1, C_1\varepsilon/3\}$. Let $T \in (0, T^*]$. Let $\eta := \min\{\eta_1, T^{2n+1}\varepsilon/3\}$. If $\|u\|_{H^{2n+2}(0,T)} \leq \eta$, these choices imply that

$$\left|\langle z(T),\varphi_0\rangle - \delta + (-1)^n a\|u_n\|^2_{L^2(0,T)}\right| \leq \varepsilon\left(|\delta| + \|u_n\|^2_{L^2(0,T)}\right). \tag{3.35}$$

This concludes the proof of Theorem 3.

Remark 3.6. *For the previous proof of Theorem 3 to work, it is not necessary to assume* (1.22)*. Indeed, we applied Proposition 3.4 for $m = n$ and thus we only assumed that*

$$\langle z(T),\varphi_j\rangle = 0, \quad \forall j \in J, \tag{3.36}$$

where J is any subset of $\mathbb{N}^*$ of cardinal n such that $\langle \mu,\varphi_j\rangle \neq 0$ for every $j \in J$.

As a consequence we prove the impossibility of any local motion from an initial condition of the form $z_0 \in \mathbb{R}^*_+\varphi_0$ (or $z_0 \in \mathbb{R}^*_-\varphi_0$) to a target $z_f \in \overline{\text{Span}}\ \{\varphi_j; j \in \mathbb{N}^* \setminus J\}$.

The constraints (3.36) can be interpreted as the infinite dimensional analogous of the fact that, to prove a quadratic obstruction of order n in finite dimension, one needs to impose that n linearly controllable components of the state have returned to zero (see [6] for more precise statements).

4 Obstructions caused by quadratic fractional drifts

The goal of this section is to prove Theorem 4.

4.1 Heuristic

We build upon the ideas used for the integer-order drifts. On the one side, we must study the asymptotic quadratic form. On the other side, we must determine if the quadratic approximation describes correctly the nonlinear state. We go through the following steps.

- First, we prove in Section 4.2 that, under assumption (1.27), there exist positive constants $\gamma(s)$ and $\beta(s)$ such that the quadratic state satisfies

$$\left|\langle z_2(T),\varphi_0\rangle - a\gamma(s)(-1)^n\|u\|^2_{H^{-s}(\mathbb{R})}\right| \lesssim \|u\|^2_{H^{-s-\beta}(\mathbb{R})}. \tag{4.1}$$

- Then, we prove in Section 4.3 that, for small-times, the H^{-s} drift is indeed the dominant phenomenon because

$$\|u\|^2_{H^{-s-\beta}(\mathbb{R})} \lesssim T^{2\beta}\|u\|^2_{H^{-s}(\mathbb{R})}. \tag{4.2}$$

- Moreover, we prove in Section 4.4 a fractional Gagliardo-Nirenberg type interpolation inequality in order to absorb the cubic residuals behind the fractional drift.

- Eventually, we gather these arguments to conclude the proof of Theorem 4 in Section 4.5.

4.2 Computation of the asymptotic quadratic form

We start with a technical result, which we state in a slightly more general setting, because we intend to reuse it in Section 5. For the fractional drifts case, we only intend to apply it to the constant function $\Theta \equiv 1$. For $\Theta : \mathbb{R} \to \mathbb{R}$, we define

$$\|\Theta\|_{W^{1,\infty}} := \|\Theta\|_{L^\infty(\mathbb{R})} + \|\Theta'\|_{L^\infty(\mathbb{R})}. \tag{4.3}$$

Lemma 4.1. *Let $s \in (0,1)$. There exists $\beta = \beta(s) > 0$ and $C = C(s) > 0$ such that, for every $\Theta \in W^{1,\infty}(\mathbb{R},\mathbb{R})$ and $\xi \in \mathbb{R}$ with $|\xi| \geq 1$,*

$$\left| \sum_{j=1}^{\infty} \frac{j^{3-4s}\,\Theta(\ln(j))}{j^4+\xi^2} - \int_0^\infty \frac{t^{3-4s}\,\Theta(\ln(t))}{t^4+\xi^2}\,\mathrm{d}t \right| \leq \frac{C\|\Theta\|_{W^{1,\infty}}}{|\xi|^{2(s+\beta)}}. \tag{4.4}$$

Proof. Let $s \in (0,1)$, $\Theta \in W^{1,\infty}(\mathbb{R},\mathbb{R})$ and $\xi \in \mathbb{R}$ with $|\xi| \geq 1$. The Taylor formula leads to

$$\begin{aligned}
&\sum_{j=1}^{\infty} \frac{j^{3-4s}\,\Theta(\ln(j))}{j^4+\xi^2} - \int_0^\infty \frac{t^{3-4s}\,\Theta(\ln(t))}{t^4+\xi^2}\,\mathrm{d}t \\
=&\sum_{j=1}^{\infty}\left(\frac{j^{3-4s}\,\Theta(\ln(j))}{j^4+\xi^2} - \int_{j-1}^{j} \frac{t^{3-4s}\,\Theta(\ln(t))}{t^4+\xi^2}\,\mathrm{d}t \right) \\
=&\sum_{j=1}^{\infty}\int_{j-1}^{j} (t-j+1)\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{t^{3-4s}\,\Theta(\ln(t))}{t^4+\xi^2}\right]\mathrm{d}t \\
=&\sum_{j=1}^{\infty}\int_{j-1}^{j} (t-j+1)\left(\frac{t^{2-4s}}{t^4+\xi^2}((3-4s)\Theta+\Theta')(\ln(t)) - \frac{4t^{6-4s}\,\Theta(\ln(t))}{(t^4+\xi^2)^2}\right)\mathrm{d}t.
\end{aligned} \tag{4.5}$$

Thus, using the bound on Θ,

$$\begin{aligned}
&\left| \sum_{j=1}^{\infty} \frac{j^{3-4s}\Theta(\ln(j))}{j^4+\xi^2} - \int_0^\infty \frac{t^{3-4s}\Theta(\ln(t))}{t^4+\xi^2}\,\mathrm{d}t \right| \\
&\leq 8\|\Theta\|_{W^{1,\infty}} \sum_{j=1}^{\infty}\int_{j-1}^{j} (t-j+1)\left(\frac{t^{2-4s}}{t^4+\xi^2} + \frac{t^{6-4s}}{(t^4+\xi^2)^2}\right)\mathrm{d}t \\
&\leq 8\|\Theta\|_{W^{1,\infty}}\left(\int_0^1 \frac{t^{3-4s}}{t^4+\xi^2}\,\mathrm{d}t + \int_1^\infty \frac{t^{2-4s}}{t^4+\xi^2}\,\mathrm{d}t + \int_0^\infty \frac{t^{6-4s}}{(t^4+\xi^2)^2}\,\mathrm{d}t\right).
\end{aligned} \tag{4.6}$$

Moreover, using the change of variable $t = y|\xi|^{\frac{1}{2}}$, we get as $|\xi| \to +\infty$,

$$\int_0^1 \frac{t^{3-4s}}{t^4+\xi^2}\,\mathrm{d}t = |\xi|^{-2s}\int_0^{|\xi|^{-\frac{1}{2}}} \frac{y^{3-4s}}{1+y^4}\,\mathrm{d}y = O(|\xi|^{-2s-2(1-s)}), \tag{4.7}$$

$$\int_0^\infty \frac{t^{6-4s}}{(t^4+\xi^2)^2}\,\mathrm{d}t = |\xi|^{-2s-\frac{1}{2}}\int_0^\infty \frac{y^{6-4s}}{(y^4+1)^2}\,\mathrm{d}y = O(|\xi|^{-2s-\frac{1}{2}}). \tag{4.8}$$

Eventually,

$$\begin{aligned}
\int_1^\infty \frac{t^{2-4s}}{t^4+\xi^2}\,\mathrm{d}t &= |\xi|^{-2s-\frac{1}{2}}\int_{|\xi|^{-\frac{1}{2}}}^\infty \frac{y^{2-4s}}{y^4+1}\,\mathrm{d}y \\
&= \begin{cases} O(|\xi|^{-2s-\frac{1}{2}}) & \text{when } (2-4s) > -1 \text{ i.e. } s < 3/4, \\ O(|\xi|^{-2s-\frac{1}{2}}\ln(|\xi|)) & \text{when } (2-4s) = -1 \text{ i.e. } s = 3/4, \\ O(|\xi|^{-2s-2(1-s)}) & \text{when } (2-4s) < -1 \text{ i.e. } s > 3/4. \end{cases}
\end{aligned} \tag{4.9}$$

This proves the claimed estimate with $\beta = \frac{1}{4}$ when $s \in (0, \frac{3}{4})$, with any $\beta < \frac{1}{4}$ when $s = \frac{3}{4}$ and with $\beta(s) := 1 - s > 0$ when $s \in (\frac{3}{4}, 1)$. □

We now turn to the main result of this paragraph.

Proposition 4.2. *Let $s \in (0,1)$ and $\alpha > -1 + 4s$. There exist constants $\gamma = \gamma(s) > 0$ and $\beta = \beta(s, \alpha) > 0$ such that, for every sequence $(c_j)_{j\in\mathbb{N}^*} \in \mathbb{R}^{\mathbb{N}^*}$ satisfying*

$$c_j = \frac{1}{j^{-1+4s}} + \underset{j\to+\infty}{O}\left(\frac{1}{j^{\alpha}}\right), \tag{4.10}$$

there exists a constant $C > 0$ such that, for every $T > 0$ and $u \in L^\infty(0,T)$, if $K : \mathbb{R} \to \mathbb{R}$ denotes the kernel defined by (3.7), one has

$$\left| \int_0^T u(t) \int_0^t u(\tau) K(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}t - \gamma \|u\|^2_{H^{-s}(\mathbb{R})} \right| \leq C \|u\|^2_{H^{-(s+\beta)}(\mathbb{R})}. \tag{4.11}$$

Proof. **Step 1: We prove that $K \in L^1(\mathbb{R})$ and, for every $T > 0$ and $u \in L^\infty(0,T)$,**

$$\int_0^T u(t) \int_0^t u(\tau) K(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}t = \frac{1}{8\pi^2} \int_{\mathbb{R}} |\hat{u}(\xi)|^2 \widehat{K}(\xi)\,\mathrm{d}\xi. \tag{4.12}$$

Clearly, $K \in C^0(\mathbb{R}^*, \mathbb{R})$. Moreover, for every $\kappa > 0$ there exists $c(\kappa) > 0$ such that, for every $y \in (0, \infty)$, $y^\kappa e^{-y} \leq c(\kappa)$. Thanks to the assumption (4.10), there exists $M > 0$ such that $|c_j - j^{1-4s}| \leq M j^{-\alpha}$. Hence $|c_j| \leq (M+1) j^{1-4s}$. Then, for every $\sigma \in \mathbb{R}^*$, we have

$$|K(\sigma)| \leq c(\kappa)(M+1) \left(\sum_{j=1}^{\infty} \frac{1}{j^{-1+4s+2\kappa}} \right) \frac{1}{|\sigma|^\kappa}. \tag{4.13}$$

By considering $\kappa \in (1-2s, 1)$ (resp. $\kappa > 1$), this inequality proves that K is integrable near $\sigma = 0$ (resp. at infinity). Then, recalling our choice of normalization for the Fourier transform (1.11), Fubini and Plancherel's theorems prove that

$$\begin{aligned} \int_0^T u(t) \int_0^t u(\tau) K(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}t &= \frac{1}{2} \int_0^T u(t) \int_0^T u(\tau) K(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}t \\ &= \frac{1}{2} \int_0^T u(t)(u * K)(t)\,\mathrm{d}t \\ &= \frac{1}{8\pi^2} \int_{\mathbb{R}} |\hat{u}(\xi)|^2 \widehat{K}(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{4.14}$$

We introduce the constant $\gamma(s) > 0$ which is defined for $s \in (0,1)$, as

$$\gamma(s) := \frac{1}{4\pi^2} \int_0^{+\infty} \frac{y^{3-4s}}{1+y^4}\,\mathrm{d}y. \tag{4.15}$$

Step 2: We prove that there exists $\beta = \beta(s, \alpha) > 0$ such that

$$\frac{1}{8\pi^2} \widehat{K}(\xi) = \frac{\gamma(s)}{|\xi|^{2s}} + \underset{|\xi|\to\infty}{O}\left(\frac{1}{|\xi|^{2(s+\beta)}}\right). \tag{4.16}$$

The function $x \mapsto e^{-|x|}$ has Fourier transform $\xi \mapsto \frac{2}{1+\xi^2}$ for our normalization (1.11). Thus,

$$\frac{1}{2} \widehat{K}(\xi) = \sum_{j=1}^{\infty} \frac{j^2 c_j}{j^4 + \xi^2}. \tag{4.17}$$

The change of variable $t = y|\xi|^{\frac{1}{2}}$ proves that

$$4\pi^2\gamma(s)|\xi|^{-2s} = \int_0^\infty \frac{t^{3-4s}}{\xi^2 + t^4}\,\mathrm{d}t. \tag{4.18}$$

Thus

$$\frac{1}{2}\widehat{K}(\xi) - 4\pi^2\gamma(s)|\xi|^{-2s} = f_1(\xi) + f_2(\xi), \tag{4.19}$$

where we define

$$f_1(\xi) := \sum_{j=1}^{\infty} \frac{j^{3-4s}}{j^4 + \xi^2} - \int_0^\infty \frac{t^{3-4s}}{\xi^2 + t^4}\,\mathrm{d}t, \tag{4.20}$$

$$f_2(\xi) := \sum_{j=1}^{\infty} \left(c_j - \frac{1}{j^{-1+4s}}\right) \frac{j^2}{j^4 + \xi^2}. \tag{4.21}$$

By applying Lemma 4.1 to the function $\Theta = 1$, we obtain $\beta_1 = \beta_1(s) > 0$ such that $f_1(\xi) = O(|\xi|^{-2(s+\beta)})$. Moreover, thanks to the assumption (4.10) and $\alpha > -1 + 4s$, there exists $s' \in (s, 1)$ and $M > 0$ such that

$$|f_2(\xi)| \leq \sum_{j=1}^{\infty} \frac{j^{3-4s'}}{j^4 + \xi^2} = 4\pi^2\gamma(s')|\xi|^{-2s'} + \sum_{j=1}^{\infty} \frac{j^{3-4s'}}{j^4 + \xi^2} - \int_0^\infty \frac{t^{3-4s'}}{\xi^2 + t^4}\,\mathrm{d}t. \tag{4.22}$$

By applying Lemma 4.1 to the function $\Theta = 1$, we obtain $\beta_2 = \beta_2(s, \alpha) > 0$ such that $f_2(\xi) = O(|\xi|^{-2s'}) + O(|\xi|^{-2s'-2\beta_2})$, which concludes the proof of (4.16).

Step 3: We recognize fractional Sobolev norms. We deduce from (4.16) the existence of a constant $C > 0$ such that, for every $\xi \in \mathbb{R}$,

$$\left| \frac{1}{8\pi^2}\widehat{K}(\xi) - \frac{\gamma(s)}{(1 + \xi^2)^s} \right| \leq \frac{C}{(1 + \xi^2)^{s+\beta}} \tag{4.23}$$

Using the definition (1.12) of the fractional Sobolev norms, we obtain, for every $T > 0$ and every $u \in L^\infty(0, T)$,

$$\left| \frac{1}{8\pi^2} \int_{\mathbb{R}} \widehat{K}(\xi)|\hat{u}(\xi)|^2\,\mathrm{d}\xi - \gamma\|u\|_{H^{-s}}^2 \right| \leq C\|u\|_{H^{-(s+\beta)}}^2, \tag{4.24}$$

which, together with (4.12), gives the conclusion of Proposition 4.2. ☐

4.3 Uncertainty principle and comparison of fractional norms

A non-null L^2 function with compact support in the time domain cannot have a compact support in the frequency domain. This idea is known as the *uncertainty principle* for Fourier transform. We will use the following quantitative version of it. This inequality can be deduced from the seminal works [2, 9]. See also [23, 31] for estimates of the best constant, or [10, 21, 22] for thorough reviews.

Proposition 4.3 (Uncertainty principle)**.** *There exists $C_{\mathrm{up}} > 0$ such that, for any $T > 0$ and for any $f \in L^2(\mathbb{R})$ satisfying $|\mathrm{supp}\ f| \leq T$, one has*

$$\int_{\mathbb{R}} |\hat{f}(\xi)|^2\,\mathrm{d}\xi \leq C_{\mathrm{up}} \int_{|\xi|\geq 1/T} |\hat{f}(\xi)|^2\,\mathrm{d}\xi. \tag{4.25}$$

From definition (1.12) of the negative fractional Sobolev norms, it was already clear that these norms were ordered, in the sense that $\|f\|_{H^{-s-\beta}(\mathbb{R})} \leq \|f\|_{H^{-s}(\mathbb{R})}$ for $s, \beta > 0$. Using the uncertainty principle, we prove in the following lemma that, for asymptotically small times, the weaker norms are negligible with respect to the stronger norms, up to some low-order term.

Proposition 4.4. *Let $\xi^\star \in i\mathbb{R}^*$, $s \in (0,1)$ and $\beta \in [0, 1-s]$. There exists $C > 0$ such that, for every $T \in (0,1]$ and every $f \in L^\infty(0,T)$,*

$$\|f\|^2_{H^{-(s+\beta)}(\mathbb{R})} \le CT^{2\beta}\|f\|^2_{H^{-s}(\mathbb{R})} + \frac{C}{T}|\widehat{f}(\xi^\star)|^2. \tag{4.26}$$

Proof. **Step 1: We start with the particular case when $s + \beta = 1$ and $\widehat{f}(\xi^\star) = 0$.** First, there exists $c = c(\xi^\star) > 0$ such that

$$\forall \xi \in \mathbb{R}, \quad \frac{1}{c}(1+\xi^2) \le |\xi - \xi^\star|^2 \le c(1+\xi^2). \tag{4.27}$$

Then, using the definition (1.12) of the negative Sobolev norm,

$$\|f\|^2_{H^{-1}(\mathbb{R})} = \int_{\mathbb{R}} \frac{|\widehat{f}(\xi)|^2}{1+\xi^2}\,\mathrm{d}\xi \le c\int_{\mathbb{R}} \left|\frac{\widehat{f}(\xi)}{\xi - \xi^\star}\right|^2 \mathrm{d}\xi = c\int_{\mathbb{R}} |\widehat{g}(\xi)|^2\,\mathrm{d}\xi, \tag{4.28}$$

where we define

$$\widehat{g}(\xi) := \frac{\widehat{f}(\xi)}{\xi - \xi^\star}. \tag{4.29}$$

Since f is supported in $[0,T]$, $\widehat{f}$ is an entire function of exponential type. There exists $C_f > 0$ such that

$$\forall z \in \mathbb{C}, \quad |\widehat{f}(z)| \le C_f e^{T|z|/2}. \tag{4.30}$$

Moreover, since we assumed that $\widehat{f}(\xi^\star) = 0$, (4.29) defines an entire function $\widehat{g}$ on $\mathbb{C}$. Thanks to (4.30), there exists $C_g > 0$ such that

$$\forall z \in \mathbb{C}, \quad |\widehat{g}(z)| \le C_g e^{T|z|/2}. \tag{4.31}$$

Thanks to the Paley-Wiener theorem (see e.g. [34, Theorem 19.3, p.375]), $\widehat{g}$ is the Fourier transform of a function $g \in L^2(\mathbb{R})$ with a support of size at most T thanks to (4.31). Thus, we can apply Proposition 4.3. From the uncertainty estimate (4.25), we obtain

$$\int_{\mathbb{R}} |\widehat{g}(\xi)|^2\,\mathrm{d}\omega \le C_{\mathrm{up}} \int_{|\xi| \ge 1/T} |\widehat{g}(\xi)|^2\,\mathrm{d}\xi\,. \tag{4.32}$$

Then, (4.28) and (4.27) lead to

$$\|f\|^2_{H^{-1}(0,T)} \le c^2 C_{\mathrm{up}} \int_{|\xi| \ge 1/T} \frac{|\widehat{f}(\xi)|^2}{1+\xi^2}\,\mathrm{d}\xi. \tag{4.33}$$

Moreover, for $|\xi| \ge 1/T$, one has

$$(1+\xi^2) \ge T^{-2(1-s)}(1+\xi^2)^s \tag{4.34}$$

thus

$$\|f\|^2_{H^{-1}(\mathbb{R})} \le c^2 C_{\mathrm{up}} T^{2(1-s)} \|f\|^2_{H^{-s}(\mathbb{R})}. \tag{4.35}$$

Step 2: We consider the case when $s + \beta \in (s, 1)$ and $\widehat{f}(\xi^\star) = 0$. We introduce $\theta := (1-s-\beta)/(1-s) \in (0,1)$. Using Hölder's inequality with exponents $1/\theta$ and $1/(1-\theta)$, we obtain

$$\begin{aligned} \|f\|^2_{H^{-(s+\beta)}(\mathbb{R})} &= \int_{\mathbb{R}} |\hat{f}(\xi)|^{2\theta}(1+\xi^2)^{-s\theta} \cdot |\hat{f}(\xi)|^{2(1-\theta)}(1+\xi^2)^{-\beta - s(1-\theta)}\,\mathrm{d}\xi \\ &\le \|f\|^{2\theta}_{H^{-s}(\mathbb{R})} \|f\|^{2(1-\theta)}_{H^{-1}(\mathbb{R})}. \end{aligned} \tag{4.36}$$

Thanks to the previous estimate (4.35), we obtain with $C_{s,\beta} := (cC_{\mathrm{up}}^{\frac{1}{2}})^{1-\theta}$,

$$\|f\|^2_{H^{-(s+\beta)}(\mathbb{R})} \leq C^2_{s,\beta} T^{2\beta} \|f\|^2_{H^{-s}(\mathbb{R})}. \tag{4.37}$$

Step 3: We move to the case when $\widehat{f}(\xi^\star)$ is arbitrary. Let $\xi^* = ib \in i\mathbb{R}^*$ and $\chi := \frac{\mathbf{1}_{[0,T]}}{e^{bT}-1}$. Then

$$\widehat{\chi}(\xi^\star) = \frac{1}{e^{bT}-1} \int_0^T e^{bt}\,\mathrm{d}t = 1 \tag{4.38}$$

and, for every $\sigma \in [0,1]$,

$$\|\chi\|_{H^{-\sigma}(\mathbb{R})} \leq \|\chi\|_{L^2(0,T)} = \frac{\sqrt{T}}{e^{bT}-1} \leq \frac{C}{\sqrt{T}}. \tag{4.39}$$

Applying (4.37) to the function $\widetilde{f}(t) := f(t) - \widehat{f}(\xi^\star)\chi(t)$, and the triangular inequality, we get

$$\begin{aligned} \|f\|_{H^{-(s+\beta)}(\mathbb{R})} &\leq \|\widetilde{f}\|_{H^{-(s+\beta)}(\mathbb{R})} + |\widehat{f}(\xi^\star)| \|\chi\|_{H^{-(s+\beta)}(\mathbb{R})} \\ &\leq C_{s,\beta} T^\beta \|\widetilde{f}\|_{H^{-s}(\mathbb{R})} + |\widehat{f}(\xi^\star)| \|\chi\|_{H^{-(s+\beta)}(\mathbb{R})} \\ &\leq C_{s,\beta} T^\beta \|f\|_{H^{-s}(\mathbb{R})} + |\widehat{f}(\xi^\star)| \Big(\|\chi\|_{H^{-(s+\beta)}(\mathbb{R})} + C_{s,\beta} T^\beta \|\chi\|_{H^{-s}(\mathbb{R})} \Big) \\ &\leq C_{s,\beta} T^\beta \|f\|_{H^{-s}(\mathbb{R})} + \frac{C}{\sqrt{T}} |\widehat{f}(\xi^\star)|\,. \end{aligned} \tag{4.40}$$

This concludes the proof of (4.26) in all cases. ☐

Lemma 4.5. *There exists $C > 0$ such that, for $s \in (0,1)$, $T \in (0,1]$, $u \in L^\infty(0,T)$ and $n \in \mathbb{N}$,*

$$\int_0^T |u_{n+1}(t)|^2\,\mathrm{d}t \leq C \left(T^{2(1-s)} \|u_n\|^2_{H^{-s}(\mathbb{R})} + T|u_{n+1}(T)|^2 \right). \tag{4.41}$$

Proof. It is sufficient to work with $n = 0$. The function u_1 is extended by zero outside $(0,T)$. By Plancherel's equality and Proposition 4.3, we have

$$\int_0^T |u_1(t)|^2\,\mathrm{d}t = \frac{1}{4\pi^2} \int_{\mathbb{R}} |\widehat{u_1}(\xi)|^2\,\mathrm{d}\xi \leq \frac{C_{\mathrm{up}}}{4\pi^2} \int_{|\xi|>1/T} |\widehat{u_1}(\xi)|^2\,\mathrm{d}\xi. \tag{4.42}$$

Moreover, using integration by parts, we obtain

$$\widehat{u_1}(\xi) = \int_0^T u_1(t) e^{-it\xi}\,\mathrm{d}t = \frac{\widehat{u}(\xi)}{i\xi} - \frac{u_1(T)e^{-iT\xi}}{i\xi}. \tag{4.43}$$

Thus,

$$\begin{aligned} \int_{|\xi|>1/T} |\widehat{u_1}(\xi)|^2\,\mathrm{d}\xi &\leq 2 \int_{|\xi|>1/T} \left| \frac{\widehat{u}(\xi)}{\xi} \right|^2 \mathrm{d}\xi + 2|u_1(T)|^2 \int_{|\xi|>1/T} \frac{\mathrm{d}\xi}{\xi^2} \\ &\leq 4T^{2(1-s)} \int_{|\xi|>1/T} \frac{|\widehat{u}(\xi)|}{(1+\xi^2)^s}\,\mathrm{d}\xi + 4T|u_1(T)|^2\,. \end{aligned} \tag{4.44}$$

Indeed, taking into account that $T \leq 1$, we have, for every $\xi \geq 1/T$,

$$\xi^2 \geq \frac{1}{2}(1+\xi)^2 \geq \frac{T^{-2(1-s)}}{2}(1+\xi)^{2s}. \tag{4.45}$$

Gathering these estimates concludes the proof of (4.41). ☐

4.4 A fractional Gagliardo-Nirenberg inequality

In order to bound the cubic terms by the quadratic drift, we will use an interpolation inequality, similar to the one stated in Lemma 3.5 for the integer-order case, but adapted to our fractional setting. We start with a weighted Young convolution inequality.

Lemma 4.6. *Let $s \in (0,1)$. There exists $C > 0$ such that, for $f \in L^2(\mathbb{R})$ and $g \in L^1((1+x^2)^{\frac{s}{2}}\,\mathrm{d}x)$,*

$$\int_{\mathbb{R}} \frac{|(f*g)(x)|^2}{(1+x^2)^s}\,\mathrm{d}x \le C \left(\int_{\mathbb{R}} \frac{|f(x)|^2}{(1+x^2)^s}\,\mathrm{d}x\right)\left(\int_{\mathbb{R}} |g(x)|(1+x^2)^{\frac{s}{2}}\,\mathrm{d}x\right)^2. \tag{4.46}$$

Proof. We use the duality caracterization of the $L^2\left(\frac{\mathrm{d}x}{(1+x^2)^s}\right)$-norm. Let $\phi \in L^2\left(\frac{\mathrm{d}x}{(1+x^2)^s}\right)$. Using the Fubini theorem and the relation

$$1+(x-y)^2 \le 1+2x^2+2y^2 \le 2(1+x^2)(1+y^2)\,, \quad \forall x,y \in \mathbb{R}, \tag{4.47}$$

we obtain

$$\begin{aligned}\int_{\mathbb{R}} |(f*g)(x)\phi(x)| \frac{\mathrm{d}x}{(1+x^2)^s} &= \int_{\mathbb{R}} \Big| \int_{\mathbb{R}} f(x-y)g(y)\,\mathrm{d}y \Big| |\phi(x)| \frac{\mathrm{d}x}{(1+x^2)^s} \\ &\le \int_{\mathbb{R}} |g(y)| \int_{\mathbb{R}} \frac{|f(x-y)|}{(1+|x-y|^2)^{\frac{s}{2}}} \frac{\phi(x)}{(1+x^2)^{\frac{s}{2}}} \left(\frac{1+|x-y|^2}{1+x^2}\right)^{\frac{s}{2}} \mathrm{d}x\,\mathrm{d}y \\ &\le 2^{\frac{s}{2}} \left(\int_{\mathbb{R}} (1+y^2)^{\frac{s}{2}} |g(y)|\,\mathrm{d}y\right)\left(\int_{\mathbb{R}} \frac{|f(z)|^2}{(1+z^2)^s}\,\mathrm{d}z\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} \frac{|\phi(x)|^2}{(1+x^2)^s}\,\mathrm{d}x\right)^{\frac{1}{2}},\end{aligned} \tag{4.48}$$

which gives the conclusion. □

Now we can prove the following fractional Gagliardo-Nirenberg interpolation inequality. For a recent reference tackling the fractional case with optimal norms and constants, we refer to [30]. Although it is not optimal, we will use the following statement for which we provide a detailed proof, because it mixes Sobolev norms on $(0,T)$ and Sobolev norms on $\mathbb{R}$.

Proposition 4.7. *Let $n \in \mathbb{N}$, $s \in (0,1)$ and $\nu > 0$. There exists a constant $C > 0$ such that, for $T \in (0,1]$ and $v \in H^{3n+2s+\frac{3}{2}+\nu}(0,T)$, there holds*

$$\|v^{(n)}\|^3_{L^\infty(0,T)} \le \frac{C}{T^{3(n+1)}} \|v\|^2_{H^{-s}(\mathbb{R})} \|v\|_{H^{3n+2s+\frac{3}{2}+\nu}(0,T)}. \tag{4.49}$$

Proof. **Step 1:** We prove the existence of $C_1 > 0$ such that, for every $f \in H^{3n+\frac{3}{2}+2s+\nu}(\mathbb{R})$,

$$\|f^{(n)}\|^3_{L^\infty(\mathbb{R})} \le C_1 \|f\|^2_{H^{-s}(\mathbb{R})} \|f\|_{H^{3n+\frac{3}{2}+2s+\nu}(\mathbb{R})}. \tag{4.50}$$

By using the inverse Fourier transform, the Cauchy-Schwarz inequality, and Hölder inequality (with

$p = 3/2$ and $p' = 3$) we obtain, for every $t \in \mathbb{R}$,

$$
\begin{aligned}
|f^{(n)}(t)|^2 &= \left| \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f^{(n)}}(\xi) e^{it\xi} \, \mathrm{d}\xi \right|^2 \\
&\leq C \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \xi^{2n} (1+\xi^2)^{\frac{1}{2}+\frac{\nu}{3}} \, \mathrm{d}\xi \\
&\leq C \int_{\mathbb{R}} \frac{|\widehat{f}(\xi)|^{\frac{4}{3}}}{(1+\xi^2)^{\frac{2s}{3}}} (1+\xi^2)^{\frac{1}{2}+\frac{\nu}{3}+\frac{2s}{3}} \xi^{2n} |\widehat{f}(\xi)|^{\frac{2}{3}} \, \mathrm{d}\xi \\
&\leq C \left(\int_{\mathbb{R}} \frac{|\widehat{f}(\xi)|^2}{(1+\xi^2)^s} \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}} (1+\xi^2)^{\frac{3}{2}+\nu+2s} \xi^{6n} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi \right)^{\frac{1}{3}} \\
&\leq C \|f|^{\frac{4}{3}}_{H^{-s}(\mathbb{R})} \left(\int_{\mathbb{R}} (1+\xi^2)^{\frac{3}{2}+\nu+2s} |\xi^{3n} \widehat{f}(\xi)|^2 \, \mathrm{d}\xi \right)^{\frac{1}{3}} \\
&\leq C \|f\|^{\frac{4}{3}}_{H^{-s}(\mathbb{R})} \|f^{(3n)}\|^{\frac{2}{3}}_{H^{\frac{3}{2}+\nu+2s}(\mathbb{R})}
\end{aligned}
\tag{4.51}
$$

Step 2: We construct a continuous extension operator $P : L^2(0,1) \to L^2(\mathbb{R})$ that maps continuously $H^{3n+4}(0,1)$ into $H^{3n+4}(\mathbb{R})$:

$$
\|P(v)\|_{H^{3n+4}(\mathbb{R})} \leq C_2 \|v\|_{H^{3n+4}(0,1)} , \quad \forall v \in H^{3n+4}(0,1). \tag{4.52}
$$

To simplify the notations, we denote by m the integer $3n + 4$ in this step. We also identify a function $v \in L^2(0,1)$ with its extension by zero to the whole real line. Let $(\alpha_1, ..., \alpha_m) \in \mathbb{R}^m$ be the solution of the Vandermonde linear system

$$
\sum_{\ell=1}^{m} \alpha_\ell \left(-\frac{\ell}{m} \right)^k = 1 , \quad \forall k \in \{0, ..., m-1\} . \tag{4.53}
$$

We define an extension operator $\widetilde{P} : L^2(0,1) \to L^2(-2,3)$ by

$$
\widetilde{P}(v)(t) := \begin{cases} v(t) & \text{if } t \in [0,1] , \\ \displaystyle\sum_{\ell=1}^{m} \alpha_\ell v \left(-\frac{\ell\, t}{m} \right) & \text{if } t \in (-1,0) , \\ \displaystyle\sum_{\ell=1}^{m} \alpha_\ell v \left(1 - \frac{\ell(t-1)}{m} \right) & \text{if } t \in (1,2) . \end{cases} \tag{4.54}
$$

Clearly, $\widetilde{P}$ maps continuously $L^2(0,1)$ into $L^2(-1,2)$. The relation (4.53) ensures, for every $k \in \{0, ..., m-1\}$, the continuity of $\widetilde{P}(v)^{(k)}$ at $t = 0$ and $t = T$ when $v \in C^k([0,T], \mathbb{R})$. Thus $\widetilde{P}$ maps continuously $H^m(0,1)$ into $H^m(-1,2)$. Let $\chi \in C_c^\infty(\mathbb{R}, \mathbb{R})$ be such that $\chi = 1$ on $[0,1]$ and $\mathrm{supp}(\chi) \subset (-1,2)$. Then the operator P defined by $P(v) := \chi\, \widetilde{P}(v)$ gives the conclusion of Step 2.

Then, by interpolation, the extension operator P maps continuously $H^a(0,T)$ into $H^a(\mathbb{R})$ for every $a \in (0, 3n+4)$:

$$
\|P(v)\|_{H^a(\mathbb{R})} \leq C_2 \|v\|_{H^a(0,1)} , \quad \forall v \in H^a(0,1) , \forall a \in [0, 3n+4]. \tag{4.55}
$$

Step 3: We prove the existence of $C_3 > 0$ such that, for every $v \in L^2(0,1)$,

$$
\|P(v)\|_{H^{-s}(\mathbb{R})} \leq C_3 \|v\|_{H^{-s}(\mathbb{R})}. \tag{4.56}
$$

Identifying v with its extension to $\mathbb{R}$ by zero, we obtain

$$P(v) - v = \chi \sum_{\ell=1}^{m} \alpha_\ell \, v_{0,\ell} + \chi \sum_{\ell=1}^{m} \alpha_\ell \, v_{1,\ell}, \tag{4.57}$$

where, we set, for every $t \in \mathbb{R}$,

$$v_{0,\ell}(t) := v\left(-\frac{\ell\, t}{m}\right) \quad \text{ and } \quad v_{1,\ell}(t) := v\left(1 - \frac{\ell(t-1)}{m}\right). \tag{4.58}$$

Thus

$$\widehat{P(v)}(\xi) - \widehat{v}(\xi) = \sum_{\ell=1}^{m} \alpha_\ell \, \widehat{\chi} * \widehat{v_{0,\ell}} + \sum_{\ell=1}^{m} \alpha_\ell \, \widehat{\chi} * \widehat{v_{1,\ell}}, \tag{4.59}$$

where

$$\widehat{v_{0,\ell}}(\xi) = \widehat{v}\left(-\frac{m\xi}{\ell}\right) \quad \text{ and } \quad \widehat{v_{1,\ell}}(\xi) = \widehat{v}\left(-\frac{m\xi}{\ell}\right) e^{i\left(1+\frac{m}{\ell}\right)\xi}. \tag{4.60}$$

Taking into account that

$$\int_{\mathbb{R}} (1+\xi^2)^{\frac{s}{2}} |\widehat{\chi}(\xi)| \,\mathrm{d}\xi \le \left(\int_{\mathbb{R}} (1+\xi^2)^{1+\frac{s}{2}} |\widehat{\chi}(\xi)|^2 \,\mathrm{d}\xi\right)^{1/2} \left(\int_{\mathbb{R}} \frac{\mathrm{d}\xi}{1+\xi^2}\right)^{1/2} < \infty, \tag{4.61}$$

Lemma 4.6 proves the existence of a constant $C_3 = C_3(\chi) > 0$ such that

$$\int_{\mathbb{R}} \frac{|\widehat{P(v)}(\xi) - \widehat{v}(\xi)|^2}{(1+\xi^2)^s} \,\mathrm{d}\xi \le C_3 \int_{\mathbb{R}} \frac{|\widehat{v}(\xi)|^2}{(1+\xi^2)^s} \,\mathrm{d}\xi, \tag{4.62}$$

which gives the conclusion.

Step 4: We prove inequality (4.49) when $T = 1$. For $v \in H^{3n+2s+\frac{3}{2}+\nu}(0,1)$, we have

$$\begin{aligned} \|v\|^3_{L^\infty(0,1)} &\le \|P(v)\|^3_{L^\infty(\mathbb{R})} \\ &\le C_1 \|P(v)\|_{H^{-s}(\mathbb{R})} \|P(v)\|_{H^{3n+2s+\frac{3}{2}+\nu}(\mathbb{R})} \text{ by Step 1} \\ &\le C_1 C_2 C_3^2 \|v\|^2_{H^{-s}(\mathbb{R})} \|v\|_{H^{3n+2s+\frac{3}{2}+\nu}(0,1)} \text{ by Steps 2 and 3.} \end{aligned} \tag{4.63}$$

Step 5: We prove inequality (4.49) with an arbitrary $T \in (0,1)$. Let $v \in H^{3n+2s+\frac{3}{2}+\nu}(0,T)$. We consider the function $w : (0,1) \to \mathbb{R}$ defined by $w(\theta) = v(T\theta)$. Then, for every $T \in (0,1]$,

$$\|w^{(n)}\|^3_{L^\infty(0,1)} = T^{3n} \|v^{(n)}\|^3_{L^\infty(0,T)}, \tag{4.64}$$

$$\|w\|_{H^{3n+2s+\frac{3}{2}+\nu}(0,1)} \le \|v\|_{H^{3n+2s+\frac{3}{2}+\nu}(0,T)} \tag{4.65}$$

and, taking into account that $\widehat{w}(\xi) = \frac{1}{T}\widehat{v}\left(\frac{\xi}{T}\right)$,

$$\|w\|^2_{H^{-s}(\mathbb{R})} = \frac{1}{T}\int_{\mathbb{R}} \frac{|\widehat{v}(\eta)|^2}{(1+T^2\eta^2)^s} d\eta \le \frac{1}{T^{1+2s}} \int_{\mathbb{R}} \frac{|\widehat{v}(\eta)|}{(1+\eta^2)^s} d\eta = \frac{\|v\|^2_{H^{-s}(\mathbb{R})}}{T^{1+2s}}. \tag{4.66}$$

Finally, applying Step 4 to w gives

$$\|v^{(n)}\|^3_{L^\infty(0,T)} \le \frac{C_1 C_2 C_3^2}{T^{3n+2s+1}} \|v\|^2_{H^{-s}(\mathbb{R})} \|v\|_{H^{3n+2s+\frac{3}{2}+\nu}(0,T)}. \tag{4.67}$$

This concludes the proof of (4.49) in the general case. □

4.5 Proof of the fractional drift theorem

Let $n \in \mathbb{N}$, $s \in (0,1)$, $\nu > 0$, Γ be such that (1.6) and (1.17) hold and the coefficients c_j defined by (1.18) satisfy (1.27) for some $a \in \mathbb{R}^*$ and $\alpha > -1 + 4s$. We also assume that (1.28) holds.

Working as in Section 3.4, we obtain

$$\langle z(T), \varphi_0 \rangle = \delta + \langle z_2(T), \varphi_0 \rangle + \mathcal{O}\left(\|u\|^3_{L^\infty(0,T)} + \delta \|u\|_{L^\infty(0,T)} \right). \tag{4.68}$$

By Proposition 3.3 and assumption (1.28), we obtain

$$\langle z_2(T), \varphi_0 \rangle =(-1)^n \int_0^T u_n(t) \int_0^t u_n(\tau) K^{(2n)}(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}t + \mathcal{O}\left(\sum_{\ell=1}^n |\alpha_\ell|^2 + |u_\ell(T)|^2 \right). \tag{4.69}$$

Moreover, one can check (see Lemma 4.8 below) that there exists $C, \beta > 0$ such that

$$\sum_{\ell=1}^n |\alpha_\ell|^2 \leq C \|u_n\|^2_{H^{-s-\beta}(\mathbb{R})}. \tag{4.70}$$

Up to choosing a smaller β, we deduce from Proposition 4.2 that

$$\langle z_2(T), \varphi_0 \rangle = (-1)^n a \gamma(s) \|u_n\|^2_{H^{-s}(\mathbb{R})} + \mathcal{O}\left(\|u_n\|^2_{H^{-s-\beta}(\mathbb{R})} + \sum_{\ell=1}^n |u_\ell(T)|^2 \right). \tag{4.71}$$

Applying Proposition 4.4 to $f = u_n$ and $\xi^* = i$, we obtain

$$\langle z_2(T), \varphi_0 \rangle = (-1)^n a \gamma(s) \|u_n\|^2_{H^{-s}(\mathbb{R})} + \mathcal{O}\left(T^{2\beta} \|u_n\|^2_{H^{-s}(\mathbb{R})} + \frac{|\widehat{u_n}(i)|^2}{T} + \sum_{\ell=1}^n |u_\ell(T)|^2 \right). \tag{4.72}$$

Moreover, an integration by parts and the Cauchy-Schwarz formula prove that

$$\frac{1}{T} |\widehat{u_n}(i)|^2 = \frac{1}{T} \left| u_{n+1}(T) e^T - \int_0^T u_{n+1}(t) e^t \,\mathrm{d}t \right|^2 = \mathcal{O}\left(\frac{|u_{n+1}(T)|^2}{T} + \|u_{n+1}\|^2_{L^2(0,T)} \right). \tag{4.73}$$

By Proposition 3.4 applied with $m = n+1$ (assumption (1.27) implies that an infinite number of coefficients $\langle \mu, \varphi_j \rangle$ are non-zero),

$$\frac{1}{T} \sum_{\ell=1}^{n+1} |u_\ell(T)|^2 = \mathcal{O}\left(\|u_{n+1}\|^2_{L^2(0,T)} + \frac{\|u\|^4_{L^\infty(0,T)}}{T} + \frac{\delta^2 \|u\|^2_{L^\infty(0,T)}}{T} \right) \tag{4.74}$$

and by Lemma 4.5, we have

$$\|u_{n+1}\|^2_{L^2(0,T)} = \mathcal{O}\left(T^{2(1-s)} \|u_n\|^2_{H^{-s}(\mathbb{R})} + T |u_{n+1}(T)|^2 \right). \tag{4.75}$$

Combining (4.74) and (4.75) with the properties (3.9) and (3.10) of the definition of the notation $\mathcal{O}$, which includes smallness assumptions on T and on u, we deduce

$$\|u_{n+1}\|^2_{L^2(0,T)} + \frac{1}{T} \sum_{\ell=1}^{n+1} |u_\ell(T)|^2 = \mathcal{O}\left(T^{2(1-s)} \|u_n\|^2_{H^{-s}(\mathbb{R})} + \|u\|^3_{L^\infty(0,T)} + \delta \|u\|_{L^\infty(0,T)} \right). \tag{4.76}$$

Finally, incorporating (4.73) and (4.76) into (4.72), we get

$$\begin{aligned} \langle z(T), \varphi_0 \rangle &= \delta + (-1)^n a \gamma(s) \|u_n\|^2_{H^{-s}(\mathbb{R})} \\ &\quad + \mathcal{O}\left(T^\rho \|u_n\|^2_{H^{-s}(\mathbb{R})} + \|u\|^3_{L^\infty(0,T)} + \delta \|u\|_{L^\infty(0,T)} \right). \end{aligned} \tag{4.77}$$

where $\rho := \min\{2\beta, 2(1-s)\} > 0$. Applying the Gagliardo-Nirenberg inequality of Proposition 4.7 to $v = u_n$ we get

$$\|u\|_{L^\infty(0,T)}^3 = \|u_n^{(n)}\|_{L^\infty(0,T)}^3 = \mathcal{O}\left(\frac{1}{T^{3(n+1)}}\|u_n\|_{H^{-s}(\mathbb{R})}^2\|u_n\|_{H^{3n+2s+\frac{3}{2}+\nu}(0,T)}\right). \tag{4.78}$$

Moreover, for every $k \in \{0, ..., n-1\}$,

$$\|u_n^{(k)}\|_{L^2(0,T)} = \|u_{n-k}\|_{L^2(0,T)} \leq T^{n-k}\|u\|_{L^2(0,T)} \tag{4.79}$$

thus

$$\|u_n\|_{H^{3n+2s+\frac{3}{2}+\nu}(0,T)} = \mathcal{O}\left(\|u\|_{H^{2n+2s+\frac{3}{2}+\nu}(0,T)}\right) \tag{4.80}$$

and

$$\|u\|_{L^\infty(0,T)}^3 = \mathcal{O}\left(\frac{1}{T^{3(n+1)}}\|u_n\|_{H^{-s}(\mathbb{R})}^2\|u\|_{H^{2n+2s+\frac{3}{2}+\nu}(0,T)}\right). \tag{4.81}$$

Incorporating the previous relation in (4.77), we get

$$\begin{aligned}\langle z(T), \varphi_0\rangle = \delta + (-1)^n a\gamma(s)\|u_n\|_{H^{-s}(\mathbb{R})}^2 \\ + \mathcal{O}\left(\left(T^\rho + \frac{\|u\|_{H^{2n+2s+\frac{3}{2}+\nu}(0,T)}}{T^{3(n+1)}}\right)\|u_n\|_{H^{-s}(\mathbb{R})}^2 + \delta\|u\|_{L^\infty(0,T)}\right).\end{aligned} \tag{4.82}$$

We conclude as in the integer-order drift case, thanks to the definition of the $\mathcal{O}$ notation.

Lemma 4.8. *Under the decay assumption* (1.27) *on the coefficients c_j, there exists $C, \beta > 0$ such that, for any $\ell \in \{1, ...n\}$, one has*

$$\left|\int_0^T u_n(t)K^{(n+\ell-1)}(T-t)\,\mathrm{d}t\right| \leq C\|u_n\|_{H^{-s-\beta}(\mathbb{R})}. \tag{4.83}$$

Proof. We extend u_n by zero to $\mathbb{R}$. Seeing the integral as an $L^2(\mathbb{R})$ scalar product, we obtain, thanks to Plancherel's theorem

$$\left|\int_0^T u_n(t)K^{(n+\ell-1)}(T-t)\,\mathrm{d}t\right| \leq \frac{1}{4\pi^2}\sum_{j\in\mathbb{N}^*}|c_j|j^{2(n+\ell-1)}\int_{\mathbb{R}}|\widehat{u_n}(\xi)|\frac{2j^2}{j^4+\xi^2}\,\mathrm{d}\xi. \tag{4.84}$$

Let $\beta \in (0, 1-s)$. Estimate (4.83) holds with

$$C := \frac{1}{2\pi^2}\sum_{j\in\mathbb{N}^*}|c_j|j^{4n}\left(\int_{\mathbb{R}}\frac{(1+\xi^2)^{s+\beta}}{(j^4+\xi^2)^2}\,\mathrm{d}\xi\right)^{\frac{1}{2}}. \tag{4.85}$$

Thanks to assumption (1.27), there exists $M_1 > 0$ such that $|c_j| \leq M_1 j^{1-4n-4s}$. Moreover, there exists $M_2 > 0$ such that

$$\left(\int_{\mathbb{R}}\frac{(1+\xi^2)^{s+\beta}}{(j^4+\xi^2)^2}\,\mathrm{d}\xi\right)^{\frac{1}{2}} \leq M_2 j^{2s+2\beta-3}. \tag{4.86}$$

Hence, the constant C is finite when $\beta < s + \frac{1}{2}$. ∎

4.6 A methodological remark on more complex systems

Thanks to our particular choice of nonlinear system (1.1), the quadratic integral operators that we manipulated in this section are associated with kernels of the form $K(t-\tau)$. Hence, we were able to interpret the quadratic operators in the Fourier time frequency domain and perform explicit computations. We chose this setting for the ease of presentation. However, the same kind of results can be obtained if the kernel is of the form $K(t,\tau)$. For such kernels, it is harder to perform the computations in the frequency domain. Instead, one can study the degeneracy of $K(t,\tau)$ near the diagonal $t=\tau$ to compute the associated coercivity. The residues can then be estimated using the theory of *weakly singular integral operators* (see e.g. [35, 36]). Such a study was performed by the second author in [27] in the case of a nonlinear Burgers equation, establishing a drift in $H^{-5/4}$ norm.

5 Controllability stemming from the second order

We prove Theorem 5, which illustrates that, for scalar-input parabolic systems, small-time null controllability can sometimes be recovered from the quadratic approximation.

5.1 Construction of a magic system

We construct a nonlinearity Γ satisfying (1.6) with good properties. We perform this construction as a very first step, to stress that the choice of Γ does not depend on the control time T. From the previous sections concerning the fractional obstructions to controllability, we guess that we must build a quadratic kernel whose Fourier transform takes both positive and negative values up to infinity. We start with the following elementary lemma.

Lemma 5.1. *Let $s\in(0,1)$ and $\epsilon>0$. There exists a constant $L=L(\epsilon,s)\geq 1$ such that*

$$\int_{\mathbb{R}_+\setminus[e^{-L},e^{L}]} \frac{y^{3-4s}}{1+y^4}\,\mathrm{d}y \leq \epsilon. \tag{5.1}$$

Proof. For $s\in(0,1)$ the integral over $\mathbb{R}_+$ is finite. □

We can now turn to the construction of the system. Let $\Theta_1\in W^{1,\infty}(\mathbb{R})$ be a function of period 4 defined by its values for $x\in[0,4]$:

$$\Theta_1(x) := \begin{cases} +1 & \text{for } x\in[0,1],\\ +1-2(x-1) & \text{for } x\in[1,2],\\ -1 & \text{for } x\in[2,3],\\ -1+2(x-3) & \text{for } x\in[3,4]. \end{cases} \tag{5.2}$$

By construction $\|\Theta_1\|_{W^{1,\infty}} := \|\Theta_1\|_{L^\infty} + \|\Theta_1'\|_{L^\infty} = 3$ and there exist infinitely many plateaus where $\Theta_1=\pm 1$. Let $s\in(0,\frac{1}{4})$. Let $L\geq 1$ be given by Lemma 5.1 for $\epsilon=\pi\gamma(s)/10$, where $\gamma(s)$ is defined in (4.15). We set

$$\Theta(x) := \Theta_1\left(\frac{x}{2L}\right). \tag{5.3}$$

Since $L\geq 1$, there holds $\|\Theta\|_{W^{1,\infty}}\leq 2$. We define, for $z\in H^1_N(0,\pi)$,

$$\Gamma_\Theta(z) := \sum_{k\in\mathbb{N}^*} k^{\frac{1}{2}-2s}\varphi_k + \left(\sum_{j\in\mathbb{N}^*} \Theta(\ln j) j^{\frac{1}{2}-2s}\langle z,\varphi_j\rangle\right)\varphi_0 \tag{5.4}$$

Elementary computations prove that definition (5.4) ensures that Γ satisfies the regularity assumptions (1.6). The first direction φ_0 is lost since $\langle \Gamma_\Theta[0], \varphi_0 \rangle = 0$. Moreover, for $j \in \mathbb{N}^*$, the coefficients c_j defined by (1.18) satisfy

$$c_j = \frac{\Theta(\ln j)}{j^{-1+4s}}. \tag{5.5}$$

In the following paragraphs, we prove that the nonlinear system (1.1) is small-time locally null controllable with quadratic cost for the nonlinearity Γ_Θ.

Remark 5.2. *The shape $\Theta(\ln j)$ guarantees that the sequence c_j oscillates between positive and negative values, up to infinity, at an increasingly slower pace. This property is, at least on an heuristic level, mandatory to obtain small-time controllability.*

Remark 5.3. *For the ease of presentation, we chose a system which is homogeneous in the sense that, if z denotes the trajectory to* (1.1) *with $z^0 = 0$, one has, for $j \in \mathbb{N}^*$,*

$$\langle z(T), \varphi_j \rangle = \int_0^T u(t) e^{-j^2(T-t)} \,\mathrm{d}t \tag{5.6}$$

and

$$\langle z(T), \varphi_0 \rangle = \int_0^T u(t) \int_0^t u(\tau) K_\Theta(t-\tau) \,\mathrm{d}\tau \,\mathrm{d}t, \tag{5.7}$$

where K_Θ is defined as in (3.7) *for the coefficients* (5.5)*. The first component is purely quadratic whereas the others are purely linear. This choice simplifies the proof of Theorem 5, but it would be possible to prove the same result for other systems.*

5.2 Construction of rough elementary controls

Let $T > 0$. We prove that there exists elementary controls $v^\pm \in L^2(0,T)$ such that the associated trajectories starting from $z^0 = 0$ satisfy $\langle z(T), \varphi_0 \rangle = \pm 1$. In this paragraph, we do not require that $\langle z(T), \varphi_j \rangle = 0$ for $j \in \mathbb{N}^*$. We adapt the controls accordingly in Section 5.3.

For $\omega > 2\pi/T$, we consider the function $v_\omega : \mathbb{R} \to \mathbb{R}$ defined by

$$v_\omega(t) := \cos(\omega t) \mathbf{1}_{(0,\tau(\omega))}(t), \tag{5.8}$$

where $\tau(\omega)$ is the largest element of $[0, T/4] \cap \left(\frac{\pi}{2\omega} + \frac{\pi}{\omega}\mathbb{N}\right)$. Then, v_ω is supported on $(0, T/4)$,

$$0 \le \frac{T}{4} - \tau(\omega) \le \frac{\pi}{\omega} \quad \text{and} \quad \cos(\omega \tau(\omega)) = 0. \tag{5.9}$$

Lemma 5.4. *Let $\theta \in (0, \frac{1}{2})$. There exists $C > 0$ such that, for $T \in (0,1)$ and $\omega > 2\pi/T$,*

$$\left| \int_{\mathbb{R}} \frac{|\widehat{v_\omega}(\xi)|^2}{(1+\xi^2)^\theta} \,\mathrm{d}\xi - \frac{\pi T/4}{(1+\omega^2)^\theta} \right| \le \frac{C}{(1+\omega^2)^{\frac{1}{2}}}. \tag{5.10}$$

Proof. Let $\theta \in (0, \frac{1}{2})$, $T \in (0,1)$ and $\omega > 2\pi/T$. To lighten the notations, in this proof, we write τ instead of $\tau(\omega)$. Simple calculations lead to

$$\widehat{v_\omega}(\xi) = \frac{\tau}{2} \Big(\operatorname{sinc}\left[(\omega - \xi)\frac{\tau}{2}\right] e^{i(\omega-\xi)\frac{\tau}{2}} + \operatorname{sinc}\left[(\omega + \xi)\frac{\tau}{2}\right] e^{-i(\omega+\xi)\frac{\tau}{2}} \Big). \tag{5.11}$$

Thus, thanks to relation (5.9),

$$|\widehat{v_\omega}(\xi)|^2 = \frac{\tau^2}{4} \Big(\operatorname{sinc}^2\left[(\omega - \xi)\frac{\tau}{2}\right] + \operatorname{sinc}^2\left[(\omega + \xi)\frac{\tau}{2}\right] \Big) \tag{5.12}$$

and, for parity reasons,

$$\int_{\mathbb{R}} \frac{|\widehat{v_\omega}(\xi)|^2}{(1+\xi^2)^\theta}\,\mathrm{d}\xi = \frac{\tau^2}{2}\int_{\mathbb{R}} \frac{\operatorname{sinc}^2\left[(\omega-\xi)\frac{\tau}{2}\right]}{(1+\xi^2)^\theta}\,\mathrm{d}\xi. \tag{5.13}$$

A change of variable proves that

$$\frac{\tau^2}{2}\int_{\mathbb{R}} \operatorname{sinc}^2\left[\eta\frac{\tau}{2}\right]\mathrm{d}\eta = \tau\int_{\mathbb{R}}\operatorname{sinc}^2(x)\,\mathrm{d}x = \pi\tau. \tag{5.14}$$

To establish the lemma, we estimate the difference as

$$\left|\int_{\mathbb{R}} \frac{|\widehat{v_\omega}(\xi)|^2}{(1+\xi^2)^\theta}\,\mathrm{d}\xi - \frac{\pi\tau}{(1+\omega^2)^\theta}\right| \le \frac{\tau^2}{2}\int_{\mathbb{R}}\operatorname{sinc}^2\left[(\omega-\xi)\frac{\tau}{2}\right]\left|\frac{1}{(1+\xi^2)^\theta}-\frac{1}{(1+\omega^2)^\theta}\right|\mathrm{d}\xi. \tag{5.15}$$

On the one hand, there exists $C_1(\theta)>0$ such that, for every $\xi\in\left[\frac{\omega}{2},\frac{3\omega}{2}\right]$,

$$\left|\frac{1}{(1+\xi^2)^\theta}-\frac{1}{(1+\omega^2)^\theta}\right| \le |\xi-\omega|\max\left\{\frac{2\theta\eta}{(1+\eta^2)^{\theta+1}};\eta\in[\xi,\omega]\right\} \le \frac{C_1(\theta)|\omega-\xi|}{(1+\omega^2)^{\theta+\frac12}}. \tag{5.16}$$

On the other hand, this difference is also bounded by 1. Hence, for any $\sigma\in(0,1)$,

$$\left|\frac{1}{(1+\xi^2)^\theta}-\frac{1}{(1+\omega^2)^\theta}\right| \le \frac{C_1^\sigma(\theta)|\omega-\xi|^\sigma}{(1+\omega^2)^{\theta\sigma+\frac{\sigma}{2}}}. \tag{5.17}$$

Hence, splitting the integral and applying this estimate yields

$$\begin{aligned}
&\frac{\tau^2}{2}\int_{\mathbb{R}}\operatorname{sinc}^2\left[(\omega-\xi)\frac{\tau}{2}\right]\left|\frac{1}{(1+\xi^2)^\theta}-\frac{1}{(1+\omega^2)^\theta}\right|\mathrm{d}\xi\\
\le&\frac{\tau^2}{2}\int_{\omega/2}^{3\omega/2}|\omega-\xi|^\sigma\operatorname{sinc}^2\left[(\omega-\xi)\frac{\tau}{2}\right]\frac{C_1^\sigma(\theta)}{(1+\omega^2)^{\theta\sigma+\frac{\sigma}{2}}}\,\mathrm{d}\xi+\frac{\tau^2}{2}\int_{|\xi-\omega|>\frac{\omega}{2}}\operatorname{sinc}^2\left[(\omega-\xi)\frac{\tau}{2}\right]\mathrm{d}\xi\\
\le&\frac{2^\sigma\tau^{1-\sigma}C_1^\sigma(\theta)}{(1+\omega^2)^{\theta\sigma+\frac{\sigma}{2}}}\int_{\mathbb{R}_+}|\eta|^\sigma\operatorname{sinc}^2(\eta)\,\mathrm{d}\eta+2\int_{|x|>\frac{\omega}{2}}\frac{\mathrm{d}x}{x^2}.
\end{aligned} \tag{5.18}$$

This proves the claimed estimates by using $\sigma=1/(2\theta+1)$, because, thanks to the choice of $\tau(\omega)$ (see (5.9)), $|T/4-\tau|\le\pi/\omega$. ☐

Proposition 5.5. *Let $s\in(0,\frac12)$ and $\epsilon>0$. Let $L(s,\epsilon)$ be given by Lemma 5.1. There exist $C,\beta>0$ such that, for any $\Theta\in W^{1,\infty}(\mathbb{R})$, $T\in(0,1)$ and $\omega>2\pi/T$ such that $\ln(\sqrt{\omega})$ is in the middle of a plateau of length $2L$ on which Θ is constant, i.e.,*

$$\forall x\in[\ln\sqrt{\omega}-L,\ln\sqrt{\omega}+L],\quad \Theta(x)=\Theta(\ln\sqrt{\omega}), \tag{5.19}$$

if K_Θ denotes the kernel associated to Θ by (3.7) *and* (5.5)*, then,*

$$\left|\int_0^T v_\omega(t)\int_0^t v_\omega(\tau)K_\Theta(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}t-\frac{\pi T\gamma(s)\Theta(\ln\sqrt{\omega})}{2\omega^{2s}}\right|\le\left(\frac{\epsilon T}{\omega^{2s}}+\frac{C}{\omega^{2s+2\beta}}\right)\|\Theta\|_{W^{1,\infty}}, \tag{5.20}$$

where the constant $\gamma(s)$ is defined in (4.15).

Proof. By working as in the proof of Proposition 4.2 (Step 1), we see that $K_\Theta\in L^1(\mathbb{R})$ and, thanks to (4.12) and the parity of $\widehat{v_\omega}$ and $\widehat{K_\Theta}$,

$$\int_0^T v_\omega(t)\int_0^t v_\omega(\tau)K_\Theta(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}t=\frac{1}{4\pi^2}\int_0^\infty|\widehat{v_\omega}(\xi)|^2\widehat{K_\Theta}(\xi)\,\mathrm{d}\xi. \tag{5.21}$$

On the one hand, by Lemma 4.1, there exists $C_1, \beta > 0$ independent of Θ such that, for $|\xi| \geq 1$,

$$\left| \sum_{j=1}^{\infty} \frac{j^{3-4s}\,\Theta(\ln(j))}{j^4+\xi^2} - \frac{1}{\xi^{2s}} \int_0^\infty \frac{y^{3-4s}}{1+y^4}\,\Theta\left(\ln(y\sqrt{\xi})\right)\,\mathrm{d}y \right| \leq \frac{C_1\|\Theta\|_{W^{1,\infty}}}{|\xi|^{2s+2\beta}}. \tag{5.22}$$

On the other hand, for every $\xi \in \mathbb{R}$,

$$|\widehat{K_\Theta}(\xi)| = 2\left| \sum_{j\in\mathbb{N}^*} \frac{j^{3-4s}\Theta(\ln(j))}{j^4+\xi^2} \right| \leq 2\|\Theta\|_{L^\infty} \sum_{j\in\mathbb{N}^*} j^{-1-4s}. \tag{5.23}$$

Thus, there exists $C_2 > 0$ independent of Θ, such that, for every $\xi \in (0,\infty)$

$$\left| \widehat{K_\Theta}(\xi) - \frac{2}{(1+\xi^2)^s} \int_0^\infty \frac{y^{3-4s}}{1+y^4}\,\Theta\left(\ln(y\sqrt{\xi})\right)\,\mathrm{d}y \right| \leq \frac{C_2\|\Theta\|_{W^{1,\infty}}}{(1+\xi^2)^{s+\beta}}. \tag{5.24}$$

Then, by Lemma 5.4, there exists $C_3 > 0$ independent of Θ, such that

$$\left| \int_0^\infty |\widehat{v_\omega}(\xi)|^2 \widehat{K_\Theta}(\xi)\,\mathrm{d}\xi - \int_0^\infty \frac{2|\widehat{v_\omega}(\xi)|^2}{(1+\xi^2)^s} \int_0^\infty \frac{y^{3-4s}}{1+y^4}\Theta\left(\ln(y\sqrt{\xi})\right)\,\mathrm{d}y\,\mathrm{d}\xi \right| \leq \frac{C_3\|\Theta\|_{W^{1,\infty}}}{(1+\omega^2)^{s+\beta}}. \tag{5.25}$$

Let us focus on the main term, which we decompose as

$$\int_0^\infty \frac{|\widehat{v_\omega}(\xi)|^2}{(1+\xi^2)^s} \int_0^\infty \frac{y^{3-4s}}{1+y^4}\Theta\left(\ln(y\sqrt{\xi})\right)\,\mathrm{d}y\,\mathrm{d}\xi = I_1 + I_2 + I_3, \tag{5.26}$$

where we introduce, recalling (5.12),

$$I_1 := \int_{\omega/2}^{3\omega/2} |\widehat{v_\omega}(\xi)|^2 \frac{1}{(1+\xi^2)^s} \int_{e^{-L/2}}^{e^{L/2}} \frac{y^{3-4s}}{1+y^4}\,\Theta\left(\ln(y\sqrt{\xi})\right)\,\mathrm{d}y\,\mathrm{d}\xi, \tag{5.27}$$

$$I_2 := \int_{\omega/2}^{3\omega/2} |\widehat{v_\omega}(\xi)|^2 \frac{1}{(1+\xi^2)^s} \int_{\mathbb{R}_+\setminus(e^{-L/2},e^{L}/2)} \frac{y^{3-4s}}{1+y^4}\,\Theta\left(\ln(y\sqrt{\xi})\right)\,\mathrm{d}y\,\mathrm{d}\xi, \tag{5.28}$$

$$I_3 := \int_{\mathbb{R}_+\setminus(\frac{\omega}{2},\frac{3\omega}{2})} |\widehat{v_\omega}(\xi)|^2 \frac{1}{(1+\xi^2)^s} \int_0^\infty \frac{y^{3-4s}}{1+y^4}\,\Theta\left(\ln(y\sqrt{\xi})\right)\,\mathrm{d}y\,\mathrm{d}\xi. \tag{5.29}$$

The first term I_1 gives the claimed asymptotic, while the others are easily bounded.

Estimate of I_1. Since $L \geq 1 > \ln(2)$, one has $(\frac{\omega}{2}, \frac{3\omega}{2}) \subset (\omega e^{-L}, \omega e^{L})$. Thus, for every $\xi \in (\frac{\omega}{2}, \frac{3\omega}{2})$, we have

$$|\ln(\sqrt{\omega}) - \ln(\sqrt{\xi})| \leq \frac{L}{2} \tag{5.30}$$

and, for every $y \in (e^{-L/2}, e^{L/2})$,

$$|\ln(\sqrt{\omega}) - \ln(\sqrt{\xi}) - \ln(y)| < L, \tag{5.31}$$

which implies $\Theta(\ln(y\sqrt{\xi})) = \Theta(\ln(\sqrt{\omega}))$, thanks to the assumption that $\ln(\sqrt{\omega})$ lies in the middle of a plateau of length $2L$. This relation proves that

$$I_1 = \Theta(\ln(\sqrt{\omega})) \left(\int_{\omega/2}^{3\omega/2} |\widehat{v_\omega}(\xi)|^2 \frac{\mathrm{d}\xi}{(1+\xi^2)^s} \right) \left(\int_{e^{-L/2}}^{e^{L/2}} \frac{y^{3-4s}}{1+y^4}\,\mathrm{d}y \right). \tag{5.32}$$

Moreover, proceeding as in the proof of Lemma 5.4, there exists $C_4 > 0$ such that, for $\omega > 2\pi$,

$$\left| \int_{\omega/2}^{3\omega/2} |\widehat{v_\omega}(\xi)|^2 \frac{\mathrm{d}\xi}{(1+\xi^2)^s} - \frac{\pi T/4}{\omega^{2s}} \right| \leq \frac{C_4}{\omega}. \tag{5.33}$$

Finally, we obtain, a constant $C_5 > 0$, independent of Θ, such that

$$\left| I_1 - \frac{4\pi^2 \gamma(s) \pi T}{4\omega^{2s}} \Theta(\ln(\sqrt{\omega})) \right| \leq \left(\frac{\pi \epsilon T}{4\omega^{2s}} + \frac{C_5}{\omega} \right) \|\Theta\|_{L^\infty}. \tag{5.34}$$

Estimate of I_2. Using Lemma 5.4, there exists $C_6 > 0$, such that

$$|I_2| \leq \epsilon \|\Theta\|_{L^\infty} \int_{\mathbb{R}_+} \frac{|\widehat{v_\omega}(\xi)|^2 \,\mathrm{d}\xi}{(1+\xi^2)^s} \leq \epsilon \left(\frac{\pi T}{4\omega^{2s}} + \frac{C_6}{\omega} \right) \|\Theta\|_{L^\infty}. \tag{5.35}$$

Estimate of I_3. We have, for every $T > 0$ and $\omega > 2\pi/T$, proceeding as in the proof of Lemma 5.4, there exists $C_7 > 0$ such that

$$|I_3| \leq \frac{C_7}{\omega} \|\Theta\|_{L^\infty}. \tag{5.36}$$

These estimates conclude the proof of the proposition. □

5.3 Construction of controls leaving the linear order invariant

For large enough ω and an appropriate position of $\ln \sqrt{\omega}$, v_ω will allow to enter the half-spaces $\{z(T);\ \pm\langle z(T), \varphi_0 \rangle > 0\}$. However, to obtain small-time local null controllability for the full system, it is necessary to build controls realizing the elementary motions in the directions $\pm\varphi_0$ without moving the other components.

Let $T > 0$ and $\omega > 2\pi/T$. The sequence $(d_k(v_\omega))_{k\in\mathbb{N}}$ defined by

$$d_k(v_\omega) := \int_0^T v_\omega(t) e^{-k^2(T-t)} \,\mathrm{d}t \tag{5.37}$$

belongs to D_T because, by (5.9), for every $k \in \mathbb{N}$,

$$\begin{aligned} |d_k(v_\omega)| &= \left| e^{-k^2(T-\tau(\omega))} \int_0^{\tau(\omega)} v_\omega(t) e^{-k^2(\tau(\omega)-t)} \,\mathrm{d}t \right| \\ &\leq e^{-3k^2 T/4} \left| \int_0^{\tau(\omega)} \cos(\omega t) e^{-k^2(\tau(\omega)-t)} \,\mathrm{d}t \right| \\ &\leq e^{-3k^2 T/4} \frac{2}{\omega}. \end{aligned} \tag{5.38}$$

Thus, by Proposition 2.4, we can consider a corrected control that leaves the linear order invariant

$$\widetilde{v_\omega} := v_\omega - \mathfrak{L}_1^T(d(v_\omega)) \tag{5.39}$$

and there exists $C > 0$ such that

$$\|\mathfrak{L}_1^T(d(v_\omega))\|_{L^2(0,T)} \leq \frac{C_T}{\omega}. \tag{5.40}$$

Moreover, since $K_\Theta \in L^1$, K defines a continuous bilinear form on $L^2(0,T)$. Hence, thanks to Proposition 5.5, there exists $\omega_+ = (1+8k_+)L > 2\pi/T$, with k_+ large enough and $\omega_- = (5+8k_-)L$ with k_- large enough, such that

$$\pm \int_0^T \widetilde{v_{\omega_\pm}}(t) \int_0^t \widetilde{v_{\omega_\pm}}(\tau) K_\Theta(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}t > 0. \tag{5.41}$$

Up to a rescaling, this allows to construct $v^\pm \in L^2(0,T)$ such that the associated trajectories to (1.1) starting from $z_0 = 0$ satisfy

$$z(T) = \pm\varphi_0. \tag{5.42}$$

5.4 Conclusion of the proof

Theorem 5 is a direct consequence of the existence of the controls $v^\pm \in L^2(0,T)$ realizing elementary movements in the directions $\pm\varphi_0$. The argument is classical (we refer to [15, Chapter 8]).

6 Perspectives

In view of the results proved in this work, the following open questions seem interesting.

- The integer-order drifts are related to Lie brackets of the vector fields defining the dynamics. Is it possible to find a geometric interpretation of the fractional drifts?
- We proved that one can recover the small-time local null controllability with quadratic cost when one direction is lost at the linear order with controls which are small in L^2. Is it also possible to recover one direction with more regular controls? Is it possible to recover a finite number of lost directions? Or even an infinite number of lost directions?
- This work concerns scalar-input control systems governed by parabolic equations. Can analogous results be obtained for time-reversible systems, like Schrödinger or Korteweg-de-Vries control systems? A difficulty might be that the quadratic kernels could be less regular.

References

[1] Farid Ammar Khodja, Assia Benabdallah, Manuel González-Burgos, and Luz de Teresa. Minimal time for the null controllability of parabolic systems: the effect of the condensation index of complex sequences. *J. Funct. Anal.*, 267(7):2077–2151, 2014.

[2] Werner Amrein and Anne-Marie Berthier. On support properties of L^p-functions and their Fourier transforms. *J. Functional Analysis*, 24(3):258–267, 1977.

[3] Karine Beauchard. Local controllability of a 1-D Schrödinger equation. *J. Math. Pures Appl. (9)*, 84(7):851–956, 2005.

[4] Karine Beauchard. Controllability of a quantum particle in a 1D variable domain. *ESAIM Control Optim. Calc. Var.*, 14(1):105–147, 2008.

[5] Karine Beauchard and Jean-Michel Coron. Controllability of a quantum particle in a moving potential well. *J. Funct. Anal.*, 232(2):328–389, 2006.

[6] Karine Beauchard and Frédéric Marbach. Quadratic obstructions to small-time local controllability for scalar-input systems. *Journal of Differential Equations*, 2017.

[7] Karine Beauchard and Morgan Morancey. Local controllability of 1D Schrödinger equations with bilinear control and minimal time. *Math. Control Relat. Fields*, 4(2):125–160, 2014.

[8] Assia Benabdallah, Franck Boyer, Manuel González-Burgos, and Guillaume Olive. Sharp estimates of the one-dimensional boundary control cost for parabolic systems and application to the N-dimensional boundary null controllability in cylindrical domains. *SIAM J. Control Optim.*, 52(5):2970–3001, 2014.

[9] Michael Benedicks. On Fourier transforms of functions supported on sets of finite Lebesgue measure. *J. Math. Anal. Appl.*, 106(1):180–183, 1985.

[10] Aline Bonami and Bruno Demange. A survey on uncertainty principles related to quadratic forms. *Collect. Math.*, (Vol. Extra):1–36, 2006.

[11] Eduardo Cerpa. Exact controllability of a nonlinear Korteweg-de Vries equation on a critical spatial domain. *SIAM J. Control Optim.*, 46(3):877–899, 2007.

[12] Eduardo Cerpa and Emmanuelle Crépeau. Boundary controllability for the nonlinear Korteweg-de Vries equation on any critical domain. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(2):457–475, 2009.

[13] Jean-Michel Coron. On the controllability of 2-D incompressible perfect fluids. *J. Math. Pures Appl. (9)*, 75(2):155–188, 1996.

[14] Jean-Michel Coron. Local controllability of a 1-D tank containing a fluid modeled by the shallow water equations. *ESAIM Control Optim. Calc. Var.*, 8:513–554, 2002. A tribute to J. L. Lions.

[15] Jean-Michel Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.

[16] Jean-Michel Coron and Emmanuelle Crépeau. Exact boundary controllability of a nonlinear KdV equation with critical lengths. *J. Eur. Math. Soc. (JEMS)*, 6(3):367–398, 2004.

[17] Belhassen Dehman and Gilles Lebeau. Analysis of the HUM control operator and exact controllability for semilinear waves in uniform time. *SIAM J. Control Optim.*, 48(2):521–550, 2009.

[18] Sylvain Ervedoza and Enrique Zuazua. A systematic method for building smooth controls for smooth data. *Discrete Contin. Dyn. Syst. Ser. B*, 14(4):1375–1401, 2010.

[19] Hector Fattorini and David Russell. Exact controllability theorems for linear parabolic equations in one space dimension. *Arch. Rational Mech. Anal.*, 43:272–292, 1971.

[20] Hector Fattorini and David Russell. Uniform bounds on biorthogonal functions for real exponentials with an application to the control theory of parabolic equations. *Quart. Appl. Math.*, 32:45–69, 1974/75.

[21] Gerald Folland and Alladi Sitaram. The uncertainty principle: a mathematical survey. *J. Fourier Anal. Appl.*, 3(3):207–238, 1997.

[22] Victor Havin and Burglind Jöricke. *The uncertainty principle in harmonic analysis*, volume 28 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1994.

[23] Philippe Jaming. Nazarov's uncertainty principles in higher dimension. *J. Approx. Theory*, 149(1):30–41, 2007.

[24] Gilles Lebeau and Maëlle Nodet. Experimental study of the HUM control operator for linear waves. *Experiment. Math.*, 19(1):93–120, 2010.

[25] Pierre Lissy. The cost of the control in the case of a minimal time of control: the example of the one-dimensional heat equation. *J. Math. Anal. Appl.*, 451(1):497–507, 2017.

[26] Yuning Liu, Takéo Takahashi, and Marius Tucsnak. Single input controllability of a simplified fluid-structure interaction model. *ESAIM Control Optim. Calc. Var.*, 19(1):20–42, 2013.

[27] Frédéric Marbach. An obstruction to small time local null controllability for a viscous Burgers' equation. *ArXiv e-prints*, November 2015.

[28] Frédéric Marbach. Obstructions quadratiques à la contrôlabilité, de la dimension finie à la dimension infinie. In *Séminaire Laurent Schwartz—Équations aux dérivées partielles et applications. Année 2016–2017*, page Exp. No. XVI. Ed. Éc. Polytech., Palaiseau, 2017.

[29] Morgan Morancey. Simultaneous local exact controllability of 1D bilinear Schrödinger equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 31(3):501–529, 2014.

[30] Carlo Morosi and Livio Pizzocchero. On the constants for some fractional gagliardo–nirenberg and sobolev inequalities. *Expositiones Mathematicae*, 2017.

[31] Fedor Nazarov. Local estimates for exponential polynomials and their applications to inequalities of the uncertainty principle type. *Algebra i Analiz*, 5(4):3–66, 1993.

[32] Louis Nirenberg. On elliptic partial differential equations. *Ann. Scuola Norm. Sup. Pisa (3)*, 13:115–162, 1959.

[33] Amnon Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[34] Walter Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[35] Rodolfo Torres. Boundedness results for operators with singular kernels on distribution spaces. *Mem. Amer. Math. Soc.*, 90(442):viii+172, 1991.

[36] Abdellah Youssfi. Regularity properties of singular integral operators. *Studia Math.*, 119(3):199–217, 1996.