Quadratic obstructions to small-time local controllability for scalar-input differential systems

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Abstract

We consider nonlinear scalar-input differential control systems in the vicinity of an equilibrium. When the linearized system at the equilibrium is controllable, the nonlinear system is smoothly small-time locally controllable, i.e., whatever $m > 0$ and $T > 0$, the state can reach a whole neighborhood of the equilibrium at time T with controls arbitrary small in C^m-norm. When the linearized system is not controllable, we prove that small-time local controllability cannot be recovered from the quadratic expansion and that the following quadratic alternative holds.

Either the state is constrained to live within a smooth strict invariant manifold, up to a cubic residual, or the quadratic order adds a signed drift in the evolution with respect to this manifold. In the second case, the quadratic drift holds along an explicit Lie bracket of length $(2k+1)$, it is quantified in terms of an H^{-k}-norm of the control, it holds for controls small in $W^{2k,\infty}$-norm. These spaces are optimal for general nonlinear systems and are slightly improved in the particular case of control-affine systems.

Unlike other works based on Lie-series formalism, our proof is based on an explicit computation of the quadratic terms by means of appropriate transformations. In particular, it does not require that the vector fields defining the dynamic are smooth. We prove that C^3 regularity is sufficient for our alternative to hold.

This work underlines the importance of the norm used in the smallness assumption on the control: depending on this choice of functional setting, the same system may or may not be small-time locally controllable, even though the state lives within a finite dimensional space.

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1 Introduction

1.1 Scalar-input differential systems

Let $n \in \mathbb{N}^*$. Throughout this work, we consider differential control systems where the state $x(t)$ lives in $\mathbb{R}^n$ and the control is a scalar input $u(t) \in \mathbb{R}$. For $f \in C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$, we consider the nonlinear control system:

$$\dot{x} = f(x, u). \tag{1.1}$$

Definition 1. *Let $T > 0$ and $x^* \in \mathbb{R}^n$ be a given initial data. We say that a couple $(x, u) \in C^0([0, T], \mathbb{R}^n) \times L^\infty((0, T), \mathbb{R})$ is a* trajectory *of* (1.1) *associated with x^* when:*

$$\forall t \in [0, T], \quad x(t) = x^* + \int_0^t f(x(s), u(s))\mathrm{d}s. \tag{1.2}$$

Proposition 1. *Let $T > 0$, $x^* \in \mathbb{R}^n$ and $u \in L^\infty((0, T), \mathbb{R})$. System* (1.1) *admits a unique maximal trajectory defined on $[0, T_u)$ for some $T_u \in (0, T]$.*

Proof. Once a control $u \in L^\infty((0, T), \mathbb{R})$ is fixed, system (1.1) can be seen as $\dot{x}(t) = g(t, x(t))$, where we introduce $g(t, x) := f(x, u(t))$. The function g is not continuous with respect to time. Hence, we cannot apply usual Cauchy-Lipschitz-Picard-Lindelöf existence and uniqueness theorem. However, the existence and uniqueness of a solution in the sense of Definition 1 holds for such functions g (see e.g. [34, Theorem 54, page 476]). The proof relies on a fixed-point theorem applied to the integral formulation (1.2). □

In the particular case of control-affine systems, we will work with a slightly different functional framework. Let $f_0, f_1 \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. A control-affine system takes the form:

$$\dot{x} = f_0(x) + u f_1(x). \tag{1.3}$$

Such systems are both important from the point of view of applications and mathematically as a first-order Taylor expansion with respect to a small control of a nonlinear dynamic.

Definition 2. *Let $T > 0$ and $x^* \in \mathbb{R}^n$ be a given initial data. We say that a couple $(x, u) \in C^0([0, T], \mathbb{R}^n) \times L^1((0, T), \mathbb{R})$ is a* trajectory *of* (1.3) *associated with x^* when:*

$$\forall t \in [0, T], \quad x(t) = x^* + \int_0^t \left(f_0(x(s)) + u(s) f_1(x(s))\right) \mathrm{d}s. \tag{1.4}$$

Proposition 2. *Let $T > 0$, $x^* \in \mathbb{R}^n$ and $u \in L^1((0, T), \mathbb{R})$. System* (1.3) *admits a unique maximal trajectory defined on $[0, T_u)$ for some $T_u \in (0, T]$.*

Proof. Here again, one applies a fixed-point theorem to the integral formulation (1.4). □

In the sequel and where not explicitly stated, it is implicit that we handle well-defined trajectories of our differential systems, either by restricting to small enough times, small enough controls or sufficiently nice dynamics preventing blow-up.

Moreover, we will often need to switch point of view between nonlinear and control-affine systems. Given a control-affine system characterized by f_0 and f_1, one can always see it as a particular case of a nonlinear system by defining:

$$f(x, u) := f_0(x) + u f_1(x). \tag{1.5}$$

Conversely, given a nonlinear system characterized by f, its dynamic can be approximated using $f(x, u) \approx f_0(x) + u f_1(x) + O(u^2)$ where we define:

$$f_0(x) := f(x, 0) \quad \text{and} \quad f_1(x) := \partial_u f(x, 0). \tag{1.6}$$

1.2 Small-time local controllability

Multiple definitions of small-time local controllability can be found in the mathematical literature. Here, we put the focus on the smallness assumption made on the control. This notion is mostly relevant in the vicinity of an equilibrium. We use the following definitions:

Definition 3. *We say that* $(x_e, u_e) \in \mathbb{R}^n \times \mathbb{R}$ *is an* equilibrium *of system* (1.1) *when* $f(x_e, u_e) = 0$ *and an equilibrium of system* (1.3) *when* $f_0(x_e) + u_e f_1(x_e) = 0$*. Up to a translation, one can always assume that* $x_e = 0$ *and* $u_e = 0$*. Thus, in the sequel, it is implicit that we consider systems for which the couple* $(0, 0)$ *is an equilibrium.*

Definition 4. *Let* $\left(E_T, \|\cdot\|_{E_T}\right)$ *be a family of normed vector spaces of scalar functions defined on* $[0, T]$*, for* $T > 0$*. We say that a scalar-input differential system is* E small-time locally controllable *when the following property holds: for any* $T > 0$*, for any* $\eta > 0$*, there exists* $\delta > 0$ *such that, for any* $x^*, x^\dagger \in \mathbb{R}^n$ *with* $|x^*| + |x^\dagger| \leqslant \delta$*, there exists a trajectory* (x, u) *of the differential system defined on* $[0, T]$ *with* $u \in E_T$ *satisfying:*

$$\|u\|_{E_T} \leqslant \eta \quad \textit{and} \quad x(0) = x^* \quad \textit{and} \quad x(T) = x^\dagger. \tag{1.7}$$

Here and in the sequel, it is implicit that this notion of local controllability refers to local controllability in the vicinity of the null equilibrium $x_e = 0$ *and* $u_e = 0$ *of our system. The translation of our results to other equilibriums is left to the reader.*

It could be thought that, in a finite dimensional setting with smooth dynamics, the notion of small-time local controllability should not depend on the smallness assumption made on the control. However, it is not the case. This fact plays a key role in this work. We will see that the relevance of the quadratic approximation depends on the chosen norm. We will also use the following notion:

Definition 5. *We say that a scalar-input differential system is* smoothly small-time locally controllable *when it is* C^m *small-time locally controllable for any* $m \in \mathbb{N}$*.*

1.3 Linear theory and the Kalman rank condition

The natural approach to investigate the local controllability of system (1.1) near an equilibrium is to study the controllability of the linearized system, which is given by:

$$\dot{y} = H_0 y + ub, \tag{1.8}$$

where $H_0 := \partial_x f(0,0)$ and $b := \partial_u f(0,0)$. It is well known (see works [31] of Pontryagin, [25, Theorem 6] of LaSalle or [22, Theorem 10] of Kalman, Ho and Narendra) that such linear control systems are controllable, independently on the allowed time T, if and only if they satisfy the Kalman rank condition:

$$\text{Span } \left\{H_0^k b, \ k \in \{0, \dots n-1\}\right\} = \mathbb{R}^n. \tag{1.9}$$

It is also classical to prove that the controllability of the linearized system implies small-time local controllability for the nonlinear system (see [28, Theorem 3] by Markus or [26, Theorem 1] by Lee and Markus). For example, we have:

Theorem 1. *Assume that the Kalman rank condition* (1.9) *holds. Then the nonlinear system* (1.1) *is smoothly small-time locally controllable. Moreover, one can choose controls compactly supported within the interior of the allotted time interval.*

Proof. **Smooth controllability of the linearized system with compactly supported controls.** Let $T > 0$. We start by proving that we can use regular compactly supported controls to achieve controllability for the linear system (1.8). We introduce the controllability Gramian:

$$\mathfrak{C}_T := \int_0^T e^{(T-t)H_0} b b^{\mathrm{tr}} e^{(T-t)H_0^{\mathrm{tr}}} \mathrm{d}t. \tag{1.10}$$

It is well-known that $\mathfrak{C}_T$ is invertible if and only if the Kalman rank condition holds (see [13, Section 1.2]). Let $(\rho_\epsilon)_{\epsilon>0}$ be a family of functions in $C_c^\infty((0,T),\mathbb{R})$ with $\mathrm{supp}(\rho_\epsilon) \subset [\epsilon, T-\epsilon]$, such that $\rho_\epsilon(t) \underset{\epsilon\to 0}{\to} 1$ for every $t \in (0,T)$ and $|\rho_\epsilon(t)| \leqslant 1$ for every $(\epsilon, t) \in \mathbb{R}_+^* \times (0,T)$. Let:

$$\mathfrak{C}_{T,\epsilon} := \int_0^T \rho_\epsilon(t) e^{(T-t)H_0} b b^{\mathrm{tr}} e^{(T-t)H_0^{\mathrm{tr}}} \mathrm{d}t. \tag{1.11}$$

By the dominated convergence theorem, $\mathfrak{C}_{T,\epsilon}$ converges to $\mathfrak{C}_T$ in $\mathcal{M}_n(\mathbb{R})$. Thus, for $\epsilon > 0$ small enough, $\mathfrak{C}_{T,\epsilon}$ belongs to $GL_n(\mathbb{R})$, because this is an open subset of $\mathcal{M}_n(\mathbb{R})$. From now on, such an ϵ is fixed. Let $y^*, y^\dagger \in \mathbb{R}^n$. Using an optimal control or "Hilbert Uniqueness Method" approach (see [13, Section 1.4]), we define for $t \in [0,T]$:

$$u(t) := \rho_\epsilon(t) b^{\mathrm{tr}} e^{(T-t)H_0^{\mathrm{tr}}} p, \tag{1.12}$$

where $p \in \mathbb{R}^n$ is defined by

$$p := \mathfrak{C}_{T,\epsilon}^{-1} \left(y^\dagger - e^{TH_0} y^* \right). \tag{1.13}$$

Using a Duhamel formula for (1.8), combined with (1.11), (1.12) and (1.13), one checks that the solution of (1.8) with initial condition $y(0) = y^*$ satisfies:

$$y(T) = e^{TH_0} y^* + \int_0^T u(t) e^{(T-t)H_0} b \mathrm{d}t = e^{TH_0} y^* + \mathfrak{C}_{T,\epsilon} p = y^\dagger. \tag{1.14}$$

From (1.12) and (1.13), one checks that:

$$\forall m \geqslant 0, \exists \eta_{T,m} > 0, \quad \|u\|_{C^m(0,T)} \leqslant \eta_{T,m} \left(|y^*| + |y^\dagger| \right). \tag{1.15}$$

Smooth controllability for the nonlinear system, with compactly supported controls. We move on to the nonlinear system using the approach followed in [13, Theorem 3.6]. Let $m \in \mathbb{N}$ and define:

$$C_\epsilon^m((0,T),\mathbb{R}) := \{ f \in C^m((0,T),\mathbb{R}), \ \mathrm{supp}(f) \subset [\epsilon, T-\epsilon] \}, \tag{1.16}$$

endowed with the norm $\|.\|_{C^m(0,T)}$, which is a Banach space. We introduce the nonlinear mapping:

$$\mathcal{F} : \begin{cases} \mathbb{R}^n \times C_\epsilon^m((0,T),\mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}^n, \\ \qquad\qquad (x^*, u) \mapsto (x^*, x(T)), \end{cases} \tag{1.17}$$

where x is the solution to (1.1) with initial data x^* and control u. It is well known (see [34, Theorem 1, page 57]), that $\mathcal{F}$ defines a C^1 map. Moreover, its differential $\mathcal{F}'(0,0)$ is the following linear map:

$$\mathcal{F}'(0,0) : \begin{cases} \mathbb{R}^n \times C_\epsilon^m((0,T),\mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}^n, \\ \qquad\qquad (y^*, u) \mapsto (y^*, y(T)), \end{cases} \tag{1.18}$$

where y is the solution to (1.8) with initial data y^*. From (1.15), we know that this differential has a bounded right-inverse. The inverse function theorem yields the existence of a

C^1 right-inverse $\mathcal{G}$ to $\mathcal{F}$, defined in a small neighborhood of $(0,0)$. Hence, for any $\eta > 0$, there exists $\delta > 0$ such that:

$$|x^*| + |x^\dagger| \leqslant \delta \quad \Rightarrow \quad \|u\|_{C^m(0,T)} \leqslant \left|\mathcal{G}(x^*, x^\dagger)\right| \leqslant \eta. \tag{1.19}$$

From (1.19), the nonlinear system (1.1) is C^m small-time locally controllable, with controls supported in $[\epsilon, T-\epsilon]$. This holds for any $m \in \mathbb{N}$, thus system (1.1) is smoothly small-time locally controllable, with compactly supported controls. □

When the linear test fails, it is necessary to continue the expansion further on to determine whether small-time local controllability holds or not. Indeed, some systems are (smoothly) small-time locally controllable despite failing the linear test (1.9).

Example 1. *Let $n = 2$. Consider the following scalar-input control-affine system:*

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1^3. \end{cases} \tag{1.20}$$

The linearized system of (1.20) around the null equilibrium is not controllable because the second direction is left invariant. However, let us explain why system (1.20) is smoothly small-time locally controllable. We start by introducing a smooth even function $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$, such that:

$$\varphi(x) = 0 \quad \text{for } |x| \geqslant 1/4, \tag{1.21}$$

$$\varphi(x) = 1 \quad \text{for } |x| \leqslant 1/8, \tag{1.22}$$

$$\int_{\mathbb{R}_-} \varphi^3 = \int_{\mathbb{R}_+} \varphi^3 = 1. \tag{1.23}$$

In particular, from (1.22), $\varphi(0) = 1$. Let $x^*, x^\dagger \in \mathbb{R}^2$ be given initial and final data. Let $T > 0$. For $t \in [0, T]$, we define:

$$u(t) := \frac{1}{T} x_1^* \varphi' \left(\frac{t}{T}\right) + \frac{1}{T} \lambda \varphi' \left(\frac{2t - T}{2T}\right) + \frac{1}{T} x_1^\dagger \varphi' \left(\frac{t-T}{T}\right), \tag{1.24}$$

where $\lambda \in \mathbb{R}$ is a constant to be chosen later on. From (1.20), (1.21), (1.22) and the initial condition $x_1(0) = x_1^*$, we deduce that, for $t \in [0, T]$:

$$x_1(t) = x_1^* \varphi\left(\frac{t}{T}\right) + \lambda \varphi\left(\frac{2t-T}{2T}\right) + x_1^\dagger \varphi\left(\frac{t-T}{T}\right). \tag{1.25}$$

From (1.21), (1.22) and (1.25), we deduce that $x_1(T) = x_1^\dagger$. From (1.20), (1.21), (1.23) and (1.25), since the three contributions in u have disjoint supports, we have:

$$x_2(T) = x_2^* + T(x_1^*)^3 + T(x_1^\dagger)^3 + 2T\lambda^3. \tag{1.26}$$

Hence, from (1.26), the constructed trajectory satisfies $x_2(T) = x_2^\dagger$ if and only if:

$$2\lambda^3 = \frac{x_2^\dagger - x_2^*}{T} - (x_1^*)^3 - (x_1^\dagger)^3. \tag{1.27}$$

Let $m \in \mathbb{N}$. Thanks to (1.21) and (1.24), we have:

$$\|u\|^2_{C^m(0,T)} = T^{-2m-1} \|\varphi\|^2_{C^m(\mathbb{R})} \left(\frac{1}{2}|x_1^*|^2 + \frac{1}{2}|x_1^\dagger|^2 + |\lambda|^2\right). \tag{1.28}$$

Let $\eta > 0$. From (1.27) and (1.28), there exists $\delta = \delta_{m,T,\eta} > 0$ such that:

$$|x^*| + |x^\dagger| \leqslant \delta \quad \Rightarrow \quad \|u\|_{C^m(0,T)} \leqslant \eta. \tag{1.29}$$

Hence, we have constructed a control small in C^m-norm driving the state from x^* to $x^\dagger$. This holds for every $T > 0$ and $m \in \mathbb{N}$, thus system (1.20) is smoothly small-time locally controllable. Moreover, thanks to (1.22), our construction yields controls which are compactly supported within $(0, T)$. Hence, there is no "control-jerk" near the initial or the final time.

1.4 Iterated Lie brackets

The main tool to study the controllability of nonlinear systems beyond the linear test is the notion of iterated Lie brackets. Many works have investigated the link between Lie brackets and controllability with the hope of finding necessary or sufficient conditions. We refer to [13, Sections 3.2 and 3.4] by Coron and [23] by Kawski for surveys on this topic. Let us recall elementary definitions from geometric control theory that will be useful in the sequel.

Definition 6. *Let X and Y be smooth vector fields on $\mathbb{R}^n$. The Lie bracket $[X, Y]$ of X and Y is the smooth vector field defined by:*

$$[X, Y](x) := Y'(x)X(x) - X'(x)Y(x). \tag{1.30}$$

Moreover, we define by induction on $k \in \mathbb{N}$ the notations:

$$\mathrm{ad}_X^0(Y) := Y, \tag{1.31}$$

$$\mathrm{ad}_X^{k+1}(Y) := \left[X, \mathrm{ad}_X^k(Y)\right]. \tag{1.32}$$

In addition to these special brackets with a particular nesting structure, we define the following classical linear subspaces of $\mathbb{R}^n$ for smooth control-affine systems.

Definition 7. *Let f_0 and f_1 be smooth vector fields on $\mathbb{R}^n$. For $k \geqslant 1$, we define $\mathcal{S}_k$ as the non decreasing sequence of linear subspaces of $\mathbb{R}^n$ spanned by the iterated Lie brackets of f_0 and f_1 (with any possible nesting structure), containing f_1 at most k times, evaluated at the null equilibrium. For nonlinear systems, we extend these definitions thanks to (1.6).*

The spaces $\mathcal{S}_1$ and $\mathcal{S}_2$ play a key role in this paper; the former describes the set of controllable directions for the linearized system while the latter describes the directions involved at the quadratic order. When the Kalman rank condition is not fulfilled, the quadratic obstructions to small-time local controllability will come from the components of the state living in the orthogonal of the controllable space.

Definition 8. *Let $\langle \cdot, \cdot \rangle$ denote the usual euclidian scalar product on $\mathbb{R}^n$. We introduce $\mathbb{P} : \mathbb{R}^n \to \mathcal{S}_1$ the orthogonal projection on $\mathcal{S}_1$ with respect to $\langle \cdot, \cdot \rangle$. Similarly, we define $\mathbb{P}^\perp := \mathrm{Id} - \mathbb{P} : \mathbb{R}^n \to \mathcal{S}_1^\perp$ the orthogonal projection on $\mathcal{S}_1^\perp$.*

1.5 The first known quadratic obstruction

At the quadratic order, the situation is more involved than at the linear order and very little is known. Proposing a classification of the possible quadratic behaviors for scalar-input systems is the main motivation of this work. Historically, the following conjecture due to Hermes was proved by Sussmann in [37] for control-affine systems (1.3).

Proposition 3. *Let f_0, f_1 be analytic vector fields over $\mathbb{R}^n$ with $f_0(0) = 0$. Assume that:*

$$\{g(0);\ \ g \in \mathrm{Lie}(f_0, f_1)\} = \mathbb{R}^n, \tag{1.33}$$

$$\mathcal{S}_{2k+2} \subset \mathcal{S}_{2k+1} \quad \textit{for any } k \in \mathbb{N}. \tag{1.34}$$

Then system (1.3) is L^∞ small-time locally controllable.

Reciprocally, for analytic vector fields, hypothesis (1.33) is a necessary condition for small-time local controllability (for a proof, see [37, Proposition 6.2]). Sussmann was mostly interested in investigating whether (1.34) was also a necessary condition, in particular for $k = 0$, the condition:

$$\mathcal{S}_2 \subset \mathcal{S}_1. \tag{1.35}$$

The first violation of (1.35) occurs when $[f_1, [f_0, f_1]](0) \notin \mathcal{S}_1$. The following important known result is due to Sussmann (see [37, Proposition 6.3, page 707]).

Proposition 4. *Let f_0, f_1 be analytic vector fields over $\mathbb{R}^n$ with $f_0(0) = 0$. Assume that:*

$$[f_1, [f_0, f_1]](0) \notin \mathcal{S}_1. \tag{1.36}$$

Then system (1.3) is not L^∞ small-time locally controllable.

Although Sussmann does not insist on the smallness assumption made on the control, it can be seen that his assumption is linked to the $W^{-1,\infty}$-norm of the control. Indeed, he works with arbitrary small-times T and controls u such that $|u|_{L^\infty(0,T)} \leqslant A$, with a fixed constant $A > 0$. This guarantees that $|u_1|_{L^\infty(0,T)} \leqslant AT$ is arbitrary small, where $u_1(t) := \int_0^t u(s)\mathrm{d}s$. We prove in Theorem 3 that the smallness in $W^{-1,\infty}(0,T)$ is in fact the correct assumption for this first quadratic obstruction. We also prove that the analyticity assumption is not necessary: it suffices that $f_0 \in C^3$ and $f_1 \in C^2$ (see Corollary 3).

Example 2. *Let $n = 2$. We consider the following control-affine system:*

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1^2. \end{cases} \tag{1.37}$$

Around the null equilibrium, we have $\mathcal{S}_1 = \mathbb{R}e_1$ and:

$$[f_1, [f_0, f_1]](0) = -2e_2 \notin \mathcal{S}_1. \tag{1.38}$$

Equation (1.38) causes a drift in the direction e_2, quantified by the H^{-1}-norm of the control. Indeed, if the initial state is $x(0) = 0$, we have:

$$x_1(t) = u_1(t) \quad \textit{and} \quad x_2(t) = \int_0^t u_1^2(s)\mathrm{d}s, \tag{1.39}$$

where $u_1(t) := \int_0^t u(s)\mathrm{d}s$. Thus $x_2(t) \geqslant 0$ and the system is not locally controllable.

1.6 The first Lie bracket paradox

Sussmann also attempted to study further violations of condition (1.35). In particular, when $[f_1, [f_0, f_1]](0) \in \mathcal{S}_1$, the next violation is $[f_1, [f_0, [f_0, [f_0, f_1]]]](0) \notin \mathcal{S}_1$. The intermediate violation involving two times f_0 never happens. Indeed, from the Jacobi identity, we have:

$$[f_1, [f_0, [f_0, f_1]]](0) = -[[f_0, f_1], [f_1, f_0]](0) - [f_0, [[f_0, f_1], f_1]](0). \tag{1.40}$$

The first term in the right-hand side of (1.40) vanishes because of the antisymmetry of the Lie bracket operator. Moreover, when $[f_1,[f_0,f_1]](0) \in \mathcal{S}_1$, the second term in the right-hand side of (1.40) belongs to $\mathcal{S}_1$ because this subspace is stable with respect to bracketing by f_0. Thus, the second simplest violation of (1.35) occurs when:

$$\left[f_1, \mathrm{ad}^3_{f_0}(f_1)\right](0) \notin \mathcal{S}_1. \tag{1.41}$$

However, Sussmann exhibits the following example which indicates that the violation (1.41) does not prevent a system from being L^∞ small-time locally controllable.

Example 3. *Let $n = 3$ and consider the following control-affine system:*

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1, \\ \dot{x}_3 = x_1^3 + x_2^2. \end{cases} \tag{1.42}$$

Around the null equilibrium, $\mathcal{S}_1 = \mathbb{R}e_1 + \mathbb{R}e_2$. One checks that:

$$[f_1, \mathrm{ad}^1_{f_0}(f_1)](0) = 0, \tag{1.43}$$

$$[f_1, \mathrm{ad}^3_{f_0}(f_1)](0) = 2e_3. \tag{1.44}$$

Hence, this system exhibits the violation (1.41). However, it is L^∞ small-time locally controllable (see [37, pages 711-712]).

Historically, Example 3 stopped the investigation of whether condition (1.35) was a necessary condition for small-time local controllability. One of the motivations of our work is to understand in what sense (1.35) can be seen as a necessary condition. We give further comments on Example 3 in Subsection 2.3.

1.7 A short survey of related results

The search for a necessary and sufficient condition for the small-time local controllability of differential control systems has a long history and many related references. We only provide here a short overview of some results connected to our work.

1.7.1 Lie algebra rank condition

A well known necessary condition for the L^∞ small-time local controllability of analytic systems is the Lie algebra rank condition (1.33) due to Hermann and Nagano [16, 29, 36]. By considering the example $\dot{x}_1 = ue^{-1/u^2}$ one sees that the analyticity assumption on f cannot be removed.

By the Frobenius theorem (see [21, Theorem 4] for a proof), if there exists $k \in \{1, \dots n\}$ such that, for every x in a neighborhood of 0, Span $\{h(x);\ \ h \in \mathrm{Lie}(f_0, f_1)\}$ is of dimension k, then the reachable set is (locally) contained in a submanifold of $\mathbb{R}^n$ of dimension k.

In the particular case of C^∞ driftless control-affine systems $\dot{x} = \sum_{i=1}^m u_i f_i(x)$, the Lie algebra rank condition is also a sufficient condition for the L^∞ small-time local controllability; this is the Rashevski-Chow theorem proved in [12]. However, the case of control-affine systems with drift is still widely open, even in the scalar-input case as in system (1.3).

1.7.2 Lie brackets necessary and sufficient conditions

Some necessary conditions and some sufficient conditions for L^∞ small-time local controllability were proved by means of Lie-series formalism (Chen-Fliess series), for analytic control-affine systems. First, necessary conditions:

- the Sussmann condition (1.36), proved in [37, Proposition 6.3, page 707],
- the Stefani condition [35]: $\mathrm{ad}_{f_1}^{2m}(f_0)(0) \in \mathcal{S}_{2m-1}$ for every $m \in \mathbb{N}^*$.

Then, sufficient conditions:

- the Hermes condition [17], recalled in Proposition 3 and proved by Sussmann in [37],
- the Sussman $S(\theta)$ condition, introduced in [38] (see also [13, Theorem 3.29]): there exists $\theta \in [0,1]$, such that every bracket involving f_0 an odd number l of times and f_1 an even number k of times must be a linear combination of brackets involving k_i times f_1 and l_i times f_0 and such that $\theta l_i + k_i < \theta l + k$. In some sense, this condition says that bad brackets may be neutralized by good ones. The weight θ has to be the same for all the bad brackets.
- the Kawski condition (see [23, Theorem 3.7]): there exists $\theta \in [0,1]$ such that every bracket involving f_0 an odd number l of times and f_1 an even number k of times is a linear combination of brackets of the form $\mathrm{ad}_{f_0}^{\nu_i}(h_i)$ where $\nu_i \geqslant 0$ and h_i is a bracket involving k_i times f_1 and l_i times f_0 with $\theta l_i + k_i < \theta l + k$.

In the particular case of control-affine systems that are homogeneous with respect to a family of dilatations (corresponding to time scalings in the control, not amplitude scalings), a necessary and sufficient condition for L^∞ small-time local controllability was proved by Aguilar and Lewis in [3, Theorem 4.1].

1.7.3 Control of bilinear systems without a priori bound on the control

A scalar-input bilinear system $\dot{g} = dL_g(h_0 + uh_1)$ on a semi simple compact Lie group U is (globally) controllable in large time if and only if the Lie algebra generated by h_0 and h_1 is equal to the compact semi-simple Lie algebra on U (see [4, 33, 39]). In [2], Agrachev and Chambrion use geometric control theory to estimate the minimal time needed for the global controllability of such systems. Contrary to the present article, these works do not impose any *a priori* bound on the control.

In [5, 6], Beauchard, Coron and Teisman propose classes of Schrödinger PDEs (infinite dimensional bilinear control systems) for which approximate controllability in L^2 is impossible in small time, even with large controls.

1.7.4 Quadratic approximations

In [1] and [18], Agrachev and Hermes proceed to local investigations of mappings of type input-state $F_t : u \mapsto x(t)$ (with fixed initial condition $x(0) = x^*$) near a fixed critical point $\bar{u}$ (i.e. for which the linearized system is not controllable) to determine whether $F_t(\bar{u})$ belongs to the interior or the boundary of the image of F_t, in the particular case when the linearized system misses only one direction.

In [18], Hermes proposes a sufficient condition for 0 to be an interior point in any time and a necessary condition for 0 to be a boundary point in small-time.

In [1], Agrachev proves that, when the quadratic form $F_t''(\bar{u})$ on Ker $F_t'(\bar{u})$ is definite on a subspace of Ker $F_t'(\bar{u})$ with finite codimension, then it is enough to find its inertia index (which is either a nonnegative integer, or $+\infty$) to answer the question. He describes flexible explicit formulas for the inertia index of F_t'' and uses them for a general study of the quadratic mapping F_t''.

Finally, Brockett proposes in [9] sufficient conditions for controllability in large time and in small time for systems with quadratic drifts:

$$\dot{y} = Ay + Bu \quad \text{and} \quad \dot{z} = y^{\mathrm{tr}} Q y, \tag{1.45}$$

where $y \in \mathbb{R}^d$ and $z \in \mathbb{R}$, with $Q \in \mathcal{M}_d(\mathbb{R})$ (satisfying some specific structural assumptions). In the case of a single scalar control, he proves that such systems are never small-time locally controllable (see [9, Lemma 4.1, page 444]). More precisely, he establishes that, if $Q \neq 0$ is a symmetric matrix contained within a specific subspace of dimension d, then there exists a $0 \leqslant k < d$ such that $(A^k B)^{\mathrm{tr}} Q (A^k B) \neq 0$ and such that z shares the same sign for trajectories starting from $y(0) = 0$ and for small enough times. The sign argument prevents small-time local controllability.

Our results can be seen as stemming from this sign argument. Indeed, we extend it to any matrix Q and more generally to any second-order expansion. Then, we improve the argument by proving that the positive quantity is in fact coercive with respect to some specific norm of the control. Last, we use this coercivity to overwhelm higher-order terms coming from Taylor expansions of general nonlinear systems.

2 Main results and examples

This section is organized as follows. We state our main results in Subsection 2.1. Then, we give comments in Subsection 2.2. We propose examples illustrating these statements and capturing the essential phenomena in Subsection 2.3, and examples proving the optimality of our functional framework in Subsection 2.4.

As stated in the abstract, the functional setting plays a key role in our results. For $T > 0$ and $m \in \mathbb{N}$, we consider the usual Sobolev spaces $W^{m,\infty}(0,T)$ equipped with their natural norm. We also use the subspaces $W^{m,\infty}_0(0,T)$ corresponding to functions $\varphi \in W^{m,\infty}(0,T)$ which satisfy $\varphi(0) = \varphi'(0) = \ldots = \varphi^{(m-1)}(0) = 0$ (no boundary condition is required at the right boundary for our proofs to hold). For $j \geqslant 0$, we define by induction the iterated primitives of u, denoted $u_j : (0,T) \to \mathbb{R}$ and defined by:

$$u_0 := u \quad \text{and} \quad u_{j+1}(t) := \int_0^t u_j(s) \mathrm{d}s. \tag{2.1}$$

For $p \in [1, +\infty]$ we let:

$$\|u\|_{W^{-1,p}(0,T)} := \|u_1\|_{L^p(0,T)}. \tag{2.2}$$

By convention, we set $W^{-1,\infty}_0(0,T) := W^{-1,\infty}(0,T)$ to avoid singling out this particular case. Eventually, the adjective *smooth* is used as a synonym of C^∞ throughout the text.

2.1 Statement of the main theorems

For a nonlinear system (1.1), our main result is the following quadratic alternative.

Theorem 2. *Let $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ with $f(0,0) = 0$. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be as in Definition 7. We define $d := \dim \mathcal{S}_1$ and the vector:*

$$d_0 := \partial_u^2 f(0,0) \in \mathbb{R}^n. \tag{2.3}$$

There exists a map $G \in C^\infty(\mathcal{S}_1, \mathcal{S}_1^\perp)$ with $G(0) = 0$ and $G'(0) = 0$, such that the following alternative holds:

- ***When** $\mathcal{S}_2 + \mathbb{R} d_0 = \mathcal{S}_1$, up to a cubicly small error in the control, the state lives within the smooth manifold $\mathcal{M} \subset \mathbb{R}^n$ of dimension d given by the graph of G:*

$$\mathcal{M} := \{ p_\parallel + G(p_\parallel);\ \ p_\parallel \in \mathcal{S}_1 \}. \tag{2.4}$$

More precisely, for every $T > 0$, there exists $C, \eta > 0$ such that, for any trajectory $(x, u) \in C^0([0,T], \mathbb{R}^n) \times L^\infty((0,T), \mathbb{R})$ of system (1.1) with $x(0) = 0$ and satisfying $\|u\|_{L^\infty} \leqslant \eta$, one has:

$$\forall t \in [0,T], \quad \left|\mathbb{P}^\perp x(t) - G\left(\mathbb{P}x(t)\right)\right| \leqslant C \|u\|_{L^3}^3 . \tag{2.5}$$

- ***When*** $d_0 \notin \mathcal{S}_1$*, for sufficiently small-times and regular controls, the state drifts with respect to the invariant manifold $\mathcal{M}$ in the direction $\mathbb{P}^\perp d_0$. More precisely:*
  - *System (1.1) is not L^∞ small-time locally controllable.*
  - *There exists $T^* > 0$ such that, for any $\mathcal{T} \in (0, T^*)$, there exists $\eta > 0$ such that, for any $T \in (0, \mathcal{T}]$ and any trajectory $(x, u) \in C^0([0,T], \mathbb{R}^n) \times L^\infty((0,T), \mathbb{R})$ of system (1.1) with $x(0) = 0$ and satisfying $\|u\|_{L^\infty} \leqslant \eta$, one has:*

$$\forall t \in [0,T], \quad \left\langle \mathbb{P}^\perp x(t) - G\left(\mathbb{P}x(t)\right), d_0 \right\rangle \geqslant 0. \tag{2.6}$$

- ***When*** $d_0 \in \mathcal{S}_1$ ***and*** $\mathcal{S}_2 \not\subset \mathcal{S}_1$*, for sufficiently small-times and regular controls, the state drifts with respect to the invariant manifold $\mathcal{M}$ in a fixed direction. More precisely:*
  - *There exists $k \in \{1, \dots, d\}$ such that*

$$[\mathrm{ad}_{f_0}^{j-1}(f_1), \mathrm{ad}_{f_0}^{j}(f_1)](0) \in \mathcal{S}_1 \quad \text{for } 1 \leqslant j < k, \tag{2.7}$$

$$[\mathrm{ad}_{f_0}^{k-1}(f_1), \mathrm{ad}_{f_0}^{k}(f_1)](0) \notin \mathcal{S}_1. \tag{2.8}$$

  - *System (1.1) is not $W^{2k,\infty}$ small-time locally controllable.*
  - *There exists $T^* > 0$ such that, for any $\mathcal{T} \in (0, T^*)$, there exists $\eta > 0$ such that, for any $T \in (0, \mathcal{T}]$ and any trajectory $(x, u) \in C^0([0,T], \mathbb{R}^n) \times L^\infty((0,T), \mathbb{R})$ of system (1.1) with $x(0) = 0$ and $u \in W_0^{2k,\infty}$ satisfying $\|u\|_{W^{2k,\infty}} \leqslant \eta$, one has:*

$$\forall t \in [0,T], \quad \left\langle \mathbb{P}^\perp x(t) - G\left(\mathbb{P}x(t)\right), d_k \right\rangle \geqslant 0, \tag{2.9}$$

  where the drifting direction $d_k \neq 0$ is defined as:

$$d_k := -\mathbb{P}^\perp[\mathrm{ad}_{f_0}^{k-1}(f_1), \mathrm{ad}_{f_0}^{k}(f_1)](0). \tag{2.10}$$

Corollary 1. *Assume that $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ with $f(0,0) = 0$ and system (1.1) is smoothly small-time locally controllable. Then $\mathcal{S}_2 + \mathbb{R}d_0 = \mathcal{S}_1$.*

Proof. Assume that $\mathcal{S}_2 + \mathbb{R}d_0 \not\subset \mathcal{S}_1$. From Theorem 2, either $d_0 \notin \mathcal{S}_1$ and system (1.1) is not C^0 small-time locally controllable, or $d_0 \in \mathcal{S}_1$ and $\mathcal{S}_2 \not\subset \mathcal{S}_1$ and there exists $1 \leqslant k \leqslant d < n$ such that it is not C^{2k} small-time locally controllable. Both cases contradict Definition 5. ☐

In the particular case of control-affine systems (1.3), the optimal functional framework for the conclusions of Theorem 2 to hold can be improved. In the first case, it is sufficient that the control be small in $W^{-1,\infty}$-norm (instead of L^∞-norm), whereas in the third case, it is sufficient that the control be small in $W^{2k-3,\infty}$-norm (instead of $W^{2k,\infty}$-norm).

Theorem 3. *Let $f_0, f_1 \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $f_0(0) = 0$ and $d := \dim \mathcal{S}_1$. There exists a map $G \in C^\infty(\mathcal{S}_1, \mathcal{S}_1^\perp)$ with $G(0) = 0$ and $G'(0) = 0$, such that the following alternative holds:*

- ***When*** $\mathcal{S}_2 = \mathcal{S}_1$*, up to a cubicly small error in the control, the state lives within a smooth manifold $\mathcal{M} \subset \mathbb{R}^n$ of dimension d given by the graph of G (see (2.4)). More precisely, for every $\mathcal{T} > 0$, there exists $C, \eta > 0$ such that, for any $T \in (0, \mathcal{T}]$, for any trajectory $(x, u) \in C^0([0,T], \mathbb{R}^n) \times L^1((0,T), \mathbb{R})$ of system (1.3) with $x(0) = 0$ which satisfies $\|u\|_{W^{-1,\infty}} \leqslant \eta$, one has:*

$$\forall t \in [0,T], \quad \left|\mathbb{P}^\perp x(t) - G\left(\mathbb{P}x(t)\right)\right| \leqslant C \|u\|_{W^{-1,3}}^3 . \tag{2.11}$$

- ***When*** $\mathcal{S}_2 \not\subset \mathcal{S}_1$, *for sufficiently small-times and small regular controls, the state drifts with respect to the invariant manifold* $\mathcal{M}$ *in a fixed direction. More precisely:*
  - *There exists* $1 \leqslant k \leqslant d$ *such that* (2.7) *and* (2.8) *hold.*
  - *System* (1.3) *is not* $W^{2k-3,\infty}$ *small-time locally controllable.*
  - *There exists* $T^* > 0$ *such that, for any* $\mathcal{T} \in (0, T^*)$, *there exists* $\eta > 0$ *such that, for any* $T \in (0, \mathcal{T}]$ *and any trajectory* $(x, u) \in C^0([0,T], \mathbb{R}^n) \times L^1((0,T), \mathbb{R})$ *of system* (1.3) *with* $x(0) = 0$ *such that* $u \in W_0^{2k-3,\infty}$ *with* $\|u\|_{W^{2k-3,\infty}} \leqslant \eta$, *inequality* (2.9) *holds.*

2.2 Comments on the main theorems

- The linear subspace $\mathcal{S}_1$ is the tangent space to the manifold $\mathcal{M}$ at 0 because $G'(0) = 0$.
- The optimality of our result, in terms of the norms used in the smallness assumption on the control, is illustrated in Subsection 2.4. Even in the case of bilinear systems (for which f_0 and f_1 are linear in x), the norms involved in Theorem 3 are optimal.
- We give a characterization of the time T^* in paragraph 5.4.1: it is the maximal time for which some coercivity property holds for an appropriate second-order approximation of the system under study. Therefore, it does not depend on higher-order terms.
- When $\mathcal{S}_2 + \mathbb{R}d_0 \not\subset \mathcal{S}_1$, the drift relations (2.6) and (2.9) can be used to exhibit impossible motions. In particular, they imply that there exists $T, \eta > 0$ such that, for any $x^\dagger \in \mathcal{S}_1^\perp$ satisfying $\langle x^\dagger, d_k \rangle < 0$, no motion from $x(0) = 0$ to $x(T) = x^\dagger$ is possible with controls of $W^{2k,\infty}$-norm smaller then η (or $W^{2k-3,\infty}$-norm smaller then η for control-affine systems). Other impossible motions can also be exhibited, by time reversibility. For instance, no motion from an initial state $x^* \in \mathcal{S}_1^\perp$ such that $\langle x^*, d_k \rangle > 0$ to $x(T) = 0$ is possible for small times and small controls. More generally, motions going against the drift direction are impossible for small-times and small controls.
- For a particular class of systems ("well-prepared systems", i.e. that satisfy (5.1)), inequalities (2.6) and (2.9) hold in the following stronger form:

$$\forall t \in [0,T], \quad \left\langle \mathbb{P}^\perp x(t) - G\left(\mathbb{P}x(t)\right), d_k \right\rangle \geqslant C(\mathcal{T}) \int_0^t u_k(s)^2 \mathrm{d}s, \tag{2.12}$$

  where u_k is defined in (2.1). Such an estimate also holds for general (i.e. not well-prepared) systems but with a value of T^* that may be strictly smaller than the one explicitly given in paragraph 5.4.1 (see paragraph 6.2.4).
- The drift relations (2.6) and (2.9) hold for controls in $W_0^{m,\infty}$ (with $m = 2k$ for nonlinear systems and $m = 2k-3$ for control-affine systems). For controls which only belong to $W^{m,\infty}$, they are still true at the final time, but the bound η depends on T (and $\eta(T) \to 0$ as $T \to 0$). The persistence of the drift at the final time under this weaker assumption will be clear from the proof (see paragraph 5.4.2) and allows to deny $W^{m,\infty}$ small-time local controllability, and not only $W_0^{m,\infty}$ small-time local controllability (see also paragraph 2.4.4 for more insight on this topic).
- To lighten the statement of the theorems, we assumed that the vector fields are smooth. This guarantees that the spaces $\mathcal{S}_1$ and $\mathcal{S}_2$ are well-defined. However, it is in fact possible to give a meaning to the considered objects when $f \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ (see Subsection 3.3). Moreover, the conclusions of Theorem 2 and 3 persist for nonsmooth dynamics (see Corollary 2 and 3 in Subsection 6.3). In particular, these corollaries imply that, even in the case of smooth vector fields, the constants in our main theorems only depend on low-order derivatives of the dynamic.

2.3 Illustrating toy examples

2.3.1 Evolution within an invariant manifold

In the absence of drift (i.e. when $\mathcal{S}_2 = \mathcal{S}_1$), our theorems imply that the state must live within an invariant manifold, up to the cubic order. A simple case for which the manifold is not trivial (i.e. $\mathcal{M} \neq \mathcal{S}_1$) is the following toy model, for which the state stays exactly within the manifold, without any remainder.

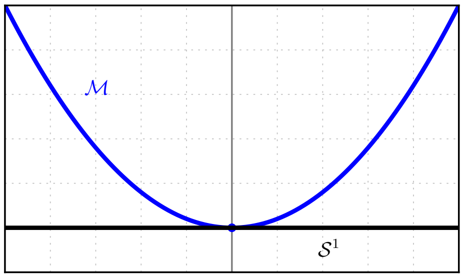

Figure 1: The toy invariant manifold (2.14) for system (2.13).

Example 4. *Let $n = 2$. We consider the following control-affine system:*

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = 2ux_1. \end{cases} \tag{2.13}$$

Here, $\mathcal{S}_1 = \mathcal{S}_2 = \mathbb{R}e_1$. Using the notation u_1 introduced in (2.1), system (2.13) can be integrated as $x_1(t) = u_1(t)$ and $x_2(t) = u_1^2(t)$. For any control and any time $t \in [0, T]$, $x(t) \in \mathcal{M}$ (see Figure 1), where:

$$\mathcal{M} = \{(x_1, x_2) \in \mathbb{R}^2, \ \ x_2 = x_1^2\}. \tag{2.14}$$

One checks that $\mathcal{S}_1$ is indeed the tangent space to $\mathcal{M}$ at the origin.

2.3.2 Quadratic drifts and coercivity

The simplest kind of quadratic drift for control-affine systems was exposed in Example 2. Let us study two new examples.

Example 5. *Let $n = 1$. We consider the following control-affine system:*

$$\dot{x}_1 = u + x_1^2. \tag{2.15}$$

Here, $\mathcal{S}_1 = \mathbb{R}e_1$ and Theorem 1 asserts that system (2.15) is smoothly small-time locally controllable. The potential drift direction $[f_1, [f_0, f_1]] = -2e_1 \in \mathcal{S}_1$ and is actually absorbed by the linear controllability in the vicinity of the null equilibrium.

Example 6. *Let $n = 3$. We consider the following control-affine system, which is also proposed by Brockett in [9, page 445]:*

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1, \\ \dot{x}_3 = x_1^2 - x_2^2. \end{cases} \tag{2.16}$$

Here, $\mathcal{S}_1 = \mathbb{R}e_1 + \mathbb{R}e_2$ and $[f_1, [f_0, f_1]] = -2e_3 \notin \mathcal{S}_1$. Thus, Theorem 3 states that, in small-time, there will be a quadratic drift preventing the system to be $W^{-1,\infty}$ small-time locally controllable. One can actually compute explicitly the minimal time and prove that:

- *if $T \leqslant \pi$, system (2.16) is not locally controllable in time T (even with large controls),*
- *if $T > \pi$, system (2.16) is smoothly locally controllable in time T.*

Here, there is a quadratic competition, and one of the terms is dominant in small-time:

$$x_3(T) = x_3(0) + \int_0^T \left(\dot{x}_2(t)^2 - x_2(t)^2\right) \mathrm{d}t. \tag{2.17}$$

The proof of both statements is related to the Poincaré-Wirtinger inequality (see [15, p. 47]), that holds for any $T > 0$ and any function $\varphi \in H^1_0((0,T),\mathbb{R})$:

$$\int_0^T \varphi'(t)^2 \mathrm{d}t \geqslant \left(\frac{\pi}{T}\right)^2 \int_0^T \varphi(t)^2 \mathrm{d}t. \tag{2.18}$$

The constant in the right-hand side is optimal and achieved by the function $\varphi(t) := \sin\left(\frac{\pi t}{T}\right)$.

First case: *$T \leqslant \pi$. Let $(x,u) \in C^0([0,T],\mathbb{R}^3) \times L^1((0,T),\mathbb{R})$ be a trajectory of system (2.16) such that $x_2(0) = x_2(T) = 0$. Combining (2.17) and (2.18) yields:*

$$x_3(T) - x_3(0) \geqslant \left(\left(\frac{\pi}{T}\right)^2 - 1\right) \int_0^T x_2(t)^2 \mathrm{d}t \geqslant 0. \tag{2.19}$$

Thus the system is not locally controllable in time T.

Second case: *$T > \pi$. We prove that, for any $m \in \mathbb{N}$ and any $\eta > 0$, there exists $\delta > 0$ such that, for any $x^*, x^\dagger \in \mathbb{R}^3$ with $|x^*| + |x^\dagger| \leqslant \delta$, there exists a trajectory of (2.16) $(x,u) \in C^0([0,T],\mathbb{R}^3) \times L^1((0,T),\mathbb{R})$ such that $x(0) = x^*$, $x(T) = x^\dagger$ and $\|u\|_{C^m} \leqslant \eta$.*

By an argument that relies on the Brouwer fixed point theorem (see [13, Chapter 8]), it suffices to prove the existence of $u^\pm \in C^\infty([0,T],\mathbb{R})$ such that the associated solution of system (2.16) with initial conditions $x^\pm(0) = 0$ satisfy $x^\pm(T) = (0,0,\pm 1)$.

Thanks to the previous case, one can obtain u^+ by rescaling any smooth function supported on an interval of length less than π and such that $\int_0^T u(t)\mathrm{d}t = \int_0^T tu(t)\mathrm{d}t = 0$.

Now, we construct u^-. To that end, we introduce $(\rho_\epsilon)_{\epsilon>0}$ a family of functions in $C^\infty_c((0,T),[0,1])$ with $\mathrm{supp}(\rho_\epsilon) \subset [\epsilon, T-\epsilon]$, even with respect to $T/2$, such that, for any $t \in (0,T)$, $\rho_\epsilon(t) \to 1$ as $\epsilon \to 0$. By the dominated convergence theorem, one has:

$$\int_0^T \left|\rho_\epsilon(t)\frac{\pi}{T}\cos\left(\frac{\pi t}{T}\right)\right|^2 \mathrm{d}t \underset{\epsilon\to 0}{\longrightarrow} \int_0^T \left|\frac{\pi}{T}\cos\left(\frac{\pi t}{T}\right)\right|^2 \mathrm{d}t = \frac{\pi^2}{2T}, \tag{2.20}$$

$$\int_0^T \left|\int_0^t \rho_\epsilon(\tau)\frac{\pi}{T}\cos\left(\frac{\pi\tau}{T}\right) d\tau\right|^2 \mathrm{d}t \underset{\epsilon\to 0}{\longrightarrow} \int_0^T \left|\sin\left(\frac{\pi t}{T}\right)\right|^2 \mathrm{d}t = \frac{T}{2}. \tag{2.21}$$

Thus, for $\epsilon > 0$ small enough, since $T > \pi$:

$$\zeta_\epsilon := \int_0^T \left|\rho_\epsilon(t)\frac{\pi}{T}\cos\left(\frac{\pi t}{T}\right)\right|^2 \mathrm{d}t - \int_0^T \left|\int_0^t \rho_\epsilon(\tau)\frac{\pi}{T}\cos\left(\frac{\pi\tau}{T}\right) \mathrm{d}\tau\right|^2 \mathrm{d}t < 0. \tag{2.22}$$

Let us fix such an ϵ. The solution of (2.16) with initial condition $x^-(0) = 0$ and control:

$$u^-(t) := \frac{1}{\sqrt{-\zeta_\epsilon}}\partial_t\left[\rho_\epsilon(t)\frac{\pi}{T}\cos\left(\frac{\pi t}{T}\right)\right] \tag{2.23}$$

satisfies:

$$x_1^-(t) = \frac{1}{\sqrt{-\zeta_\epsilon}} \rho_\epsilon(t) \frac{\pi}{T} \cos\left(\frac{\pi t}{T}\right), \tag{2.24}$$

$$x_2^-(t) = \frac{1}{\sqrt{-\zeta_\epsilon}} \int_0^t \rho_\epsilon(\tau) \frac{\pi}{T} \cos\left(\frac{\pi \tau}{T}\right) \mathrm{d}\tau, \tag{2.25}$$

$$x_3^-(T) = \int_0^T \left(x_1^-(t)^2 - x_2^-(t)^2 \right) \mathrm{d}t = \left(\frac{1}{\sqrt{-\zeta_\epsilon}} \right)^2 \zeta_\epsilon = -1. \tag{2.26}$$

Hence $x_1^-(T) = 0$ because $\rho_\epsilon(T) = 0$ and $x_2^-(T) = 0$ as the integral over $(0, T)$ of a function which is odd with respect to $T/2$.

2.3.3 Resolution of Sussmann's paradox example

We turn back to Sussmann's historical Example 3. Taking a trajectory satisfying $x(0) = 0$, explicit integration of (1.42) yields:

$$x_1(t) = u_1(t), \quad x_2(t) = u_2(t) \quad \text{and} \quad x_3(t) = \int_0^t \left(u_1^3(s) + u_2^2(s) \right) \mathrm{d}s. \tag{2.27}$$

Up to the second-order, the component $x_3(T)$ is a coercive quadratic form with respect to $\|u\|_{H^{-2}} = \|u_2\|_{L^2}$. However, the sum of the orders 0+1+2 is not a good approximation of the nonlinear solution for the same norm $\|u\|_{H^{-2}}$ when u is small in L^∞. Indeed:

$$\int_0^T u_1^3(t)\mathrm{d}t \neq \underset{\|u\|_{L^\infty} \to 0}{o} \left(\int_0^T u_2^2(t)\mathrm{d}t \right). \tag{2.28}$$

Nevertheless, if the smallness assumption on the control in strengthened into $\|u\|_{W^{1,\infty}} \ll 1$, then the quadratic approximation becomes dominant. Considering a trajectory such that $x_1(T) = x_2(T) = 0$ and using integration by parts yields:

$$\int_0^T u_1^3(t)\mathrm{d}t = -2 \int_0^T u_2(t) u_1(t) u(t) \mathrm{d}t = \int_0^T u_2^2(t) \dot{u}(t) \mathrm{d}t. \tag{2.29}$$

Combining (2.27) and (2.29) yields:

$$x_3(T) = \int_0^T u_2^2(t) \left(1 + \dot{u}(t)\right) \mathrm{d}t \geqslant \frac{1}{2} \|u_2\|_{L^2}^2, \tag{2.30}$$

provided that $\|u\|_{W^{1,\infty}(0,T)} \leqslant 1/2$. From (2.30), we deduce that it is impossible to reach states of the form $(0, 0, -\delta)$ with $\delta > 0$. Hence, system (1.42) is not $W^{1,\infty}$ small-time locally controllable. Our point of view is that condition (1.41) allows to deny small-time controllability with small $W^{1,\infty}$ controls. We think that this notion is pertinent because it highlights that the quadratic drift can only be avoided with highly oscillating controls. We recover here the conclusion of Theorem 3 with $k = 2$, in the particular case of Example 3.

2.3.4 Drifts with respect to an invariant manifold

In Examples 2, 3 and 6, the drifts hold relatively to the invariant manifold $\mathcal{M} = \mathcal{S}_1$. However, the drift can also hold with respect to a bent manifold.

Example 7. *Let $n = 2$ and $\lambda \in \mathbb{R}$. We consider the following control-affine system:*

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = 2ux_1 + \lambda x_1^2. \end{cases} \tag{2.31}$$

Straightforward explicit integration of (2.31) *yield, for any* $t \geqslant 0$*:*

$$x_2(t) = x_1^2(t) + \lambda \int_0^t x_1^2(s)\mathrm{d}s. \tag{2.32}$$

Equation (2.32) *illustrates the idea that, when* $\lambda \neq 0$*, the state* (x_1, x_2) *endures a quadratic drift with respect to the manifold* (2.14)*. The sign of the drift depends on* λ*: when it is positive, the state is above the manifold and reciprocally.*

The direction of the drift is not related in any way with the curvature of the manifold at the origin. Elementary adaptations of (2.31) in a three dimensional context can be built so that the drift holds along a direction which is orthogonal to the curvature of the manifold.

2.3.5 Higher-order behaviors

When $\mathcal{S}_2 = \mathcal{S}_1$, Theorem 3 asserts that the state can only leave the manifold at cubic order with respect to the control. One must then continue the expansion further on, as multiple different behaviors are possible (see Figure 2). We propose examples exhibiting different higher-order properties.

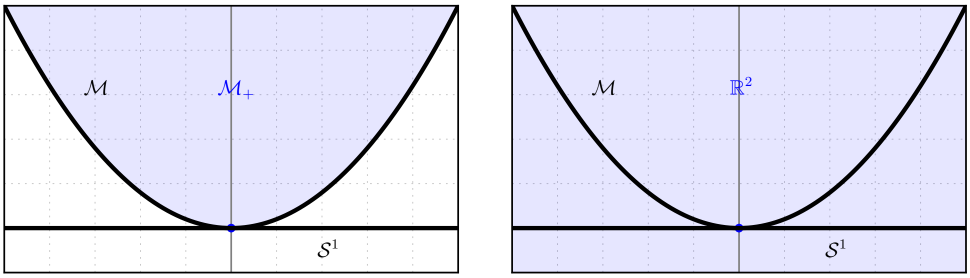

Figure 2: Influence of higher-order terms on the reachable set. *Left*: a higher-order drift pushes the state over the manifold. *Right*: small-time controllability is recovered.

Example 8. *To recover the situation exposed in Figure 2, left, where a higher-order drift pushes the state over the manifold, we perturb system* (2.13) *into:*

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = 2ux_1 + x_1^4. \end{cases} \tag{2.33}$$

This modification changes neither $\mathcal{S}_1$ *nor* $\mathcal{S}_2$*. For any* $t \geqslant 0$*:*

$$x_2(t) = x_1^2(t) + \int_0^t x_1^4(s)\mathrm{d}s. \tag{2.34}$$

Hence, the state is constrained to involve within:

$$\mathcal{M}_+ := \left\{(x_1, x_2) \in \mathbb{R}^2, \ \ x_2 - x_1^2 \geqslant 0\right\}. \tag{2.35}$$

Example 9. *Higher-order terms can also help to recover small-time local controllability, as in Figure 2, right. One possibility is to introduce* $\beta \in \mathbb{R}$ *and perturb system* (2.13) *into:*

$$\begin{cases} \dot{x}_1 = u + \beta x_2, \\ \dot{x}_2 = 2ux_1. \end{cases} \tag{2.36}$$

Here again, $\mathcal{S}_1$ and $\mathcal{S}_2$ are preserved. However, one checks that:

$$[f_1, [f_1, [f_0, f_1]]] = 12\beta e_2. \tag{2.37}$$

Hence, when $\beta \neq 0$, the Hermes local controllability condition is satisfied since $\mathcal{S}_2 = \mathcal{S}_1$ and $\mathcal{S}_3 = \mathbb{R}^2$ from (2.37). Thus, one obtains that system (2.36) is L^∞ small-time locally controllable (see [37, Theorem 2.1, page 688]).

Example 10. *The following control-affine example was introduced by Kawski in [23, Example 4.1], where it is proved that it is L^∞ small-time locally controllable. Let $n = 4$ and consider:*

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1, \\ \dot{x}_3 = x_1^3, \\ \dot{x}_4 = x_2 x_3. \end{cases} \tag{2.38}$$

One checks that $\mathcal{M} = \mathcal{S}_2 = \mathcal{S}_1 = \mathbb{R}e_1 + \mathbb{R}e_2$. Therefore, Theorem 3 does not raise an obstruction to controllability (at best, the lost directions will be recovered at the cubic order).

2.3.6 Examples for nonlinear systems

Regarding our work, nonlinear systems exhibit two new features with respect to control-affine systems: the possibility of a u^2 term and a u^3 term in the vector field.

Example 11. *Let $n = 2$ and $\lambda \in \mathbb{R}$. We consider the following non-linear system:*

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1^2 + \lambda u^2 + u^3. \end{cases} \tag{2.39}$$

When $\lambda \neq 0$, there is a new strong drift, which involves the L^2-norm of the control and exceeds any other possible drift. It holds along the direction $\partial_u^2 f(0,0)$ (here, $2\lambda e_2$) as long as u is small in L^∞-norm (so that the cubic term can be ignored). When $\lambda = 0$, the strongest possible drift is the one involving the H^{-1}-norm of the control and is similar to the one described for control-affine systems. We recover the obstruction corresponding to the first bad Lie bracket $[f_1, [f_0, f_1]](0) = -2e_2$. However, the smallness assumption involves the $W^{2,\infty}$-norm of the control in order to ensure that the u^3 term can be ignored (which is not the case with smallness in $W^{-1,\infty}$).

2.4 Optimality of the norm hypothesis

We give examples of systems proving that the smallness assumptions used in Theorems 2 and 3 cannot be improved (at least in the range of Sobolev spaces).

2.4.1 Optimality for control-affine systems

Example 12. *Let $k \in \mathbb{N}^*$. We consider the following control-affine system set in $\mathbb{R}^{k+1}$:*

$$\begin{cases} \dot{x}_1 = u, & \\ \dot{x}_{j+1} = x_j, & \text{for } 1 \leqslant j < k, \\ \dot{x}_{k+1} = x_k^2 + x_1^3. & \end{cases} \tag{2.40}$$

The controllable space is such that $\mathcal{S}_1^\perp = \mathbb{R}e_{k+1}$. Moreover, one checks that:

$$[\mathrm{ad}_{f_0}^{j-1}(f_1), \mathrm{ad}_{f_0}^{j}(f_1)](0) = 0, \qquad \text{for } 1 \leqslant j < k, \tag{2.41}$$

$$[\mathrm{ad}_{f_0}^{k-1}(f_1), \mathrm{ad}_{f_0}^{k}(f_1)](0) = -2e_{k+1}. \tag{2.42}$$

Hence, we are in the setting of the quadratic obstruction of order k. Straightforward integration of (2.40) for trajectories with $x(0) = 0$ yields $x_j = u_j$ for $1 \leqslant j \leqslant k$ and:

$$x_{k+1}(T) = \int_0^T u_k^2(t)\mathrm{d}t + \int_0^T u_1^3(t)\mathrm{d}t. \tag{2.43}$$

Hence, the existence of drift amounts to the existence of a Sobolev embedding relation.

Let us prove that our functional setting is optimal. More precisely, we wish to construct controls realizing both signs in relation (2.43) and which are small in the most regular spaces. We look directly for u_k. We choose a function $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$, compactly supported in $(0, T)$, such that $\int_0^T \varphi^2 = 1$ and $a := \int_0^T \left(\varphi^{(k-1)}\right)^3 \neq 0$. Let $\lambda \in \mathbb{R}$ and $\mu \geqslant 1$. We define the family of dilatations:

$$\varphi_{\lambda,\mu}(t) := \lambda\varphi(\mu t). \tag{2.44}$$

For $m \in [-1, +\infty)$, straightforward scaling arguments lead to:

$$\left\| \varphi_{\lambda,\mu}^{(k)} \right\|_{W^{m,\infty}} = |\lambda| \mu^{m+k} \left\| \varphi^{(k)} \right\|_{W^{m,\infty}}. \tag{2.45}$$

Hence, the control $u := \varphi_{\lambda,\mu}^{(k)}$ is small in $W^{m,\infty}$ when $|\lambda|\mu^{m+k} \ll 1$. Moreover,

$$\int_0^T \varphi_{\lambda,\mu}^2 = \lambda^2 \int_0^T \varphi^2(\mu t)\mathrm{d}t = \lambda^2\mu^{-1} \int_0^{\mu T} \varphi^2 = \lambda^2\mu^{-1} \tag{2.46}$$

and:

$$\begin{aligned} \int_0^T \left(\varphi_{\lambda,\mu}^{(k-1)}\right)^3 &= \lambda^3\mu^{3k-3} \int_0^T \left(\varphi^{(k-1)}\right)^3 (\mu t)\mathrm{d}t \\ &= \lambda^3\mu^{3k-4} \int_0^{\mu T} \left(\varphi^{(k-1)}\right)^3 \\ &= a\lambda^3\mu^{3k-4}. \end{aligned} \tag{2.47}$$

Hence, the cubic term dominates the quadratic term when $|\lambda|\mu^{3k-3} \gg 1$. Since $\mu \geqslant 1$, this relation is compatible with smallness of the control in $W^{m,\infty}$ if and only if $m + k < 3k - 3$. Thus, the lost direction can be recovered with controls which are small in $W^{m,\infty}$ for any $m < 2k-3$. Indeed, sign λ can take both signs, and since φ is compactly supported in $(0, T)$, we can build controls $u^\pm$ and associated trajectories $x^\pm$ of (2.40) with $x^\pm(0) = 0$ satisfying $x^\pm(T) = (0, \ldots, 0, \pm 1)$. Using an argument based on the Brouwer fixed point theorem as in [13, Chapter 8] enables us to conclude that system (2.40) is $W^{m,\infty}$ small-time locally controllable for $m < 2k - 3$. From an homogeneity point of view, our theorem involving a smallness assumption in $W^{2k-3,\infty}(0, T)$ cannot be improved using Sobolev spaces.

When $k = 2$, system (2.40) corresponds to Sussmann's Example 3 and we conclude that this system is $W^{m,\infty}$ small-time locally controllable for any $m \in [-1, 1)$. In particular, we recover the fact, proved by Sussmann, that it is L^∞ small-time locally controllable.

2.4.2 Optimality for nonlinear systems

Example 13. *Let $k \in \mathbb{N}^*$. We consider the following nonlinear system set in $\mathbb{R}^{k+1}$:*

$$\begin{cases} \dot{x}_1 = u, & \\ \dot{x}_{j+1} = x_j, & \text{for } 1 \leqslant j < k, \\ \dot{x}_{k+1} = x_k^2 + u^3. & \end{cases} \tag{2.48}$$

From Theorem 2, this system is not is not $W^{2k,\infty}$ small-time locally controllable. The arguments used in the previous paragraph can be adapted to prove that the smallness assumption in $W^{2k,\infty}(0, T)$ cannot be improved using Sobolev spaces.

2.4.3 Optimality for bilinear systems

One could think that our assumption can be enhanced for some better-behaved classes of systems, e.g. bilinear control systems, for which the pathologic cubic term cannot be so easily injected. It turns out to be false. The same kind of counterexample can be built within the bilinear class. For brevity, we give an example only for the first obstruction (but examples for higher-order obstructions are also possible).

Example 14. *We work with $n = 5$. We consider the bilinear system:*

$$\begin{cases} \dot{x}_1 = 0, \\ \dot{x}_2 = ux_1, \\ \dot{x}_3 = 2ux_2, \\ \dot{x}_4 = 3ux_3, \\ \dot{x}_5 = x_5 + ux_2 + x_4, \end{cases} \tag{2.49}$$

around its equilibrium $x_e = (1, 0, 0, 0, 0)$. One checks that $d = 1$, that $\mathcal{S}_1 = \mathbb{R}e_2$ and that $\mathcal{S}_2 \not\subset \mathcal{S}_1$. Indeed, $[f_1, [f_0, f_1]](x_e) = -2e_5$. Starting from the initial state $x(0) = x_e$, we have $x_1 = 1$, $x_2 = u_1$, $x_3 = u_1^2$ and $x_4 = u_1^3$. Using the Duhamel formula, we obtain:

$$x_5(T) = \int_0^T e^{T-t} \left(u(t)u_1(t) + u_1^3(t)\right) \mathrm{d}t. \tag{2.50}$$

Assuming that we are looking for trajectories satisfying $x_2(T) = 0$, we have $u_1(T) = 0$. Hence, integration by parts leads to:

$$x_5(T)e^{-T} = \frac{1}{2} \int_0^T e^{-t} u_1^2(t) \mathrm{d}t + \int_0^T e^{-t} u_1^3(t) \mathrm{d}t. \tag{2.51}$$

For singular controls and short times, the exponential multiplier in the integrand does not play an important role. Thus, we recover the key balance of (2.43) and the smallness assumption of our theorem cannot be improved even within the favorable class of bilinear systems.

2.4.4 Optimality of the trace hypothesis

We give an example illustrating why the drift relation holds for any time only when the controls have enough vanishing traces at the initial time. We consider once more Sussmann's Example 3 and we compute the solution associated with a constant control $u(t) := \eta$, starting from $x(0) = 0$. One has:

$$x_1(t) = \eta t, \quad x_2(t) = \frac{1}{2}\eta t^2 \quad \text{and} \quad x_3(t) = \frac{1}{20}\eta^2 t^5 + \frac{1}{4}\eta^3 t^4. \tag{2.52}$$

Hence, for any $T > 0$, if $|\eta|$ is small enough, then $x_3(T) \geqslant 0$. However, for any $\eta < 0$, there exists t small enough such that $x_3(t) < 0$. This illustrates the comment announced in Subsection 2.2:

- the drift relation holds at the final time without any additional assumption on the traces of the control at the initial time,
- the drift relation also holds for any time provided that a sufficient number of traces of the control vanish at the initial time.

Indeed, if we consider $u(t) = \eta t$ (a control for which $u(0) = 0$), then:

$$x_1(t) = \frac{1}{2}\eta t^2, \quad x_2(t) = \frac{1}{6}\eta t^3 \quad \text{and} \quad x_3(t) = \frac{1}{252}\eta^2 t^7 + \frac{1}{56}\eta^3 t^7. \tag{2.53}$$

Hence, if $|\eta|$ is small enough, $x_3(t) \geqslant 0$ for any $t \geqslant 0$. Of course, similar limiting examples can be built at higher orders. We refer to paragraph 5.4.2 for a detailed proof of the existence of a drift, either at the final time or at any time.

2.5 Plan of the paper

The remainder of the paper is organized in the following way:

- In Section 3, we prove preliminary algebraic and analytic results concerning the Lie brackets involved in the spaces $\mathcal{S}_1$ and $\mathcal{S}_2$ that shed light on their structure.
- In Section 4, we introduce auxiliary systems useful in the proof of our alternative. They are obtained by iterated application of appropriate flows.
- In Section 5, we prove our quadratic alternative for a particular class of well-prepared smooth systems, whose linear behavior is a nilpotent iterated integrator.
- In Section 6, we show that the quadratic behavior both of general control-affine and nonlinear systems can be reduced to this particular class of well-prepared systems. We also extend our alternative to non-smooth dynamics.
- In Section 7, we introduce an appropriate second order approximation of nonlinear systems that stays exactly within a smooth quadratic manifold which can be explicitly computed and corresponds to the second-order approximation of the manifold constructed in the smooth case.
- In Section 8, we explore an alternative definition of small-time local controllability.

3 Algebraic properties of the Lie spaces

We explore the structure of the Lie spaces $\mathcal{S}_1$ and $\mathcal{S}_2$ to gain some insight on these objects and to prove useful lemmas. We compute the involved Lie brackets explicitly and we extend the definition of these spaces to nonsmooth vector fields.

3.1 Computations modulo higher-order terms

With a view to computing explicity the Lie brackets of $\mathcal{S}_1$ and $\mathcal{S}_2$, we will need to carry out computations modulo higher-order terms. Indeed, higher-order terms do not impact the Lie brackets we want to consider. We use the following concept.

Definition 9. *Let $m, p \in \mathbb{N}^*$ and $\varphi, \tilde{\varphi} \in C^\infty(\mathbb{R}^p, \mathbb{R}^p)$. We say that φ and $\tilde{\varphi}$ are* equal modulo m-th order terms *and write $\varphi =_{[m]} \tilde{\varphi}$ when there exists $\theta \in C^\infty(\mathbb{R}^p, \mathcal{T}^{m,1}_{p,p})$, where $\mathcal{T}^{m,1}_{p,p}$ denotes the space of symmetric multilinear maps from $(\mathbb{R}^p)^m$ to $\mathbb{R}^p$, such that:*

$$\forall x \in \mathbb{R}^p, \quad \varphi(x) = \tilde{\varphi}(x) + \theta(x)[x, \dots, x]. \tag{3.1}$$

Remark 1. *In particular, when $\varphi =_{[m]} \tilde{\varphi}$, since θ is locally bounded near 0, we have:*

$$\varphi(x) = \tilde{\varphi}(x) + \underset{x \to 0}{O}\left(|x|^m\right). \tag{3.2}$$

However, (3.2) is a weaker hypothesis than (3.1) because it does not allow differentiation, which is necessary in our context for the computation of Lie brackets.

Lemma 1. *Let $\varphi, \tilde{\varphi}, \psi, \tilde{\psi} \in C^\infty(\mathbb{R}^p, \mathbb{R}^p)$. We assume that:*

$$\varphi =_{[1]} 0, \quad \varphi =_{[3]} \tilde{\varphi}, \quad \text{and} \quad \psi =_{[2]} \tilde{\psi}. \tag{3.3}$$

Then, there holds:

$$[\varphi, \psi] =_{[2]} [\tilde{\varphi}, \tilde{\psi}]. \tag{3.4}$$

Proof. From hypothesis (3.3), there exists functions $\theta_0 \in C^\infty(\mathbb{R}^p, \mathcal{T}^{1,1}_{p,p})$, $\theta_\varphi \in C^\infty(\mathbb{R}^p, \mathcal{T}^{3,1}_{p,p})$ and $\theta_\psi \in C^\infty(\mathbb{R}^p, \mathcal{T}^{2,1}_{p,p})$ such that, for any $x \in \mathbb{R}^p$:

$$\varphi(x) = \theta_0(x)[x], \tag{3.5}$$

$$\varphi(x) = \tilde{\varphi}(x) + \theta_\varphi(x)[x, x, x], \tag{3.6}$$

$$\psi(x) = \tilde{\psi}(x) + \theta_\psi(x)[x, x]. \tag{3.7}$$

Differentiating (3.6) and (3.7) with respect to x yields that, for any $x, h \in \mathbb{R}^p$:

$$\varphi'(x) \cdot h = \tilde{\varphi}'(x) \cdot h + (\theta'_\varphi(x) \cdot h)[x, x, x] + 3\theta_\varphi(x)[h, x, x], \tag{3.8}$$

$$\psi'(x) \cdot h = {}^{\tilde{}\prime}\psi(x) \cdot h + (\theta'_\psi(x) \cdot h)[x, x] + 2\theta_\psi(x)[h, x]. \tag{3.9}$$

From (3.7) and (3.8), we have $\varphi' \cdot \psi =_{[2]} \tilde{\varphi}' \cdot \tilde{\psi}$. Indeed, for any $x \in \mathbb{R}^p$:

$$\begin{aligned} \varphi'(x) \cdot \psi(x) = \tilde{\varphi}'(x) \cdot \tilde{\psi}(x) + \tilde{\varphi}'(x) \cdot \theta_\psi(x)[x, x] \\ + (\theta'_\varphi(x) \cdot \psi(x))[x, x, x] + 3\theta_\varphi(x)[\psi(x), x, x] \end{aligned} \tag{3.10}$$

From (3.5), (3.6) and (3.9), we have $\psi' \cdot \varphi =_{[2]} \tilde{\psi}' \cdot \tilde{\varphi}$. Indeed, for any $x \in \mathbb{R}^p$:

$$\begin{aligned} \psi'(x) \cdot \varphi(x) = \tilde{\psi}'(x) \cdot \tilde{\varphi}(x) + \tilde{\psi}'(x) \cdot \theta_\varphi(x)[x, x, x] \\ + (\theta'_\psi(x) \cdot \varphi(x))[x, x] + 2\theta_\psi(x)[\theta_0(x)[x], x]. \end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11) yields (3.4). ☐

Lemma 2. *Let $\varphi, \tilde{\varphi}, \psi, \tilde{\psi} \in C^\infty(\mathbb{R}^p, \mathbb{R}^p)$. We assume that:*

$$\varphi =_{[2]} \tilde{\varphi} \quad \text{and} \quad \psi =_{[2]} \tilde{\psi}. \tag{3.12}$$

Then, there holds:

$$[\varphi, \psi](0) = [\tilde{\varphi}, \tilde{\psi}](0). \tag{3.13}$$

Proof. We proceed as above. Thanks to (3.12), we introduce functions $\theta_\varphi \in C^\infty(\mathbb{R}^p, \mathcal{T}^{2,1}_{p,p})$ and $\theta_\psi \in C^\infty(\mathbb{R}^p, \mathcal{T}^{2,1}_{p,p})$ such that, for any $x \in \mathbb{R}^p$:

$$\varphi(x) = \tilde{\varphi}(x) + \theta_\varphi(x)[x, x], \tag{3.14}$$

$$\psi(x) = \tilde{\psi}(x) + \theta_\psi(x)[x, x]. \tag{3.15}$$

Differentiating (3.14) and (3.15) with respect to x yields that, for any $x, h \in \mathbb{R}^p$:

$$\varphi'(x) \cdot h = \tilde{\varphi}'(x) \cdot h + (\theta'_\varphi(x) \cdot h)[x, x] + 2\theta_\varphi(x)[h, x], \tag{3.16}$$

$$\psi'(x) \cdot h = \tilde{\psi}'(x) \cdot h + (\theta'_\psi(x) \cdot h)[x, x] + 2\theta_\psi(x)[h, x]. \tag{3.17}$$

From (3.15) and (3.16), we have $\varphi' \cdot \psi =_{[1]} \tilde{\varphi}' \cdot \tilde{\psi}$. From (3.14), (3.17), we have $\psi' \cdot \varphi =_{[1]} \tilde{\psi}' \cdot \tilde{\varphi}$. Thus, $[\varphi, \psi] =_{[1]} [\tilde{\varphi}, \tilde{\psi}]$, which yields (3.13) by evaluation at $x = 0$. ☐

Lemma 3. *Let $p, q \in \mathbb{N}^*$, $m \in \mathbb{N}$ and $\varphi \in C^\infty(\mathbb{R}^p, \mathbb{R}^q)$. We denote by $T_m(\varphi) \in C^\infty(\mathbb{R}^p, \mathbb{R}^q)$ the Taylor expansion of order m at the origin associated with φ. Then:*

$$\varphi =_{[m+1]} T_m(\varphi). \tag{3.18}$$

Proof. This lemma is a consequence of the Taylor formula with integral remainder for vector-valued multivariate functions:

$$\varphi(x) = T_m(\varphi)(x) + \frac{1}{m!} \int_0^1 (1-t)^m \varphi^{(m+1)}(tx)[x, \dots, x] \mathrm{d}t, \tag{3.19}$$

where $\varphi^{(m+1)}$ denotes the usual multi-linear differential map taking $m+1$ arguments in $\mathbb{R}^p$. From (3.19), we deduce (3.18). ☐

3.2 Explicit computation of the Lie brackets

We compute explicitly first and second-order Lie brackets using only low-order derivatives of the vector fields. Let $f_0, f_1 \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $f_0(0) = 0$.

3.2.1 First-order Lie brackets

We recall the following notations (already used in Subsection 1.3):

$$H_0 := f_0'(0) \in \mathcal{M}_n(\mathbb{R}) \quad \text{and} \quad b := f_1(0) \in \mathbb{R}^n. \tag{3.20}$$

Since $f_0(0) = 0$, for any smooth vector field $g \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and any $k \in \mathbb{N}$:

$$\mathrm{ad}_{f_0}^k(g)(0) = (-H_0)^k g(0) \tag{3.21}$$

In particular, applying (3.21) to f_1, we define the vectors $b_k \in \mathbb{R}^n$ for $k \in \mathbb{N}$ as:

$$b_k := \mathrm{ad}_{f_0}^k(f_1)(0) = (-H_0)^k b. \tag{3.22}$$

Hence, as these brackets span $\mathcal{S}_1$, we obtain:

$$\mathcal{S}_1 = \mathrm{Span}\ \left\{ \mathrm{ad}_{f_0}^k(f_1)(0),\ \ k \in \mathbb{N} \right\} = \mathrm{Span}\ \{ b_k,\ \ k \in \mathbb{N} \}. \tag{3.23}$$

Therefore, the Kalman rank condition (1.9) is equivalent to $\mathcal{S}_1 = \mathbb{R}^n$, thanks to (3.23) and the Cayley-Hamilton theorem applied to the matrix $H_0 \in \mathcal{M}_n(\mathbb{R})$.

3.2.2 Second-order Lie brackets

To carry on the computations at second order, we define the following matrix:

$$H_1 := f_1'(0) \in \mathcal{M}_n(\mathbb{R}) \tag{3.24}$$

We also introduce the following third order tensor, which defines a bilinear map from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$ (and can be thought of as the Hessians of the components of f_0):

$$Q_0 := \frac{1}{2} f_0''(0). \tag{3.25}$$

By induction on $k \in \mathbb{N}$, we also define the linear operators:

$$L_0 := H_1, \tag{3.26}$$

$$L_{k+1} := L_k H_0 - H_0 L_k - 2 Q_0(b_k, \cdot). \tag{3.27}$$

Thanks to these definitions, we can compute the first-order Lie brackets more precisely.

Lemma 4. *For any $k \in \mathbb{N}$:*

$$\mathrm{ad}_{f_0}^k(f_1)(x) =_{[2]} b_k + L_k x. \tag{3.28}$$

Proof. From Lemma 3, $f_0(0) = 0$, definitions (3.20), (3.24) and (3.25):

$$f_0(x) =_{[3]} H_0 x + Q_0(x, x), \tag{3.29}$$

$$f_1(x) =_{[2]} b + H_1 x. \tag{3.30}$$

For $k = 0$, (3.22), (3.26) and (3.30) prove (3.28). Proceeding by induction on $k \in \mathbb{N}$, we evaluate the next bracket using Lemma 1 with $\varphi(x) := f_0(x)$, $\tilde{\varphi}(x) := H_0 x + Q_0(x, x)$, $\psi(x) := \mathrm{ad}_{f_0}^k(f_1)(x)$ and $\tilde{\psi}(x) := b_k + L_k x$. Thus, using (3.22), (3.27) and (3.29):

$$\begin{aligned} \left[f_0, \mathrm{ad}_{f_0}^k(f_1) \right](x) &=_{[2]} L_k(H_0 x + Q_0(x, x)) - (H_0 + 2Q_0(x, \cdot))(b_k + L_k x) \\ &=_{[2]} b_{k+1} + L_{k+1} x + L_k Q_0(x, x) - 2Q_0(x, L_k x) \\ &=_{[2]} b_{k+1} + L_{k+1} x, \end{aligned} \tag{3.31}$$

since the terms which have been dropped from (3.31) are bilinear in x. $\square$

Using (3.21), (3.28) and Lemma 2, we obtain the values of the second-order Lie brackets:

$$\left[\mathrm{ad}_{f_0}^k(f_1), \mathrm{ad}_{f_0}^j(f_1)\right](0) = L_j b_k - L_k b_j, \tag{3.32}$$

$$\left(\mathrm{ad}_{f_0}^i\left(\left[\mathrm{ad}_{f_0}^k(f_1), \mathrm{ad}_{f_0}^j(f_1)\right]\right)\right)(0) = (-H_0)^i(L_j b_k - L_k b_j). \tag{3.33}$$

Since $\mathcal{S}_2$ is spanned by such brackets, we obtain:

$$\mathcal{S}_2 = \mathrm{Span}\ \left\{(-H_0)^i(L_j b_k - L_k b_j),\ \ (i,j,k) \in \mathbb{N}^3\right\}. \tag{3.34}$$

3.3 Definitions for nonsmooth vector fields

Let $f_0 \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ and $f_1 \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. For nonlinear systems, these assumptions are satisfied if $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ and f_0 and f_1 are defined by (1.6). With such regularity, the objects (3.20), (3.24) and (3.25) are well-defined. Hence, definitions (3.22), (3.26) and (3.27) make sense. Therefore, one can use (3.23) and (3.34) as definitions of the spaces $\mathcal{S}_1$ and $\mathcal{S}_2$. Moreover, relations like (3.32) can be used to give a meaning to individual brackets.

Another more flexible approach (which leads to the same spaces) is to introduce regularized vector fields $\hat{f}_0 := T_2 f_0$ and $\hat{f}_1 := T_1 f_1$, corresponding respectively to second-order and first-order Taylor expansions of f_0 and f_1 at 0. Hence $\hat{f}_0, \hat{f}_1 \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Then, one defines $\mathcal{S}_1$ and $\mathcal{S}_2$ as the usual Lie spaces associated with these regularized vector fields.

3.4 Algebraic relations between second-order brackets

The following algebraic relations highlight the fact that our theorems could actually be stated using other equivalent brackets (see (3.36)). Although these results are not new, we include the proofs (inspired from [24, pages 279-280]) for the sake of completeness.

First, for any $i, l \in \mathbb{N}$ such that $0 \leqslant l \leqslant i$, it can be proved by induction on l, using the Jacobi identity and the skew symmetry of the bracketing operation that:

$$[f_1, \mathrm{ad}_{f_0}^i(f_1)] = \sum_{j=0}^{l-1}(-1)^j[f_0, [\mathrm{ad}_{f_0}^j(f_1), \mathrm{ad}_{f_0}^{i-j-1}(f_1)]] + (-1)^l[\mathrm{ad}_{f_0}^l(f_1), \mathrm{ad}_{f_0}^{i-l}(f_1)]. \tag{3.35}$$

This formula has the following consequences.

Proposition 5. *Let $k \in \mathbb{N}^*$. We assume that $f_0, f_1 \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and (2.7) holds.*

1. *Then $[\mathrm{ad}_{f_0}^a(f_1), \mathrm{ad}_{f_0}^b(f_1)](0) \in \mathcal{S}_1$ for every $a, b \in \mathbb{N}$ such that $a + b \leqslant 2k - 2$.*
2. *Statement (2.8) is equivalent to each of the following two statements:*
   - *$[\mathrm{ad}_{f_0}^l(f_1), \mathrm{ad}_{f_0}^{2k-1-l}(f_1)](0) \notin \mathcal{S}_1$, for every $l \in \{0, ..., 2k-1\}$,*
   - *$[\mathrm{ad}_{f_0}^l(f_1), \mathrm{ad}_{f_0}^{2k-1-l}(f_1)](0) \notin \mathcal{S}_1$, for some $l \in \{0, ..., 2k-1\}$.*

   Moreover:
   $$d_k = (-1)^{k-1+l}\,\mathbb{P}^\perp\left([\mathrm{ad}_{f_0}^l(f_1), \mathrm{ad}_{f_0}^{2k-1-l}(f_1)](0)\right). \tag{3.36}$$
3. *If (2.8) holds, then the family $(f_1(0), \dots, \mathrm{ad}_{f_0}^{k-1}(f_1)(0))$ is linearly independent. Thus, it can only hold for $k \leqslant d$.*
4. *If $k > \dim \mathcal{S}_1$, then $[\mathrm{ad}_{f_0}^a(f_1), \mathrm{ad}_{f_0}^b(f_1)](0) \in \mathcal{S}_1$ for every $a, b \in \mathbb{N}$.*

Proof. **Statement 1.** We decompose the proof in three steps.

- *Step A. We prove that, if* $[\mathrm{ad}^a_{f_0}(f_1), \mathrm{ad}^b_{f_0}(f_1)](0)$ *belongs to* $\mathcal{S}_1$ *for every* $a, b \in \mathbb{N}$ *such that* $a + b \leqslant 2m - 1$ *and for some* $m \in \mathbb{N}^*$*, then* $[\mathrm{ad}^a_{f_0}(f_1), \mathrm{ad}^b_{f_0}(f_1)](0)$ *belongs to* $\mathcal{S}_1$ *for every* $a, b \in \mathbb{N}$ *such that* $a + b \leqslant 2m$.

  Let $m \in \mathbb{N}$. We assume that $[\mathrm{ad}^a_{f_0}(f_1), \mathrm{ad}^b_{f_0}(f_1)](0) \in \mathcal{S}_1$ for every $a, b \in \mathbb{N}$ such that $a + b \leqslant 2m - 1$. We deduce from the stability of $\mathcal{S}_1$ with respect to bracketing by f_0 and formula (3.35) with $i = 2m$ and $l = m$ that $[f_1, \mathrm{ad}^{2m}_{f_0}(f_1)](0)$ belongs to $\mathcal{S}_1$. Then, we deduce from the stability of $\mathcal{S}_1$ with respect to bracketing by f_0 and formula (3.35) with $i = 2m$ that $[\mathrm{ad}^l_{f_0}(f_1), \mathrm{ad}^{2m-l}_{f_0}(f_1)](0) \in \mathcal{S}_1$ for every $l \in \{0, ..., 2m\}$.

- *Step B. We prove that, if* $[\mathrm{ad}^a_{f_0}(f_1), \mathrm{ad}^b_{f_0}(f_1)](0)$ *belongs to* $\mathcal{S}_1$ *for every* $a, b \in \mathbb{N}$ *such that* $a + b \leqslant 2m$*, and* $[\mathrm{ad}^m_{f_0}(f_1), \mathrm{ad}^{m+1}_{f_0}(f_1)](0)$ *belongs to* $\mathcal{S}_1$ *for some* $m \in \mathbb{N}$*, then, for every* $a, b \in \mathbb{N}$ *such that* $a + b \leqslant 2m + 1$*,* $[\mathrm{ad}^a_{f_0}(f_1), \mathrm{ad}^b_{f_0}(f_1)](0)$ *belongs to* $\mathcal{S}_1$.

  This is direct consequence of formula (3.35) with $i = 2m + 1$ and the stability of $\mathcal{S}_1$ with respect to bracketing by f_0.

- *Step C. We prove Statement 1.*

  Using the case $j = 1$ in assumption (2.7) and Step 2 with $m = 1$ we obtain that $[\mathrm{ad}^a_{f_0}(f_1), \mathrm{ad}^b_{f_0}(f_1)](0) \in \mathcal{S}_1$ for every $a, b \in \mathbb{N}$ such that $a + b \leqslant 2$. Then, using the case $j = 2$ in assumption (2.7), Step 3 with $m = 1$ and Step 2 with $m = 2$ we obtain that $[\mathrm{ad}^a_{f_0}(f_1), \mathrm{ad}^b_{f_0}(f_1)](0) \in \mathcal{S}_1$ for every $a, b \in \mathbb{N}$ such that $a + b \leqslant 4$. The proof is carried on by iteration.

Statement 2. This statement follows from formula (3.35) with $i = 2k - 1$, Statement 1 and the stability of $\mathcal{S}_1$ with respect to bracketing by f_0.

Statement 3. We assume that (2.7) holds and the family $(f_1(0), \dots \mathrm{ad}^{k-1}_{f_0}(f_1)(0))$ is not linearly independent. If $k = 1$, then $f_1(0) = 0$. Thus $\mathcal{S}_1 = \{0\}$ and:

$$[f_1, [f_1, f_0]](0) = f_0''(0)\big(f_1(0), f_1(0)\big) + f_0'(0)f_1'(0)f_1(0) - 2f_1'(0)f_0'(0)f_1(0) = 0 \in \mathcal{S}_1. \tag{3.37}$$

Now, we assume that $k \geqslant 2$. There exists $a_0, ..., a_{k-2} \in \mathbb{R}$ such that:

$$\mathrm{ad}^{k-1}_{f_0}(f_1)(0) = \sum_{r=0}^{k-2} a_r \mathrm{ad}^r_{f_0}(f_1)(0). \tag{3.38}$$

Then:

$$\mathrm{ad}^{k}_{f_0}(f_1)(0) = \sum_{r=0}^{k-2} a_r \mathrm{ad}^{r+1}_{f_0}(f_1)(0), \tag{3.39}$$

because $\mathrm{ad}^r_{f_0}(f_1)(0) = H_0^r f_1(0)$ for every $r \in \mathbb{N}$. We deduce from (2.7) and Statement 1 that, for any $a, b \in \mathbb{N}$ such that $a + b \leqslant 2k - 2$:

$$\Big(\mathrm{ad}^a_{f_0}(f_1)\Big)'\Big(\mathrm{ad}^b_{f_0}(f_1)(0)\Big) \sim \Big(\mathrm{ad}^b_{f_0}(f_1)\Big)'\Big(\mathrm{ad}^a_{f_0}(f_1)(0)\Big), \tag{3.40}$$

where $\sim$ means equality modulo additive terms in $\mathcal{S}_1$ (equivalence classes in $\mathbb{R}^n/\mathcal{S}_1$). Using (3.38) and (3.39), we have:

$$\begin{aligned}
&[\mathrm{ad}^k_{f_0}(f_1), \mathrm{ad}^{k-1}_{f_0}(f_1)](0) \\
&= \Big(\mathrm{ad}^{k-1}_{f_0}(f_1)\Big)'\Big(\mathrm{ad}^k_{f_0}(f_1)(0)\Big) - \Big(\mathrm{ad}^k_{f_0}(f_1)\Big)'\Big(\mathrm{ad}^{k-1}_{f_0}(f_1)(0)\Big) \\
&= \sum_{r=0}^{k-2} a_r \left(\Big(\mathrm{ad}^{k-1}_{f_0}(f_1)\Big)'\Big(\mathrm{ad}^{r+1}_{f_0}(f_1)(0)\Big) - \Big(\mathrm{ad}^k_{f_0}(f_1)\Big)'\Big(\mathrm{ad}^r_{f_0}(f_1)(0)\Big) \right).
\end{aligned} \tag{3.41}$$

Then, using (3.40), we have:

$$
\begin{aligned}
&[\mathrm{ad}_{f_0}^k(f_1), \mathrm{ad}_{f_0}^{k-1}(f_1)](0) \\
&\sim \sum_{r=0}^{k-2} a_r \left(\left(\mathrm{ad}_{f_0}^{r+1}(f_1)\right)' \left(\mathrm{ad}_{f_0}^{k-1}(f_1)(0)\right) - \left(\mathrm{ad}_{f_0}^{r}(f_1)\right)' \left(\mathrm{ad}_{f_0}^{k}(f_1)(0)\right) \right) \\
&\sim \sum_{r,R=0}^{k-2} a_r a_R \left(\left(\mathrm{ad}_{f_0}^{r+1}(f_1)\right)' \left(\mathrm{ad}_{f_0}^{R}(f_1)(0)\right) - \left(\mathrm{ad}_{f_0}^{r}(f_1)\right)' \left(\mathrm{ad}_{f_0}^{R+1}(f_1)(0)\right) \right) \\
&\sim \sum_{r=0}^{k-2} a_r^2 L_r + \sum_{0\leqslant r<R\leqslant k-2} a_r a_R L_{r,R},
\end{aligned}
\tag{3.42}
$$

where:

$$
L_r := [\mathrm{ad}_{f_0}^{r+1}(f_1), \mathrm{ad}_{f_0}^{r}(f_1)](0), \tag{3.43}
$$

$$
\begin{aligned}
L_{r,R} := &\left(\mathrm{ad}_{f_0}^{r+1}(f_1)\right)' \left(\mathrm{ad}_{f_0}^{R}(f_1)(0)\right) - \left(\mathrm{ad}_{f_0}^{r}(f_1)\right)' \left(\mathrm{ad}_{f_0}^{R+1}(f_1)(0)\right) \\
&+ \left(\mathrm{ad}_{f_0}^{R+1}(f_1)\right)' \left(\mathrm{ad}_{f_0}^{r}(f_1)(0)\right) - \left(\mathrm{ad}_{f_0}^{R}(f_1)\right)' \left(\mathrm{ad}_{f_0}^{r+1}(f_1)(0)\right) \\
= &[\mathrm{ad}_{f_0}^{r+1}(f_1), \mathrm{ad}_{f_0}^{R}(f_1)](0) - [\mathrm{ad}_{f_0}^{r}(f_1), \mathrm{ad}_{f_0}^{R+1}(f_1)](0).
\end{aligned}
\tag{3.44}
$$

Since $2r+1, r+R+1 \leqslant 2k-3$, L_r and $L_{r,R}$ belong to $\mathcal{S}_1$ for every $0 \leqslant r < R \leqslant k-2$. Therefore $[\mathrm{ad}_{f_0}^{k-1}(f_1), \mathrm{ad}_{f_0}^{k}(f_1)](0)$ belong to $\mathcal{S}_1$, i.e. (2.8) does not hold.

Statement 4. By iterating Statement 3, we obtain that $[\mathrm{ad}_{f_0}^{j-1}(f_1), \mathrm{ad}_{f_0}^{j}(f_1)](0) \in \mathcal{S}_1$ for every $j \in \mathbb{N}$. Then, Statement 1 yields the conclusion. ☐

In particular, these algebraic relations lead to constraints on the different ways that one can have $\mathcal{S}_2 \not\subset \mathcal{S}_1$. More precisely, we prove:

Lemma 5. *Let $f_0, f_1 \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $f_0(0) = 0$ and $d := \dim \mathcal{S}_1$. Assume that $\mathcal{S}_2 \not\subset \mathcal{S}_1$. Then, there exists $1 \leqslant k \leqslant d$ such that (2.7) and (2.8) hold.*

Proof. By contradiction, let us assume that (2.7) holds with $k > d$. From Statement 4 of Proposition 5, we get that $[\mathrm{ad}_{f_0}^a(f_1), \mathrm{ad}_{f_0}^b(f_1)](0) \in \mathcal{S}_1$ for any $a, b \in \mathbb{N}$. Since $\mathcal{S}_1$ is stable with respect to bracketing by f_0, we also have $\mathrm{ad}_{f_0}^c([\mathrm{ad}_{f_0}^a(f_1), \mathrm{ad}_{f_0}^b(f_1)])(0) \in \mathcal{S}_1$ for any $a, b, c \in \mathbb{N}$. From (3.34), this yields that $\mathcal{S}_2 \subset \mathcal{S}_1$, which contradicts our assumption. Thus, there exists $k \leqslant d$ such that (2.8) holds. Taking the smallest such $k \geqslant 1$ such that (2.8) holds ensures that (2.7) also holds. ☐

4 Construction of auxiliary systems

We construct by induction auxiliary systems that are useful in the proof of our theorems.

4.1 Definitions and notations

Let $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ with $f(0,0) = 0$. We define the C^∞ map $G_0 : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ as:

$$
G_0(\tau, p) := \begin{cases} \dfrac{1}{\tau^2}\left(f(p,\tau) - f_0(p) - \tau f_1(p)\right) & \text{for } \tau \neq 0, \\ \dfrac{1}{2}\partial_u^2 f(p, 0) & \text{for } \tau = 0. \end{cases} \tag{4.1}
$$

Let $T > 0$ and $(x, u) \in C^0([0, T], \mathbb{R}^n) \times L^\infty((0, T), \mathbb{R})$ a trajectory of (1.1) with $x(0) = 0$. In the case of a control-affine system, the control u is only assumed to belong to $L^1((0, T), \mathbb{R})$. We introduce, for $1 \leqslant j \leqslant d$:

- the smooth vector fields:
$$f_j := (-1)^{j-1} \mathrm{ad}_{f_0}^{j-1}(f_1), \tag{4.2}$$
- the associated flows ϕ_j, that are smooth and well-defined on an open neighborhood $\Omega_j \subset \mathbb{R} \times \mathbb{R}^n$ of $(0, 0)$, which are defined as the solutions to:
$$\begin{cases} \partial_\tau \phi_j(\tau, p) = f_j\big(\phi_j(\tau, p)\big), \\ \quad \phi_j(0, p) = p. \end{cases} \tag{4.3}$$
and we will sometimes write $\phi_j^\tau(p)$ instead of $\phi_j(\tau, p)$,
- the smooth maps $F_j : \Omega_j \to \mathbb{R}^n$ defined by:
$$F_j(\tau, p) := (\partial_p \phi_j(\tau, p))^{-1} f_0\big(\phi_j(\tau, p)\big), \tag{4.4}$$
- the smooth maps $G_j : \Omega_j \to \mathbb{R}^n$, defined by:
$$G_j(\tau, p) := \begin{cases} \dfrac{1}{\tau^2}\big(F_j(\tau, p) - F_j(0, p) - \tau \partial_\tau F_j(0, p)\big) & \text{for } \tau \neq 0, \\ \dfrac{1}{2}\partial_\tau^2 F_j(0, p) & \text{for } \tau = 0, \end{cases} \tag{4.5}$$
- the auxiliary states $\xi_j : (0, T) \to \mathbb{R}^n$, defined by induction by:
$$\xi_0 := x \quad \text{and} \quad \xi_{j+1} := \phi_{j+1}\left(-u_{j+1}, \xi_j\right). \tag{4.6}$$

4.2 Evolution of the auxiliary states

We start by computing some derivatives of the utility functions F_j, from which we then deduce by induction the evolution equation for the auxiliary states.

Lemma 6. *Let $1 \leqslant j \leqslant d$. One has:*
$$\partial_\tau F_j(0, \cdot) = f_{j+1}, \tag{4.7}$$
$$\partial_\tau^2 F_j(0, \cdot) = [f_j, f_{j+1}]. \tag{4.8}$$

Proof. Differentiating definition (4.4) with respect to τ yields:
$$\partial_\tau F_j = -(\partial_p \phi_j)^{-1} \partial_{\tau p} \phi_j (\partial_p \phi_j)^{-1} f_0(\phi_j) + (\partial_p \phi_j)^{-1} f_0'(\phi_j) \partial_\tau \phi_j. \tag{4.9}$$
Applying Schwarz's theorem and using (4.3), one computes:
$$\partial_{\tau p} \phi_j = \partial_{p\tau} \phi_j = f_j'(\phi_j) \partial_p \phi_j. \tag{4.10}$$
Gathering (4.9), (4.10) and using (4.3) yields:
$$\begin{aligned} \partial_\tau F_j &= -(\partial_p \phi_j)^{-1} f_j'(\phi_j) f_0(\phi_j) + (\partial_p \phi_j)^{-1} f_0'(\phi_j) f_j(\phi_j) \\ &= (\partial_p \phi_j)^{-1} [f_j, f_0](\phi_j). \end{aligned} \tag{4.11}$$
Equation (4.11) proves (4.7) by evaluation at $\tau = 0$ thanks to definition (4.2). Applying the same proof method to $\partial_\tau F_j$ yields (4.8). ∎

Lemma 7. *Let $1 \leqslant j \leqslant d$ and ξ_j be defined by (4.6). Then $\xi_j(0) = x(0)$ and ξ_j satisfies:*

$$\dot{\xi}_j = f_0(\xi_j) + u_j f_{j+1}(\xi_j) + \sum_{l=0}^{j} u_l^2 \mathcal{K}_{l,j}, \tag{4.12}$$

where we define:

$$\mathcal{K}_{l,j} := \begin{cases} (\partial_p \phi_j(u_j,\xi_j))^{-1} \cdots (\partial_p \phi_{l+1}(u_{l+1},\xi_{l+1}))^{-1} G_l(u_l,\xi_l) & \textit{for } l < j, \\ G_j(u_j,\xi_j) & \textit{for } l = j. \end{cases} \tag{4.13}$$

Proof. Setting $\xi_0 := x$ and using (4.1) proves that (4.12) holds for $j = 0$. Let $0 \leqslant j < d$. We assume that (4.12) holds. From (4.6):

$$\xi_j = \phi_{j+1}(u_{j+1}, \xi_{j+1}). \tag{4.14}$$

Differentiating (4.14) with respect to time yields:

$$\dot{\xi}_j = \dot{u}_{j+1} \partial_\tau \phi_{j+1}(u_{j+1},\xi_{j+1}) + \partial_p \phi_{j+1}(u_{j+1},\xi_{j+1}) \dot{\xi}_{j+1}. \tag{4.15}$$

Injecting (2.1), (4.3) and (4.12) into (4.15) yields:

$$\partial_p \phi_{j+1}(u_{j+1},\xi_{j+1}) \dot{\xi}_{j+1} = f_0\big(\phi_{j+1}(u_{j+1},\xi_{j+1})\big) + \sum_{l=0}^{j} u_l^2 \mathcal{K}_{l,j}. \tag{4.16}$$

From (4.13), one has:

$$\mathcal{K}_{l,j+1} = \big(\partial_p \phi_{j+1}(u_{j+1},\xi_{j+1})\big)^{-1} \mathcal{K}_{l,j}. \tag{4.17}$$

Gathering (4.4), (4.16) and (4.17) gives:

$$\dot{\xi}_{j+1} = F_{j+1}(u_{j+1},\xi_{j+1}) + \sum_{l=0}^{j} u_l^2 \mathcal{K}_{l,j+1}. \tag{4.18}$$

Moreover, by (4.4), (4.5) and Lemma 6, one has:

$$\begin{aligned} F_{j+1}(u_{j+1},\xi_{j+1}) &= F_{j+1}(0,\xi_{j+1}) + u_{j+1}\partial_\tau F_{j+1}(0,\xi_{j+1}) + u_{j+1}^2 G_{j+1}(u_{j+1},\xi_{j+1}) \\ &= f_0(\xi_{j+1}) + u_{j+1} f_{j+2}(\xi_{j+1}) + u_{j+1}^2 G_{j+1}(u_{j+1},\xi_{j+1}). \end{aligned} \tag{4.19}$$

Hence, (4.18) and (4.19) conclude the proof of (4.12) for $j+1$. ☐

4.3 An important notation for estimates

Despite the difference in the optimal functional framework between control-affine and non-linear systems, we are going to prove Theorems 2 and 3 in a unified way. To that end, we introduce two parameters γ and q:

- $\gamma := 1$ and $q := 1$, when $\partial_u^2 f = 0$ on an open neighborhood of 0 (control-affine systems),
- $\gamma := 0$ and $q := \infty$ otherwise (general nonlinear systems).

Hence, a trajectory (x,u) belongs to $C^0([0,T],\mathbb{R}^n) \times L^q((0,T),\mathbb{R})$. Moreover, the following notation is used throughout the paper as it lightens both the statements and the proofs:

Definition 10. *Given two observables $A(x,u)$ and $B(x,u)$ of interest, we will write that $A(x,u) = \mathcal{O}_\gamma(B(x,u))$ when: for any $\mathcal{T} > 0$, there exists $C, \eta > 0$ such that, for any $T \in (0,\mathcal{T}]$ and any trajectory $(x,u) \in C^0([0,T],\mathbb{R}^n) \times L^q((0,T),\mathbb{R})$ with $x(0) = 0$ which satisfies $\|u_\gamma\|_{L^\infty} \leqslant \eta$, one has $|A(x,u)| \leqslant C|B(x,u)|$. Hence, this notation refers to the convergence $\|u_\gamma\|_{L^\infty} \to 0$ and holds uniformly with respect to the trajectories on a time interval $[0,T] \subset [0,\mathcal{T}]$. For observables depending on $t \in [0,T]$, it is implicit that the notation $A(x,u,t) = \mathcal{O}_\gamma(B(x,u,t))$ always holds uniformly with respect to $t \in [0,T] \subset [0,\mathcal{T}]$. Eventually, we will use the notation $\mathcal{O}(\cdot)$ when the estimate holds without any smallness assumption on the control.*

4.4 Estimations for the auxiliary systems

We start with an estimate of the solutions of the system (1.1), requiring little regularity on dynamic (this will be useful in the sequel).

Lemma 8. *Let $f \in C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ with $f(0,0) = 0$. One has:*

$$|x(t)| = \mathcal{O}_0(\|u\|_{L^1}). \tag{4.20}$$

Proof. Let $\mathcal{T} > 0$. We define $M := \max\{|f'(p,\tau)|;\ (p,\tau) \in \overline{B}(0,1) \times [-1,1]\}$, $C := Me^{M\mathcal{T}}$ and $\eta := \min\{1, 1/(2C\mathcal{T})\}$. Let $T \in (0,\mathcal{T}]$ and $(x,u) \in C^0([0,T],\mathbb{R}^n) \times L^\infty((0,T),\mathbb{R})$ be a trajectory of system (1.1) with $x(0) = 0$ which satisfies $\|u\|_{L^\infty} \leqslant \eta$. Let $T_1 := \sup\{t \in (0,T);\ \forall s \in [0,t], |x(s)| \leqslant 1\}$. For every $t \in [0,T_1]$, one has, by the first-order Taylor expansion:

$$|x(t)| = \left| \int_0^t f(x(s),u(s)) \mathrm{d}s \right| \leqslant M \int_0^t \left(|x(s)| + |u(s)|\right) \mathrm{d}s. \tag{4.21}$$

Thus, by Grönwall's lemma:

$$|x(t)| \leqslant Me^{Mt} \int_0^t |u(s)| \mathrm{d}s \leqslant C\|u\|_{L^1} \leqslant C\mathcal{T}\|u\|_{L^\infty} \leqslant C\mathcal{T}\eta \leqslant \frac{1}{2}. \tag{4.22}$$

This proves that $T_1 = T$ and that (4.20) holds, in the sense of Definition 10. □

In the particular case of control-affine systems (1.3), the size of the solution can be estimated by a weaker norm of the control, according to the following statement.

Lemma 9. *Let $f_0, f_1 \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ with $f_0(0) = 0$. One has:*

$$|x(t)| = \mathcal{O}_1(|u_1(t)| + \|u_1\|_{L^1}), \tag{4.23}$$

$$|\xi_1(t)| = \mathcal{O}_1(\|u_1\|_{L^1}). \tag{4.24}$$

Proof. Let $\mathcal{T} > 0$, $\phi_1 : \Omega_1 \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $G_1 : \Omega_1 \to \mathbb{R}^n$ be defined by (4.3) and (4.5) for $j = 1$. Let $r > 0$ be such that $[-r,r] \times \overline{B}(0,r) \subset \Omega_1$. Let $M_1 > 0$ be such that:

$$\forall (p,\tau) \in \overline{B}(0,r) \times [-r,r], \quad |f_0'(p)|, \ |f_2(p) + \tau G_1(\tau,p)|, \ |\phi_1'(\tau,p)| \leqslant M_1. \tag{4.25}$$

Let $C_1 := M_1 e^{M_1 \mathcal{T}}$, $\eta := \min\left(r, r/(2C_1\mathcal{T})\right)$ and $C := M_1 \max\{1, C_1\}$. Let $T \in (0,\mathcal{T}]$ and $(x,u) \in C^0([0,T],\mathbb{R}^n) \times L^1((0,T),\mathbb{R})$ be a trajectory of system (1.3) with $x(0) = 0$ which satisfies $\|u\|_{W^{-1,\infty}} \leqslant \eta$. Let u_1 be defined by (2.1), ξ_1 be defined by (4.6), which solves (4.12), and $T_1 := \sup\{t \in [0,T];\ \forall s \in [0,t], |\xi_1(s)| \leqslant r\}$. For every $t \in [0,T_1]$, one has, by a first-order Taylor expansion:

$$\begin{aligned} |\xi_1(t)| &= \left| \int_0^t \Big(f_0(\xi_1(s)) + u_1(s) \Big(f_2(\xi_1(s)) + u_1(s) G_1(u_1(s), \xi_1(s)) \Big) \Big) \mathrm{d}s \right| \\ &\leqslant M_1 \int_0^t \left(|\xi_1(s)| + |u_1(s)|\right) \mathrm{d}s. \end{aligned} \tag{4.26}$$

Thus, by Grönwall's lemma:

$$|\xi_1(t)| \leqslant M_1 e^{M_1 t} \int_0^t |u_1(s)| \mathrm{d}s \leqslant C_1 \|u_1\|_{L^1} \leqslant C_1 \mathcal{T} \|u_1\|_{L^\infty} \leqslant C_1 \mathcal{T}\eta \leqslant \frac{r}{2}. \tag{4.27}$$

This proves that $T_1 = T$ and (4.24) holds for $t \in [0,T]$. Then, for every $t \in [0,T]$, one has:

$$\begin{aligned} |x(t)| &= |\phi_1(u_1(t), \xi_1(t))| \\ &\leqslant M_1 \left(|u_1(t)| + |\xi_1(t)|\right) \\ &\leqslant M_1 |u_1(t)| + M_1 C_1 \|u_1\|_{L^1} \\ &\leqslant C \left(|u_1(t)| + \|u_1\|_{L^1}\right), \end{aligned} \tag{4.28}$$

which holds then $\|u_1\|_{L^\infty} \leqslant \eta$ and thus concludes the proof. □

Lemma 10. *Let $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ with $f(0,0)=0$. The auxiliary states are well-defined and satisfy:*

$$|\xi_j(t)| = \mathcal{O}_\gamma\Big(\|u_j\|_{L^1} + \|u_\gamma\|_{L^2}^2\Big). \tag{4.29}$$

Proof. The following estimate obtained from (2.1) for $1 \leqslant j \leqslant d$ will be useful:

$$\forall p \in [1, +\infty], \quad \|u_j\|_{L^p} = \mathcal{O}_\gamma\big(\|u_\gamma\|_{L^p}\big). \tag{4.30}$$

Well-posedness of the auxiliary systems. Let us first explain why the auxiliary states are well-defined for $t \in [0,T]$. From (4.6), for $t \in [0,T]$, one has:

$$\xi_j(t) = \phi_j^{-u_j(t)} \circ \cdots \circ \phi_1^{-u_1(t)}(x(t)). \tag{4.31}$$

The flows ϕ_j for $1 \leqslant j \leqslant d$ are well-defined on open neighborhoods Ω_j of $(0,0)$ in $\mathbb{R} \times \mathbb{R}^n$. Thanks to the continuity of the flows, one deduces from (4.31) that the auxiliary states are well-defined on $[0,T]$ provided that $|x|$ and the $|u_j|$ stay small enough. Thanks to Lemma 8 in the nonlinear case, Lemma 9 in the control-affine case and to (4.30), this is true if $\|u_\gamma\|_{L^\infty}$ is small enough. Moreover, the regularity of the flows implies that:

$$\xi_j(t) = \mathcal{O}_\gamma\big(\|u_\gamma\|_{L^1}\big). \tag{4.32}$$

Estimates of the auxiliary states. From (4.12), one has:

$$\xi_j(t) = \int_0^t \left(f_0(\xi_j(s)) + u_j(s) f_{j+1}(\xi_j(s)) + \sum_{l=\gamma}^{j} u_l(s)^2 \mathcal{K}_{l,j}(s) \right) \mathrm{d}s. \tag{4.33}$$

From (4.32), (4.33), $f_0(0)=0$ and the regularity of the functions involved, one has:

$$\xi_j(t) = \int_0^t \mathcal{O}_\gamma(|\xi_j(s)|)\mathrm{d}s + \mathcal{O}_\gamma\left(\|u_j\|_{L^1} + \sum_{l=\gamma}^{j} \|u_l\|_{L^2}^2 \right). \tag{4.34}$$

From (4.30) and (4.34):

$$\xi_j(t) = \int_0^t \mathcal{O}_\gamma(|\xi_j(s)|)\mathrm{d}s + \mathcal{O}_\gamma\Big(\|u_j\|_{L^1} + \|u_\gamma\|_{L^2}^2\Big). \tag{4.35}$$

Application of Grönwall's lemma to (4.35) proves (4.29). □

5 Alternative for well-prepared smooth systems

We prove our quadratic alternative for a particular class of well-prepared smooth systems. More precisely, we consider smooth control systems such that:

$$\left(f_0'(0)\right)^d f_1(0) = H_0^d b = 0, \tag{5.1}$$

where d is the dimension of $\mathcal{S}_1$. This condition simplifies the linear dynamic of the system as it is reduced to an iterated integrator. We will explain in Subsection 6.2 how one can use static state feedback to transform any system into such a well-prepared system.

5.1 Enhanced estimates for the last auxiliary system

Under assumption (5.1), the linear dynamic has been fully taken into account once arrived at the auxiliary state ξ_d. Indeed, let us consider a trajectory with $x(0) = 0$. Hence $\xi_d(0) = 0$ and, using (4.12), we have:

$$\dot{\xi}_d = f_0(\xi_d) + u_d f_{d+1}(\xi_d) + \sum_{l=\gamma}^{d} u_l^2 \mathcal{K}_{l,d}. \tag{5.2}$$

Recalling (3.20), the linearized system of (5.2) around the null equilibrium is:

$$\dot{y}_d = H_0 y_d + u_d f_{d+1}(0). \tag{5.3}$$

From (5.1), $f_{d+1}(0) = H_0^d b = 0$ and $y_d = 0$ since $y_d(0) = 0$. There dependence of ξ_d on the control is thus at least quadratic. The second-order approximation of (5.2) around the null equilibrium is given by:

$$\dot{z_d} = H_0 z_d + \frac{1}{2} f_0''(0).(y_d, y_d) + u_d f_{d+1}'(0) y_d + \sum_{l=\gamma}^{d} u_l^2 G_l(0,0). \tag{5.4}$$

We deduce from the relation $y_d = 0$, the initial condition $z_d(0) = 0$ and (5.4) that:

$$z_d(t) = \sum_{l=\gamma}^{d} \int_0^t u_l^2(s) e^{(t-s)H_0} G_l(0,0) \mathrm{d}s. \tag{5.5}$$

These remarks lead to the following estimates.

Lemma 11. *Let $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ with $f(0,0) = 0$ satisfying (5.1). One has:*

$$|\xi_d(t)| = \mathcal{O}_\gamma\Big(\|u_\gamma\|_{L^2}^2\Big), \tag{5.6}$$

$$|\xi_d(t) - z_d(t)| = \mathcal{O}_\gamma\Big(\|u_\gamma\|_{L^3}^3\Big). \tag{5.7}$$

Proof. The integral form of (5.2) is:

$$\xi_d(t) = \int_0^t \left(f_0(\xi_d(s)) + u_d(s) f_{d+1}(\xi_d(s)) + \sum_{l=\gamma}^{d} u_l(s)^2 \mathcal{K}_{l,d}(s) \right) \mathrm{d}s. \tag{5.8}$$

Using the integrator assumption (5.1), one has $f_{d+1}(0) = 0$. Moreover $u_d = \mathcal{O}_\gamma(1)$. Hence, thanks to the regularity of the functions involved in (5.8), one obtains:

$$\xi_d(t) = \int_0^t \mathcal{O}_\gamma(|\xi_d(s)|)\mathrm{d}s + \mathcal{O}_\gamma\Big(\|u_\gamma\|_{L^2}^2\Big). \tag{5.9}$$

Estimate (5.6) follows from the application of Grönwall's lemma to (5.9). We turn to the next order bound, using the following integral formulation:

$$\begin{aligned} \xi_d(t) - z_d(t) = & \int_0^t \Big(f_0(\xi_d(s)) - H_0 z_d(s) + u_d(s) f_{d+1}(\xi_d(s)) \Big) \mathrm{d}s \\ & + \sum_{j=\gamma}^{d} \int_0^t u_j^2(s) \Big(\mathcal{K}_{j,d}(s) - G_j(0,0) \Big) \mathrm{d}s. \end{aligned} \tag{5.10}$$

We estimate separately the different parts of the integrand in (5.10). First, using the regularity of f_0 and (5.6), one has:

$$\begin{aligned} f_0(\xi_d) - H_0 z_d &= H_0(\xi_d - z_d) + \mathcal{O}_\gamma\big(|\xi_d|^2\big) \\ &= \mathcal{O}_\gamma(|\xi_d - z_d|) + \mathcal{O}_\gamma\Big(\|u_\gamma\|_{L^2}^4\Big). \end{aligned} \tag{5.11}$$

Moreover, using once again the integrator assumption (5.1), the regularity of f_{d+1}, estimate (5.6) and Hölder's inequality, one has:

$$\begin{aligned} \int_0^t u_d(s) f_{d+1}(\xi_d(s)) \mathrm{d}s &= \mathcal{O}_\gamma\left(\int_0^t |u_d(s)||\xi_d(s)| \mathrm{d}s\right) \\ &= \mathcal{O}_\gamma\left(\|u_\gamma\|_{L^2}^2 \int_0^t |u_d(s)| \mathrm{d}s\right) = \mathcal{O}_\gamma\Big(\|u_\gamma\|_{L^3}^3\Big). \end{aligned} \tag{5.12}$$

For $1 \leqslant l \leqslant d$, the bound (4.29) yields $\xi_l = \mathcal{O}_\gamma(1)$. The C^2 regularity of ϕ_l justifies that:

$$\partial_p \phi_l(u_l(t), \xi_l(t)) = \partial_p \phi_l(0, \xi_l(t)) + \mathcal{O}_\gamma(|u_l(t)|) = \mathrm{Id} + \mathcal{O}_\gamma(|u_l(t)|). \tag{5.13}$$

Inverting (5.13) proves that, for $0 \leqslant j \leqslant d$:

$$\begin{aligned} \mathcal{K}_{j,d} - G_j(0,0) &= \big(\partial_p \phi_d(u_d, \xi_d)\big)^{-1} \cdots \big(\partial_p \phi_{j+1}(u_{j+1}, \xi_{j+1})\big)^{-1} G_j(u_j, \xi_j) - G_j(0,0) \\ &= G_j(u_j, \xi_j) - G_j(0,0) + \mathcal{O}_\gamma(|u_d| + \ldots + |u_{j+1}|) \\ &= \mathcal{O}_\gamma(|\xi_j| + |u_d| + \ldots + |u_j|). \end{aligned} \tag{5.14}$$

We deduce from (4.29), (4.30), (5.11), (5.12) and (5.14) and Hölder's inequality that (5.10) can be written:

$$\xi_d(t) - z_d(t) = \mathcal{O}_\gamma\Big(\|u_\gamma\|_{L^3}^3\Big) + \int_0^t \mathcal{O}_\gamma(|\xi_d - z_d|). \tag{5.15}$$

Applying Grönwall's lemma to (5.15) concludes the proof of estimate (5.7). ☐

5.2 Construction of the invariant manifold

We construct the smooth manifold $\mathcal{M}$, as the graph of an implicit function G. The following lemma is a key ingredient in the construction of G.

Lemma 12. *Let $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ with $f(0,0) = 0$ satisfying (5.1). There exists a smooth map:*

$$F : \begin{cases} \mathcal{S}_1 \times \mathcal{S}_1^\perp \to \mathcal{S}_1^\perp, \\ (p_\parallel, p_\perp) \mapsto F(p_\parallel, p_\perp), \end{cases} \tag{5.16}$$

such that $F(0,0) = 0$, $\partial_\perp F(0,0) = \mathrm{Id}$ on $\mathcal{S}_1^\perp$ and one has:

$$F\big(\mathbb{P}x(t), \mathbb{P}^\perp x(t)\big) - \mathbb{P}^\perp \xi_d(t) = \mathcal{O}_\gamma\Big(\|u_\gamma\|_{L^3}^3\Big). \tag{5.17}$$

Proof. **Construction of F.** We introduce the smooth map $\psi : \mathbb{R}^d \to \mathcal{S}_1$ defined by:

$$\psi(u_1, \ldots, u_d) := \mathbb{P}\left(\phi_1^{u_1} \circ \cdots \circ \phi_d^{u_d}(0)\right). \tag{5.18}$$

Using (4.3), we obtain by differentiating (5.18) that, for every $h \in \mathbb{R}^d$,

$$\psi'(0) \cdot h = h_1 f_1(0) + \ldots + h_d f_d(0). \tag{5.19}$$

From (5.19), $\psi'(0) : \mathbb{R}^d \to \mathcal{S}_1$ is bijective because $(f_1(0), \dots, f_d(0))$ is a basis of $\mathcal{S}_1$. By the inverse mapping theorem, ψ is a local C^∞-diffeomorphism around 0. We introduce the smooth map $\alpha : \Omega \subset \mathcal{S}_1 \to \mathbb{R}^d$ defined (locally around 0) by:

$$\alpha(p_\parallel) = \big(\alpha_1(p_\parallel), \dots, \alpha_d(p_\parallel)\big) = \psi^{-1}(p_\parallel). \tag{5.20}$$

In particular, $\alpha_i(0) = 0$ for $1 \leqslant i \leqslant d$. We define F as:

$$F(p_\parallel, p_\perp) := \mathbb{P}^\perp \Big(\phi_d^{-\alpha_d(p_\parallel)} \circ \dots \circ \phi_1^{-\alpha_1(p_\parallel)}(p_\parallel + p_\perp) \Big). \tag{5.21}$$

Then, $F(0,\cdot)$ is the identity on $\mathcal{S}_1^\perp$ because $\alpha(0) = 0$ and $\phi_j^0 = \mathrm{Id}$. One has:

$$F(0,0) = 0\,, \quad \frac{\partial F}{\partial p_\parallel}(0,0) = 0 \quad \text{and} \quad \frac{\partial F}{\partial p_\perp}(0,0) = \mathrm{Id}. \tag{5.22}$$

Proof of estimate (5.17). First, using (4.6) and the C^1 regularity of the flows, we have:

$$\mathbb{P}x - \psi(u_1, \dots, u_d) = \mathbb{P}\phi_1^{u_1} \circ \dots \circ \phi_d^{u_d}(\xi_d) - \mathbb{P}\phi_1^{u_1} \circ \dots \circ \phi_d^{u_d}(0) = \mathcal{O}_\gamma(|\xi_d|). \tag{5.23}$$

Hence, plugging estimate (5.6) from Lemma 11 into (5.23) and using that ψ^{-1} is locally Lipschitz-continuous proves that:

$$|\alpha_j(\mathbb{P}x) - u_j| = \mathcal{O}_\gamma\Big(\|u_\gamma\|_{L^2}^2\Big). \tag{5.24}$$

Using (5.24), we will now prove that:

$$\phi_j^{-\alpha_j(\mathbb{P}x)} \circ \dots \circ \phi_1^{-\alpha_1(\mathbb{P}x)}(x) = \xi_j + \sum_{l=1}^{j} \Big(u_l - \alpha_l(\mathbb{P}x)\Big) f_l(0) + \mathcal{O}_\gamma\Big(\|u_\gamma\|_{L^3}^3\Big). \tag{5.25}$$

Equation (5.25) with $j = d$ proves (5.17) because the projection $\mathbb{P}^\perp$ involved in (5.21) kills the d terms $f_1(0), \dots, f_d(0)$ from $\mathcal{S}_1$. We proceed by induction on $1 \leqslant j \leqslant d$ to prove (5.25).

Initialization for $j = 1$. By definition of ξ_1, we have:

$$\phi_1^{-\alpha_1(\mathbb{P}x)}(x) = \phi_1^{-\alpha_1(\mathbb{P}x)} \circ \phi_1^{u_1}(\xi_1) = \phi_1\left(u_1 - \alpha_1(\mathbb{P}x), \xi_1\right). \tag{5.26}$$

Then, using a Taylor formula with respect to the time-like variable of the flow ϕ_1, we get from (4.3), (5.24) and (5.26):

$$\begin{aligned}\phi_1^{-\alpha_1(\mathbb{P}x)}(x) &= \xi_1 + (u_1 - \alpha_1(\mathbb{P}x))\, \partial_\tau \phi_1(0, \xi_1) + \mathcal{O}_\gamma(|u_1 - \alpha_1(\mathbb{P}x)|^2) \\ &= \xi_1 + (u_1 - \alpha_1(\mathbb{P}x))\, f_1(\xi_1) + \mathcal{O}_\gamma\Big(\|u_\gamma\|_{L^2}^4\Big) \\ &= \xi_1 + (u_1 - \alpha_1(\mathbb{P}x))\, f_1(0) + \mathcal{O}_\gamma\Big(\|u_\gamma\|_{L^3}^3\Big),\end{aligned} \tag{5.27}$$

thanks to the C^1 regularity of f_1, estimate (4.29) and Hölder's inequality.

Heredity. Let $1 \leqslant j < d$ and assume that (5.25) holds. Applying $\phi_{j+1}^{-\alpha_{j+1}(\mathbb{P}x)}$ to this relation and using Taylor's formula gives:

$$\begin{aligned}&\phi_{j+1}^{-\alpha_{j+1}(\mathbb{P}x)} \circ \phi_j^{-\alpha_j(\mathbb{P}x)} \circ \dots \circ \phi_1^{-\alpha_1(\mathbb{P}x)}(x) \\ &= \phi_{j+1}^{-\alpha_{j+1}(\mathbb{P}x)} \left(\xi_j + \sum_{l=1}^{j} \Big(u_l - \alpha_l(\mathbb{P}x)\Big) f_l(0) + \mathcal{O}_\gamma\Big(\|u_\gamma\|_{L^3}^3\Big) \right) \\ &= \phi_{j+1}^{-\alpha_{j+1}(\mathbb{P}x)}(\xi_j) + \sum_{j=1}^{l} (u_l - \alpha_l(\mathbb{P}x)) \partial_p \phi_{j+1}^{-\alpha_{j+1}(\mathbb{P}x)}(\xi_j) f_l(0) + \mathcal{O}_\gamma\Big(\|u_\gamma\|_{L^3}^3\Big) \\ &= \phi_{j+1}^{u_{j+1} - \alpha_{j+1}(\mathbb{P}x)}(\xi_{j+1}) + \sum_{j=1}^{l} (u_l - \alpha_l(\mathbb{P}x)) f_l(0) + \mathcal{O}_\gamma\Big(\|u_\gamma\|_{L^3}^3\Big),\end{aligned} \tag{5.28}$$

because, thanks to (5.24), for $1 \leqslant l \leqslant j$:

$$\begin{aligned}(u_l - \alpha_l(\mathbb{P}x))\left(\partial_p \phi_{j+1}^{-\alpha_{j+1}(\mathbb{P}x)} - \mathrm{Id}\right) f_l(0) &= \mathcal{O}_\gamma\left(\|u_\gamma\|_{L^2}^2 \, |\alpha_{j+1}(\mathbb{P}x)|\right) \\ &= \mathcal{O}_\gamma\left(\|u_\gamma\|_{L^2}^2 \left(|u_{j+1}| + \|u_\gamma\|_{L^2}^2\right)\right) \\ &= \mathcal{O}_\gamma\left(\|u_\gamma\|_{L^3}^3\right).\end{aligned} \tag{5.29}$$

Moreover, the same arguments as in the initialization lead to:

$$\phi_{j+1}^{u_{j+1}-\alpha_{j+1}(\mathbb{P}x)}(\xi_{j+1}) = \xi_{j+1} + (u_{j+1} - \alpha_{j+1}(\mathbb{P}x))\, f_{j+1}(0) + \mathcal{O}_\gamma\left(\|u_\gamma\|_{L^3}^3\right). \tag{5.30}$$

Thus, we obtain (5.25) at the next order by incorporating (5.30) in (5.28). ☐

Lemma 13. *Let $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ with $f(0,0) = 0$ satisfying (5.1). There exists a smooth map $G : \mathcal{S}_1 \to \mathcal{S}_1^\perp$ with $G(0) = 0$ and $G'(0) = 0$ such that:*

$$\mathbb{P}^\perp x(t) - G(\mathbb{P}x(t)) - \mathbb{P}^\perp \xi_d(t) = \mathcal{O}_\gamma\left(\|u_\gamma\|_{L^3}^3\right). \tag{5.31}$$

Proof. We consider the smooth map:

$$\tilde{F} : \begin{cases} \mathcal{S}_1 \times \mathcal{S}_1^\perp \times \mathcal{S}_1^\perp \to \mathcal{S}_1^\perp, \\ (p_\parallel, p_\perp, \rho) \mapsto F(p_\parallel, p_\perp) - \rho. \end{cases} \tag{5.32}$$

One checks from (5.22) that $\tilde{F}(0,0,0) = 0$ and $\partial_\perp \tilde{F}(0,0,0) = \mathrm{Id}$. By the implicit function theorem, there exists an open neighborhood U of $(0,0)$ in $\mathcal{S}_1 \times \mathcal{S}_1^\perp$, an open neighborhood of V of 0 in $\mathcal{S}_1^\perp$ and a smooth map $\Theta : U \to V$ such that, for any $(p_\parallel, \rho) \in U$:

$$\left(p_\perp \in V \text{ and } \tilde{F}(p_\parallel, p_\perp, \rho) = 0\right) \Leftrightarrow \left(p_\perp = \Theta(p_\parallel, \rho)\right). \tag{5.33}$$

In particular, $\Theta(0,0) = 0$. Moreover, by differentiating the relation:

$$\tilde{F}(p_\parallel, \Theta(p_\parallel, \rho), \rho) = 0, \tag{5.34}$$

we obtain thanks to (5.22):

$$\frac{\partial \Theta}{\partial p_\parallel}(0,0) = 0 \quad \text{and} \quad \frac{\partial \Theta}{\partial \rho}(0,0) = -\mathrm{Id}. \tag{5.35}$$

We define $G : \mathcal{S}_1 \to \mathcal{S}_1^\perp$ by:

$$G(p_\parallel) := \Theta(p_\parallel, 0). \tag{5.36}$$

One checks that $G(0) = 0$ and $G'(0) = 0$ from (5.35). Moreover, using (5.17), the C^1 regularity of Θ, (5.6) and (5.35) we get:

$$\begin{aligned}\mathbb{P}^\perp x(t) &= \Theta\left(\mathbb{P}x(t), \mathbb{P}^\perp \xi_d(t) + \mathcal{O}_\gamma\left(\|u_\gamma\|_{L^3}^3\right)\right) \\ &= \Theta\left(\mathbb{P}x(t), 0\right) + \mathbb{P}^\perp \xi_d(t) + \mathcal{O}_\gamma\left(\|u_\gamma\|_{L^3}^3\right) \\ &= G\left(\mathbb{P}x(t)\right) + \mathbb{P}^\perp \xi_d(t) + \mathcal{O}_\gamma\left(\|u_\gamma\|_{L^3}^3\right).\end{aligned} \tag{5.37}$$

Equation (5.37) concludes the proof of (5.31). ☐

Remark 2. *The functions F and G constructed in this subsection are only defined in small neighborhoods of the origin. However, since we are always considering controls which are small in $W^{-\gamma,\infty}$ and since Lemmas 8 and 9 imply that, for such controls, trajectories stay in a small neighborhood of the origin, we only use the constructed functions where they are well-defined. In fact, one can also choose any smooth extension of these functions to the whole space. This procedure simplifies the statement of our theorems, avoiding the need to mention this detail.*

5.3 Invariant manifold case

When $\mathcal{S}_2 + \mathbb{R}d_0 = \mathcal{S}_1$, then, by (5.5), $z_d(t) \in \mathcal{S}_1$ for any $t \in [0, T]$ because $G_j(0,0)$ belongs to $\mathcal{S}_1$ (for $j = 0$, see (2.3) and (4.1), for $1 \leqslant j \leqslant d$, see (4.5) and (4.8)) and this space is stable by H_0. Thus $\mathbb{P}^\perp z_d(t) = 0$. Taking this fact into account and combining (5.7) from Lemma 11 with (5.31) gives:

$$\mathbb{P}^\perp x - G(\mathbb{P}x) = \mathcal{O}_\gamma\left(\|u_\gamma\|_{L^3}^3 \right). \tag{5.38}$$

Estimate (5.38) and the meaning of the notation $\mathcal{O}_\gamma(\cdot)$ yield the existence of positive constants C and η such that the conclusions (2.5) and (2.11) of Theorems 2 and 3 hold for controls that are smaller than η in $W^{-\gamma,\infty}$-norm, in the particular case of well-prepared systems satisfying (5.1).

5.4 Quadratic drift case

We consider the case $\mathcal{S}_2 + \mathbb{R}d_0 \not\subset \mathcal{S}_1$ and prove that the state drifts towards the direction d_k defined by (2.3) for $k = 0$ and (2.10) for $k \geqslant 1$.

5.4.1 Coercivity of the quadratic drift for small-times

We know from Lemma 5 that there exists $0 \leqslant k \leqslant d$ such that $G_j(0,0) \in \mathcal{S}_1$ for $0 \leqslant j < k$ and $G_k(0,0) \notin \mathcal{S}_1$. Indeed, thanks to (4.1), (4.5) and (4.8), one has $G_k(0,0) = \frac{1}{2} d_k$. The heuristic is then that:

$$\mathbb{P}^\perp z_d(t) \approx \frac{1}{2} \left(\int_0^t u_k^2(s) \mathrm{d}s \right) d_k. \tag{5.39}$$

To make this statement more precise, we define, for $t \geqslant 0$, $Q_t(u) := \langle z_d(t), d_k \rangle$ and we introduce the following set:

$$\mathbb{T} := \left\{ T > 0;\ \exists C_T > 0\,, \forall t \in (0, T]\,, \forall v \in L^2(0,t), Q_t(v) \geqslant C_T \int_0^t v_k(s)^2 \mathrm{d}s \right\}. \tag{5.40}$$

Lemma 14. *The set $\mathbb{T}$ is non empty.*

Proof. From (5.5), using that $\langle G_j(0,0), d_k \rangle = 0$ for $j < k$, we compute:

$$Q_t(u) = \sum_{j=k}^{d} \int_0^t u_j(s)^2 \langle e^{(t-s)H_0} G_j(0,0), d_k \rangle \mathrm{d}s, \tag{5.41}$$

There exists $C > 0$ such that, for T small enough and $t \in [0, T]$:

$$\left| \int_0^t u_k(s)^2 \left(e^{(t-s)H_0} - \mathrm{Id} \right) G_k(0,0) \mathrm{d}s \right| \leqslant Ct \int_0^t u_k(s)^2 \mathrm{d}s \tag{5.42}$$

and, for every $k + 1 \leqslant j \leqslant d$,

$$\left| \int_0^t u_j(s)^2 e^{(t-s)H_0} G_j(0,0) \mathrm{d}s \right| \leqslant Ct^2 \int_0^t u_k(s)^2 \mathrm{d}s, \tag{5.43}$$

because:

$$|u_j(s)| \leqslant s^{j-k-1} \int_0^t |u_k(\tau)| \mathrm{d}\tau \leqslant s^{j-k-\frac{1}{2}} \left(\int_0^s |u_k(\tau)|^2 \mathrm{d}\tau \right)^{\frac{1}{2}}. \tag{5.44}$$

Gathering (5.41), (5.42) and (5.43) one has $Q_t(u) \geqslant \frac{1}{4} \|u_k\|_{L^2(0,t)}$ for T small enough. ☐

Thanks to Lemma 14, we can define the coercivity time T^* as:

$$T^* := \sup \{T > 0;\ \ T \in \mathbb{T}\}. \tag{5.45}$$

In the general case $T^* < +\infty$ (see Example 6 where $T^* = \pi$). However, in some particular easy cases, it is also possible that $T^* = +\infty$ (see Example 2). In such cases, Theorems 2 and 3 actually lead to the conclusion that the associated systems are not even $W^{2k-3\gamma}$ *large time locally controllable* in the sense that, for any $T > 0$, there exists $\eta > 0$ such that, for any $\delta > 0$, there exists $x^\dagger \in \mathbb{R}^n$ with $|x^\dagger| \leqslant \delta$ such that there is no trajectory from $x(0) = 0$ to $x(T) = x^\dagger$ with a control such that $\|u\|_{W^{2k-3\gamma}} \leqslant \eta$.

5.4.2 Absorption of cubic residuals by interpolation

Let $\mathcal{T} \in (0, T^*)$. From (5.40) and (5.45), there exists $C_{\mathcal{T}} > 0$ such that, for any $T \in (0, \mathcal{T}]$, $v \in L^2(0,T)$, and any $t \in [0,T]$:

$$Q_t(v) \geqslant C_{\mathcal{T}} \int_0^t v(s)^2 \mathrm{d}s. \tag{5.46}$$

Thanks to (5.7), (5.31) and (5.46), one has:

$$\begin{aligned} \left\langle \mathbb{P}^\perp x(t) - G\left(\mathbb{P}x(t)\right), d_k \right\rangle &= \langle \mathbb{P}^\perp \xi_d(t), d_k\rangle + \mathcal{O}_\gamma\left(\|u_\gamma\|^3_{L^3(0,t)}\right) \\ &= \langle z_d(t), d_k\rangle + \mathcal{O}_\gamma\left(\|u_\gamma\|^3_{L^3(0,t)}\right) \\ &\geqslant C_{\mathcal{T}} \|u_k\|^2_{L^2(0,t)} + \mathcal{O}_\gamma\left(\|u_\gamma\|^3_{L^3(0,t)}\right). \end{aligned} \tag{5.47}$$

Therefore, there exists $\eta_0 > 0$ and $M > 0$ such that, if $\|u_\gamma\|_{L^\infty} \leqslant \eta_0$, one has:

$$\left\langle \mathbb{P}^\perp x(t) - G\left(\mathbb{P}x(t)\right), d_k \right\rangle \geqslant C_{\mathcal{T}} \|u_k\|^2_{L^2(0,t)} - M \|u_\gamma\|^3_{L^3(0,t)}. \tag{5.48}$$

Proposition 6. *Let $\kappa \geqslant 1$. There exists constants $C_1, C_2 > 0$ (depending on κ), such that, for any $L > 0$ and any $\psi \in W^{3\kappa-3,\infty}((0,L),\mathbb{R})$:*

$$\left\|\psi^{(\kappa-1)}\right\|^3_{L^3} \leqslant C_1 \|\psi\|^2_{L^2} \left\|\psi^{(3\kappa-3)}\right\|_{L^\infty} + C_2 L^{\frac{5}{2}-3\kappa} \|\psi\|^3_{L^2}. \tag{5.49}$$

Moreover, for any $L > 0$ and any $\psi \in W_0^{3\kappa-3}((0,L),\mathbb{R})$:

$$\left\|\psi^{(\kappa-1)}\right\|^3_{L^3} \leqslant (C_1 + C_2) \|\psi\|^2_{L^2} \left\|\psi^{(3\kappa-3)}\right\|_{L^\infty}. \tag{5.50}$$

Proof. This is a particular case of the Gagliardo-Nirenberg interpolation inequality. Indeed, for any $\kappa \geqslant 1$:

$$\frac{1}{3} = \frac{\kappa-1}{1} + \frac{1}{3}\cdot\left(\frac{1}{\infty} - \frac{3\kappa-3}{1}\right) + \left(1 - \frac{1}{3}\right)\frac{1}{2}. \tag{5.51}$$

From (5.51), we can apply [30, Theorem p.125] for functions defined on $[0,1]$. The generalization to functions defined on $[0,L]$ uses a straightforward scaling argument which gives the power of L in front of C_2 in (5.49). Then, for $\psi \in W_0^{3\kappa-3,\infty}((0,L),\mathbb{R})$, using the Cauchy-Schwarz inequality, iterated integration and the conditions $\psi^{(j)}(0) = 0$ for $j = 0, ..., 3\kappa-4$ we obtain:

$$L^{\frac{5}{2}-3\kappa} \|\psi\|_{L^2} \leqslant L^{3-3\kappa} \|\psi\|_{L^\infty} \leqslant \left\|\psi^{(3\kappa-3)}\right\|_{L^\infty}, \tag{5.52}$$

which proves (5.50). ☐

We apply Proposition 6 to $\psi := u_k$ with $\kappa = k - \gamma + 1$.

First case: $u \in W_0^{2k-3\gamma,\infty}(0,T)$. Let $t \in [0,T]$. In this case, $\psi = u_k \in W_0^{3k-3\gamma}(0,t)$ because $u \in W_0^{2k-3\gamma}(0,t)$ and $u_k^{(j)}(0) = u_{k-j}(0) = 0$ for $j = 0, ..., k-1$. Thanks to (5.50) for $L = t$ and to the equality $u_k^{(k)} = u$, one has:

$$\begin{aligned} \|u_\gamma\|_{L^3(0,t)}^3 = \left\|u_k^{(k-\gamma)}\right\|_{L^3(0,t)}^3 &\leqslant (C_1 + C_2)\|u_k\|_{L^2(0,t)}^2 \|u_k^{(3k-3\gamma)}\|_{L^\infty(0,t)} \\ &\leqslant (C_1 + C_2)\|u_k\|_{L^2(0,t)}^2 \|u^{2k-3\gamma}\|_{L^\infty(0,t)}. \end{aligned} \tag{5.53}$$

Then, we deduce from (5.48) that:

$$\left\langle \mathbb{P}^\perp x(t) - G\left(\mathbb{P}x(t)\right), d_k \right\rangle \geqslant \|u_k\|_{L^2(0,t)}^2 \left(C_{\mathcal{T}} - M(C_1 + C_2)\|u^{(2k-3\gamma)}\|_{L^\infty(0,t)} \right) \tag{5.54}$$

In particular, when u satisfies:

$$\|u\|_{W^{2k-3\gamma,\infty}} \leqslant \eta(\mathcal{T}) := \min\left(\eta_0, \frac{C_{\mathcal{T}}}{2M(C_1 + C_2)}\right), \tag{5.55}$$

then:

$$\left\langle \mathbb{P}^\perp x(t) - G\left(\mathbb{P}x(t)\right), d_k \right\rangle \geqslant \frac{1}{2} C_{\mathcal{T}} \|u_k\|_{L^2(0,t)}^2 \geqslant 0\,. \tag{5.56}$$

This gives the conclusions (2.6) and (2.9) of Theorems 2 and 3. Moreover, we recover the inequality announced in Subsection 2.2 (the drift is quantified by the H^{-k}-norm of u).

Second case: $u \in W^{2k-3\gamma,\infty}(0,T)$. When the control u only belong to $W^{2k-3\gamma,\infty}(0,T)$ instead of $W_0^{2k-3\gamma,\infty}(0,T)$, inequality (5.49) with $L = T$ yields:

$$\begin{aligned} \|u_\gamma\|_{L^3}^3 &\leqslant \|u_k\|_{L^2}^2 \left(C_1 \left\|u^{(2k-3\gamma)}\right\|_{L^\infty} + C_2 T^{\frac{5}{2} - 3(k-\gamma+1)} \|u_k\|_{L^2} \right) \\ &\leqslant \|u_k\|_{L^2}^2 \left(C_1 \left\|u^{(2k-3\gamma)}\right\|_{L^\infty} + C_2 T^{3 - 3(k-\gamma+1)} \|u_k\|_{L^\infty} \right) \\ &\leqslant \|u_k\|_{L^2}^2 \left(C_1 \left\|u^{(2k-3\gamma)}\right\|_{L^\infty} + C_2 T^{-2k+2\gamma} \|u_\gamma\|_{L^\infty} \right) \\ &\leqslant \|u_k\|_{L^2}^2 \left(C_1 + C_2 T^{-2k+2\gamma} \right) \|u\|_{W^{2k-3\gamma,\infty}}\,. \end{aligned} \tag{5.57}$$

In particular, when u satisfies:

$$\|u\|_{W^{2k-3\gamma,\infty}} \leqslant \eta(T) := \min\left(\eta_0, \frac{C_{\mathcal{T}}}{2M(C_1 + C_2 T^{-2k+2\gamma})}\right), \tag{5.58}$$

then (5.56) holds at the final time $t = T$. In particular, this proves that system (1.1) is not $W^{2k-3\gamma,\infty}$ small-time locally controllable.

6 Reduction to well-prepared smooth systems

We prove Theorems 2 and 3 for general systems by reduction to the case of well-prepared smooth systems considered in Section 5, by means of a linear static state feedback transformation. We start with smooth systems then extend our results to less regular dynamics.

6.1 Linear static state feedback and Lie brackets

We consider linear static state feedback transformations of the control. We prove that the structural properties of the Lie spaces $\mathcal{S}_1$ and $\mathcal{S}_2$ that are involved in our theorems are invariant under such transformations.

Let $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ and $\beta \in \mathbb{R}^n$. We consider the transformed control $v := u - (\beta, x)$. The state now evolves under the equation $\dot{x} = g(x, v)$ where we define:

$$g(x, v) := f(x, v + (\beta, x)). \tag{6.1}$$

As in (1.6), we use the notations $f_0(x) := f(x, 0)$, $g_0(x) := g(x, 0)$, $f_1(x) := \partial_u f(x, 0)$, $g_1(x) := \partial_v g(x, 0)$ and $d_0 := \partial_{uu} f(0, 0) = \partial_{vv} g(0, 0)$. Using (6.1), a Taylor expansion with respect to the control and Lemma 3, one has:

$$g_0(x) =_{[3]} f_0(x) + (\beta, x) f_1(x) + \frac{1}{2}(\beta, x)^2 d_0, \tag{6.2}$$

$$g_1(x) =_{[2]} f_1(x) + (\beta, x) d_0. \tag{6.3}$$

The linear controllable spaces associated with the dynamics f and g are the same. We will denote them by $\mathcal{S}_1$. Indeed, $g_1(0) = f_1(0)$ and $g_0'(0) = f_0'(0) + (\beta, \cdot) f_1(0)$. Thus, the iterated powers of $g_0'(0)$ applied to $g_1(0)$ span the same space as the iterated powers of $f_0'(0)$ applied to $f_1(0)$. We prove that second-order Lie brackets of g_0 and g_1 inherit the following behavior.

Lemma 15. *Let $k \in \mathbb{N}$. Assume that $d_0 \in \mathcal{S}_1$ and that the original system satisfies:*

$$\left[f_1, \mathrm{ad}^j_{f_0}(f_1)\right](0) \in \mathcal{S}_1, \quad \text{for any } 1 \leqslant j \leqslant k. \tag{6.4}$$

Then, the transformed system satisfies:

$$\left[g_1, \mathrm{ad}^j_{g_0}(g_1)\right](0) \in \mathcal{S}_1, \quad \text{for any } 1 \leqslant j \leqslant k. \tag{6.5}$$

Moreover, one has the following equality modulo terms in $\mathcal{S}_1$:

$$\left[g_1, \mathrm{ad}^{k+1}_{g_0}(g_1)\right](0) \sim \left[f_1, \mathrm{ad}^{k+1}_{f_0}(f_1)\right](0). \tag{6.6}$$

Proof. **We start with first-order Lie brackets.** For any $0 \leqslant m \leqslant k+1$, we prove that there exists m smooth scalar functions $\lambda_0^m, \dots, \lambda_{m-1}^m : \mathbb{R}^n \to \mathbb{R}$ and a smooth function $\mu_m : \mathbb{R}^n \to \mathcal{S}_1$ with $\mu_m(0) = 0$ such that:

$$\mathrm{ad}^m_{g_0}(g_1)(x) =_{[2]} \mathrm{ad}^m_{f_0}(f_1)(x) + \mu_m(x) + \sum_{j=0}^{m-1} \lambda_j^m(x) \mathrm{ad}^j_{f_0}(f_1)(x). \tag{6.7}$$

We proceed by induction on m. For $m = 0$, the sum in (6.7) is empty by convention. Thus, (6.7) holds for $m = 0$ with $\mu_0(x) := (\beta, x) d_0$ thanks to (6.3). Let $0 \leqslant m \leqslant k$, we compute the next Lie bracket using (6.2), (6.7) and Lemma 1:

$$\begin{aligned}
\left[g_0, \mathrm{ad}^m_{g_0}(g_1)\right] =_{[2]} & \frac{1}{2}(\beta, \cdot)^2 \left(\mathrm{ad}^m_{g_0}(g_1)\right)' d_0 + \left(\mathrm{ad}^m_{f_0}(f_1)\right)' \left(f_0 + (\beta, \cdot) f_1\right) + \mu_m' g_0 \\
& + \sum_{j=0}^{m-1} \left(\left(\lambda_j^m\right)', g_0\right) \mathrm{ad}^j_{f_0}(f_1) + \lambda_j^m \left(\mathrm{ad}^j_{f_0}(f_1)\right)' \left(f_0 + (\beta, \cdot) f_1\right) \\
& - f_0' \left(\mathrm{ad}^m_{f_0}(f_1) + \sum_{j=0}^{m-1} \lambda_j^m \mathrm{ad}^j_{f_0}(f_1) + \mu_m\right) - \left(\beta, \mathrm{ad}^m_{g_0}(g_1)\right) f_1 \\
& - (\beta, \cdot) f_1' \left(\mathrm{ad}^m_{f_0}(f_1) + \sum_{j=0}^{m-1} \lambda_j^m \mathrm{ad}^j_{f_0}(f_1) + \mu_m\right) \\
& - (\beta, \cdot) \left(\beta, \mathrm{ad}^m_{g_0}(g_1)\right) d_0.
\end{aligned} \tag{6.8}$$

Using $(\beta,\cdot)\, f_1'\mu_m =_{[2]} 0$ and $(\beta,\cdot)^2 \left(\mathrm{ad}^m_{g_0}(g_1)\right)' d_0 =_{[2]} 0$, reordering the terms in (6.8) yields:

$$\begin{aligned}\left[g_0, \mathrm{ad}^m_{g_0}(g_1)\right] =_{[2]}\ & \mathrm{ad}^{m+1}_{f_0}(f_1) + \mu_m' g_0 - f_0'\mu_m - (\beta,\cdot)\left(\beta, \mathrm{ad}^m_{g_0}(g_1)\right) d_0 \\ & + (\beta,\cdot)\left([f_1, \mathrm{ad}^m_{f_0}(f_1)] + \sum_{j=0}^{m-1} \lambda^m_j [f_1, \mathrm{ad}^j_{f_0}(f_1)]\right) \\ & - \left(\beta, \mathrm{ad}^m_{g_0}(g_1)\right) f_1 + \sum_{j=0}^{m-1} \left(\left(\lambda^m_j\right)', \tilde{f}_0\right) \mathrm{ad}^j_{f_0}(f_1) + \lambda^m_j \mathrm{ad}^{j+1}_{f_0}(f_1).\end{aligned} \tag{6.9}$$

Since $m \leqslant k$, hypothesis (6.4) yields the existence of a constant $\gamma_m \in \mathcal{S}_1$ such that:

$$[f_1, \mathrm{ad}^m_{f_0}(f_1)] + \sum_{j=0}^{m-1} \lambda^m_j [f_1, \mathrm{ad}^j_{f_0}(f_1)] =_{[1]} \gamma_m. \tag{6.10}$$

From Lemma 3, $f_0' =_{[1]} H_0$. Thus, $f_0'\mu_m =_{[2]} H_0\mu_m$. Using these remarks, we define:

$$\mu_{m+1} := \mu_m' g_0 - H_0\mu_m + (\beta,\cdot)\,\gamma_m - (\beta,\cdot)\left(\beta, \mathrm{ad}^m_{g_0}(g_1)\right) d_0. \tag{6.11}$$

Since $\mathcal{S}_1$ is stable under multiplication by H_0 and the image of μ_m is included in $\mathcal{S}_1$, so is the image of μ_{m+1} and $\mu_{m+1}(0) = 0$. Eventually, we define:

$$\lambda^{m+1}_0 := -\left(\beta, \mathrm{ad}^m_{g_0}(g_1)\right) + \left(\left(\lambda^m_0\right)', \tilde{f}_0\right), \tag{6.12}$$

$$\lambda^{m+1}_j := \lambda^m_{j-1} + \left(\left(\lambda^m_j\right)', \tilde{f}_0\right), \quad \text{for} \quad 1 \leqslant j < m, \tag{6.13}$$

$$\lambda^{m+1}_m := \lambda^m_{m-1}. \tag{6.14}$$

In the particular case $m = 0$, we only use (6.12) with the convention that $\lambda^0_0 = 0$. For $m = 1$, we use (6.12) and (6.14). For larger m, we use all three formulas including (6.13). Plugging into (6.9) the definitions (6.10) and (6.11) proves (6.7) at order $m + 1$.

We move on to second-order Lie brackets. Let $1 \leqslant m \leqslant k + 1$. Using Lemma 2 and formula (6.7), we compute:

$$\begin{aligned}\left[g_1, \mathrm{ad}^m_{g_0}(g_1)\right](0) = & \left(\mathrm{ad}^m_{f_0}(f_1)\right)'(0) f_1(0) + \mu_m'(0) f_1(0) \\ & + \sum_{j=0}^{m-1} \lambda^m_j(0) \left(\mathrm{ad}^j_{f_0}(f_1)\right)'(0) f_1(0) + ((\lambda^m_j)'(0), f_1(0)) \mathrm{ad}^j_{f_0}(f_1)(0) \\ & - f_1'(0)\mathrm{ad}^m_{f_0}(f_1) - f_1'(0)\mu_m(0) - \sum_{j=0}^{m-1} \lambda^m_j(0) f_1'(0) \mathrm{ad}^j_{f_0}(f_1)(0).\end{aligned} \tag{6.15}$$

We have $\mu_m(0) = 0$. For $j \in \mathbb{N}$, the vectors $\mathrm{ad}^j_{f_0}(f_1)(0)$ belong to $\mathcal{S}_1$ by definition of $\mathcal{S}_1$. Moreover, since the image of μ_m is contained in $\mathcal{S}_1$, $\mu_m'(0) f_1(0) \in \mathcal{S}_1$. Hence, from (6.15):

$$\left[g_1, \mathrm{ad}^m_{g_0}(g_1)\right](0) \sim \left[f_1, \mathrm{ad}^m_{f_0}(f_1)\right](0) + \sum_{j=0}^{m-1} \lambda^m_j(0) \left[f_1, \mathrm{ad}^j_{f_0}(f_1)\right](0). \tag{6.16}$$

Using (6.4) and (6.16) proves (6.5) for $1 \leqslant m \leqslant k$ and (6.6) for $m = k + 1$. □

We proved Lemma 15 for nonlinear systems. It also holds in the particular case of control-affine systems. For such systems, approximate equalities (6.2) and (6.3) become equalities which hold with $d_0 = 0 \in \mathcal{S}_1$.

6.2 Generalization of the proof using a Brunovský transformation

We prove Theorem 2 and Theorem 3 for general systems by reduction to the well-prepared class studied in Section 5. The main argument is a linear transformation first proposed by Brunovský in [11] (see [40, Theorem 2.2.7] for a modern proof).

6.2.1 A linear transformation

We consider the linearized system (1.8). Denoting by d the dimension of $\mathcal{S}_1$, there exists a matrix $R \in GL_n(\mathbb{R})$ such that $Rb = e_1$ and:

$$RH_0R^{-1} = \begin{pmatrix} \Lambda_d & * \\ 0 & * \end{pmatrix}, \quad \text{where} \quad \Lambda_d := \begin{pmatrix} -\alpha_1 & & \dots & & -\alpha_d \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & 0 \end{pmatrix} \tag{6.17}$$

and the α_i are the coefficients of the characteristic polynomial of the controllable part of H_0: $\chi(X) := X^d + \alpha_1 X^{d-1} + \ldots + \alpha_d$. We introduce α a column vector with n components whose first d components are the α_i and whose last $n-d$ components are null. Let us denote by $\beta := R^{\text{tr}}\alpha$ and $v := u - (\beta, y)$. Hence:

$$\dot{y} = \mathcal{H}_0 y + vb, \tag{6.18}$$

with $\mathcal{H}_0 := H_0 + R^{-1}e_1\alpha^{\text{tr}}R$. By construction of (6.17), one has $\mathcal{H}_0^d b = 0$, which corresponds to the well-prepared nilpotent integrator form that we studied in Section 5.

6.2.2 Generalization of the proof for nonlinear systems

We start with the following lemma which proves that the analytic notions involved in Theorem 2 are invariant under linear static state feedback transformations.

Lemma 16. *Let $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ with $f(0,0) = 0$, $\beta \in \mathbb{R}^n$ and $\mathcal{T} > 0$. There exists $C, \eta > 0$ and a family of constants $C_m, \eta_m > 0$ for $m \in \mathbb{N}$ such that, for any $T \in (0, \mathcal{T})$ and any trajectory $(x, u) \in C^0([0,T], \mathbb{R}^n) \times L^\infty((0,T), \mathbb{R})$ of system (1.1) with $x(0) = 0$, letting $v := u - (\beta, x)$, one has:*

$$\|u\|_{L^\infty} \leqslant \eta \quad \Rightarrow \quad \|v\|_{L^3} \leqslant C\,\|u\|_{L^3}, \tag{6.19}$$

$$\|u\|_{W^{m,\infty}} \leqslant \eta_m \quad \Rightarrow \quad \|v\|_{W^{m,\infty}} \leqslant C_m\,\|u\|_{W^{m,\infty}}. \tag{6.20}$$

Moreover, when $u(0) = \ldots = u^{(m-1)}(0) = 0$, then $v(0) = \ldots = v^{(m-1)}(0) = 0$.

Proof. **First**, from Lemma 8, one has $x = \mathcal{O}_0(\|u\|_{L^1})$. Thus, from Hölder's inequality:

$$\|v\|_{L^3} = \mathcal{O}_0(\|u\|_{L^3} + \|u\|_{L^1}) = \mathcal{O}_0(\|u\|_{L^3}). \tag{6.21}$$

From Definition 10, estimate (6.21) proves (6.19).

Second, we prove by induction on $m \geqslant 0$ that there exists a family of smooth functions $P_m : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, vanishing at zero, such that:

$$\partial_t^m v = \partial_t^m u + P_m(u, \partial_t u, \ldots, \partial_t^{m-1} u, x). \tag{6.22}$$

For $m = 0$, (6.22) is a rephrasing of $v = u - (\beta, x)$ with $P_0(x) := -(\beta, x)$. For a fixed $m \in \mathbb{N}$, we differentiate (6.22) with respect to time using (1.1):

$$\partial_t^{m+1} v = \partial_t^{m+1} u + (\partial_x P_m) f(x,u) + \sum_{j=0}^{m-1} \partial_t^{j+1} u (\partial_j P_m). \tag{6.23}$$

Hence, (6.23) proves (6.22) at order $m+1$ provided that we define:

$$P_{m+1}(a_0,\dots a_m,x) := \sum_{j=0}^{m-1} a_{j+1}(\partial_j P_m)(a_0,\dots a_{m-1},x) \\ + (\partial_x P_m)(a_0,\dots a_{m-1},x) f(x,a_0). \tag{6.24}$$

From (6.24) and since $f(0,0)=0$, P_{m+1} vanishes at zero. In particular, from (6.22) and the null value of the P_m at zero, the derivatives of v vanish at the initial time as soon as those of u vanish for trajectories with $x(0)=0$.

Last, we prove (6.20). From the smoothness of P_m and the null value of P_m at zero, we deduce the existence of $A_m, \rho_m > 0$ such that, for any $(a_0,\dots,a_{m-1}) \in \mathbb{R}^m$, for any $x \in \mathbb{R}^n$,

$$|x| + \sum_{j=0}^{m-1} |a_j| \leqslant \rho_m \Rightarrow |P_m(a_0,\dots,a_{m-1},x)| \leqslant A_m \left(|x| + \sum_{j=0}^{m-1} |a_j| \right). \tag{6.25}$$

From Lemma 8, $x = \mathcal{O}_0(\|u\|_{L^\infty})$. Thus, from (6.25) there exists $B_m, \eta_m > 0$ such that:

$$\|u\|_{W^{m,\infty}} \leqslant \eta_m \quad \Rightarrow \quad \left| P_m(u, \partial_t u, \dots, \partial_t^{m-1} u, x) \right| \leqslant B_m \|u\|_{W^{m,\infty}}. \tag{6.26}$$

From (6.22) and (6.26), estimate (6.20) holds with $C_m := 2B_m$. □

We finish the proof of Theorem 2 in the general case. Let $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ with $f(0,0)=0$. Let $\beta \in \mathbb{R}^n$ be defined as in paragraph 6.2.1. We apply the associated linear static state feedback transformation to the full nonlinear system by setting $v := u - (\beta, x)$ and considering the transformed system as in (6.1). The new system $\dot{x} = g(x,v)$ satisfies the integrator assumption (5.1). Hence, we know from Section 5 that Theorem 2 is true for this system. Let us check that it also holds for the initial system.

- Let us assume that $\mathcal{S}_2 + \mathbb{R}d_0 = \mathcal{S}_1$ for the initial f-system. Then, thanks to Lemma 15, this is also the case for the transformed g-system. From Theorem 2 applied to g, there exists a map $G \in C^\infty(\mathcal{S}_1, \mathcal{S}_1^\perp)$ such that, for any $T>0$, there exists $M, \eta > 0$ such that, for any trajectory $(x,v) \in C^0([0,T],\mathbb{R}^n) \times L^\infty((0,T),\mathbb{R})$ of $\dot{x} = g(x,v)$ with $x(0)=0$ and $\|v\|_{L^\infty} \leqslant \eta$:

$$\forall t \in [0,T], \quad \left| \mathbb{P}^\perp x(t) - G(\mathbb{P}x(t)) \right| \leqslant M \|v\|_{L^3}^3. \tag{6.27}$$

  Thanks to Lemma 16, if $(x,u) \in C^0([0,T],\mathbb{R}^n) \times L^\infty((0,T),\mathbb{R})$ is a trajectory of (1.1) with $x(0)=0$ and $\|u\|_{L^\infty} \leqslant \min(\eta_0, \eta/C_0)$ then $\|v\|_{L^\infty} \leqslant \eta$ and:

$$\forall t \in [0,T], \quad \left| \mathbb{P}^\perp x(t) - G(\mathbb{P}x(t)) \right| \leqslant CM \|u\|_{L^3}^3. \tag{6.28}$$

- Let us assume that $\mathcal{S}_2 + \mathbb{R}d_0 \not\subset \mathcal{S}_1$ for the initial f-system. Then there exists a smallest $0 \leqslant k \leqslant d$ such that $d_k \neq 0$. Moreover, thanks to Lemma 15, this direction is the same for the g-system (see (6.6)). Thus, there is no ambiguity. From Theorem 2 applied to g, there exits $T^* > 0$ such that, for any $\mathcal{T} < T^*$ and any $T \in (0,\mathcal{T}]$, there exists $\eta > 0$, for any trajectory $(x,v) \in C^0([0,T],\mathbb{R}^n) \times L^\infty((0,T),\mathbb{R})$ of $\dot{x} = g(x,v)$ with $x(0)=0$, $v \in W_0^{2k,\infty}(0,T)$ and $\|v\|_{W^{2k,\infty}} \leqslant \eta$:

$$\forall t \in [0,T], \quad \left\langle \mathbb{P}^\perp x(t) - G(\mathbb{P}x(t)), d_k \right\rangle \geqslant 0. \tag{6.29}$$

  Thanks to Lemma 16, if $(x,u) \in C^0([0,T],\mathbb{R}^n) \times L^\infty((0,T),\mathbb{R})$ is a trajectory of (1.1) with $x(0)=0$, such that $u \in W_0^{2k,\infty}(0,T)$ and $\|u\|_{W^{2k,\infty}} \leqslant \min(\eta_{2k}, \eta/C_{2k})$ then $v \in W^{2k,\infty}(0,T)$, $\|v\|_{W^{2k,\infty}} \leqslant \eta$ and (6.29) holds.

6.2.3 Generalization of the proof for control-affine systems

We finish the proof of Theorem 3 in the general case for control-affine systems. We proceed exactly as for nonlinear systems: using a Brunovský transformation and the invariance of the geometric and analytic notions involved.

It only remains to be checked that the weaker norms $W^{-1,3}$ and $W^{-1,\infty}$ of the control are preserved under linear static state feedback transformations. From Lemma 9, one has:

$$x(t) = \mathcal{O}_1(|u_1(t)| + \|u_1\|_{L^1}). \tag{6.30}$$

Integrating (6.30) yields:

$$\left(\beta, \int_0^t x(s)\mathrm{d}s\right) = \mathcal{O}_1(\|u_1\|_{L^1}). \tag{6.31}$$

In particular, still denoting by $v = u - (\beta, x)$, equation (6.31) and Hólder's inequality yield the missing estimates:

$$\|v_1\|_{L^3} = \mathcal{O}_1(\|u_1\|_{L^3}), \tag{6.32}$$

$$\|v_1\|_{L^\infty} = \mathcal{O}_1(\|u_1\|_{L^\infty}). \tag{6.33}$$

6.2.4 Coercivity estimate for ill-prepared systems

In paragraph 5.4.1, we proved that, for well-prepared systems satisfying (5.1), the quadratic drift along d_k is quantified by the H^{-k}-norm of the control (the L^2-norm of u_k). For ill-prepared systems we therefore obtain that the drift is quantified by the L^2-norm of v_k, with $v = u - (\beta, x)$ (where the transformation is chosen from using the Brunovský approach). In the following lines, we prove that, for small enough times, this also yields coercivity with respect to the L^2-norm of u_k, thereby justifying the comment announced in Subsection 2.2.

Obstruction of order 0 (for nonlinear systems). Let us assume that we are considering a nonlinear system for which $d_0 \notin \mathcal{S}_1$. Hence, for $\mathcal{T} < T^*$ (where T^* is defined by (5.40) and (5.45)), the drift along d_0 is quantified by the L^2-norm of v. From Lemma 8 and the Cauchy-Schwarz inequality, one has:

$$|x(t)| = \mathcal{O}_0\left(\sqrt{t}\,\|u\|_{L^2}\right). \tag{6.34}$$

Hence:

$$\|v\|_{L^2(0,t)} = \|u\|_{L^2(0,t)}\,(1 + \mathcal{O}_0(t)) \geqslant \frac{1}{2}\,\|u\|_{L^2(0,t)}\,, \tag{6.35}$$

provided that $\mathcal{T}$ is small enough and that u is small enough in L^∞. Therefore, the drift is coercive with respect to $\|u\|_{L^2}$ for small enough times and controls.

Obstruction of order $1 \leqslant k \leqslant d$. Let us assume that we are considering a system for which $d_0 \in \mathcal{S}_1$ (or $d_0 = 0$ for control-affine systems) but $d_k \notin \mathcal{S}_1$. Hence, for $\mathcal{T} < T^*$, the drift along d_k is quantified by the L^2-norm of v_k. From Lemma 10 and the Cauchy-Schwarz inequality, we have:

$$|\xi_d(t)| = \mathcal{O}_\gamma\left(\sqrt{t}\,\|u_d\|_{L^2(0,t)} + \|u_\gamma\|^2_{L^2(0,t)}\right). \tag{6.36}$$

Moreover, from the C^2 regularity of the flow $\phi_1, \ldots \phi_d$, one has:

$$x = u_1 f_1(0) + \ldots + u_d f_d(0) + \mathcal{O}_\gamma\left(|\xi_d| + |u_1|^2 + \ldots + |u_d|^2\right). \tag{6.37}$$

Integrating (6.37) with respect to time k times yields:

$$\begin{aligned} \|v_k\|_{L^2(0,t)} = \|u_k\|_{L^2(0,t)} &+ \mathcal{O}\Big(\|u_{k+1}\|_{L^2(0,t)} + \ldots + \|u_{k+d}\|_{L^2(0,t)}\Big) \\ &+ t^{k+\frac{1}{2}}\mathcal{O}_\gamma\Big(\|u_d\|_{L^2(0,t)}\Big) \\ &+ t^k\mathcal{O}_\gamma\Big(\|u_\gamma\|^2_{L^2(0,t)}\Big) \\ &+ t^{k-1}\mathcal{O}_\gamma\Big(\|u_1\|^2_{L^2(0,t)} + \ldots + \|u_d\|^2_{L^2(0,t)}\Big). \end{aligned} \tag{6.38}$$

For $1 \leqslant j \leqslant d$, $\|u_{k+j}\|_{L^2(0,t)} = \mathcal{O}\Big(t\,\|u_k\|_{L^2(0,t)}\Big)$. Hence, the error terms in the first line are small when the time is small enough. The error term of the second line is also small because $k \leqslant d$ so it is of order $\mathcal{O}_\gamma\Big(t^{d+1/2}\,\|u_k\|_{L^2(0,t)}\Big)$. In the fourth line, the first error term dominates the following ones for small enough times. Hence, it remains to be checked that the following quantity is small enough:

$$U(t) := t^k\,\|u_\gamma\|^2_{L^2(0,t)} + t^{k-1}\,\|u_1\|^2_{L^2(0,t)} \tag{6.39}$$

Thanks to the Gagliardo-Nirenberg interpolation inequality, for each $1 \leqslant k \leqslant d$, there exists $C_1, C_2 > 0$ (independent on T) such that, for $l \in \{0,1\}$:

$$\|u_l\|^2_{L^2(0,t)} \leqslant C_1\left\|u_k^{(2k-2l)}\right\|_{L^2(0,t)}\|u_k\|_{L^2(0,t)} + C_2t^{2l-2k}\,\|u_k\|^2_{L^2(0,t)}. \tag{6.40}$$

First case: $k = 1$. We start with this low-order case which is handled a little differently. For control-affine systems, we directly obtain from (6.39) that:

$$U(t) \leqslant (1+t)\,\|u_1\|^2_{L^2(0,t)} \leqslant (1+t)\sqrt{t}\,\|u_1\|_{L^\infty}\,\|u_1\|_{L^2(0,t)}. \tag{6.41}$$

For nonlinear systems, we use the interpolation inequality (6.40) for the first term:

$$\begin{aligned} t\,\|u\|^2_{L^2(0,t)} &\leqslant tC_1\,\|\dot{u}\|_{L^2(0,t)}\,\|u_1\|_{L^2(0,t)} + C_2t^{-1}\,\|u_1\|^2_{L^2(0,t)} \\ &\leqslant t^{\frac{3}{2}}C_1\,\|\dot{u}\|_{L^\infty}\,\|u_1\|_{L^2(0,t)} + C_2t^{\frac{1}{2}}\,\|u\|_{L^\infty}\,\|u_1\|_{L^2(0,t)}. \end{aligned} \tag{6.42}$$

The second term is estimated as $\|u_1\|^2_{L^2(0,t)} \leqslant t^{\frac{3}{2}}\,\|u\|_{L^\infty}\,\|u_1\|_{L^2(0,t)}$. Both estimates (6.41) and (6.42) lead to the conclusion that $U(t)$ is small with respect to $\|u_1\|_{L^2(0,t)}$ when $u_\gamma \to 0$ in L^∞. From (6.38), we conclude that:

$$\|v_k\|_{L^2(0,t)} \geqslant \frac{1}{2}\,\|u_k\|_{L^2(0,t)}, \tag{6.43}$$

for small enough times and small enough controls.

Second case: $k \geqslant 2$. Thanks to (6.40), for $l \in \{0,1\}$, one has:

$$\begin{aligned} t^{k-l}\,\|u_l\|^2_{L^2(0,t)} &\leqslant C_1t^{k-l}\left\|u_k^{(2k-2l)}\right\|_{L^2(0,t)}\|u_k\|_{L^2(0,t)} + C_2t^{l-k}\,\|u_k\|^2_{L^2(0,t)} \\ &\leqslant C_1t^{k-l+\frac{1}{2}}\left\|u^{(k-2l)}\right\|_{L^\infty}\|u_k\|_{L^2(0,t)} + C_2t^{l+\frac{1}{2}}\,\|u\|_{L^\infty}\,\|u_k\|_{L^2(0,t)}. \end{aligned} \tag{6.44}$$

Using (6.39), we conclude that:

$$U(t) = \mathcal{O}_\gamma(\|u\|_{W^{k-2\gamma,\infty}} + \|u\|_{W^{k-2,\infty}})\,\|u_k\|_{L^2}. \tag{6.45}$$

For nonlinear systems, $k < 2k$ and $k-2 < 2k$. For control-affine systems, $k-2 < 2k-3$. Hence, in both cases:

$$U(t) = \mathcal{O}_\gamma(\|u\|_{W^{2k-3\gamma,\infty}}\,\|u_k\|_{L^2}). \tag{6.46}$$

Hence, when $u \to 0$ in $W^{2k-3\gamma}$, $U(t)$ is small with respect to $\|u_k\|_{L^2(0,t)}$ and we conclude that (6.43) also holds for small enough times and small enough controls.

6.3 Persistence of results for less regular systems

Theorems 2 and 3 are stated with smooth vector fields to facilitate their understanding. However, the same conclusions can be extended to less regular dynamics and this highlights that our method only relies on the quadratic behavior of the system. We explain how one can extend the quadratic alternative to such less regular systems. As observed in Subsection 3.3, the spaces $\mathcal{S}_1$ and $\mathcal{S}_2$ can be defined as soon as $f \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$. We start with general nonlinear systems.

Corollary 2. *Let $f \in C^3(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ with $f(0,0) = 0$. The conclusions of Theorem 2 hold.*

Proof. We start by considering a truncated system with initial condition $\hat{x}(0) = 0$:

$$\dot{\hat{x}} = T_2 f(\hat{x}, u) = H_0 \hat{x} + ub + \frac{1}{2} Q_0(\hat{x}, \hat{x}) + u H_1 \hat{x} + \frac{1}{2} d_0 u^2, \tag{6.47}$$

where $T_2 f$ denotes the second-order Taylor expansion of f around the origin, with straightforward notations already used in the previous subsections. This defines a new nonlinear system, which is smooth and therefore satisfies the quadratic alternative of Theorem 2. Moreover, by definition (see Subsection 3.3), the spaces $\mathcal{S}_1$ and $\mathcal{S}_2$ for this new system are the same as those of the original system. Let us prove that x and $\hat{x}$ are close. One has:

$$\begin{aligned} \dot{x} - \dot{\hat{x}} &= f(x, u) - T_2 f(\hat{x}, u) \\ &= \big(f(x,u) - T_2 f(x,u)\big) + (H_0 + u H_1)(x - \hat{x}) + \frac{1}{2} Q_0(x - \hat{x}, x + \hat{x}). \end{aligned} \tag{6.48}$$

From Lemma 8 applied to f and $T_2 f$:

$$x(t) = \mathcal{O}_0(1), \tag{6.49}$$
$$\hat{x}(t) = \mathcal{O}_0(1), \tag{6.50}$$
$$x(t) = \mathcal{O}_0(\|u\|_{L^1}). \tag{6.51}$$

Since $u(t) = \mathcal{O}_0(1)$, one has:

$$(H_0 + u H_1)(x - \hat{x}) = \mathcal{O}_0(|x - \hat{x}|). \tag{6.52}$$

From (6.49) and (6.50), one has:

$$Q_0(x - \hat{x}, x + \hat{x}) = \mathcal{O}_0(|x - \hat{x}|). \tag{6.53}$$

Since $f \in C^3(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$:

$$|f(x,u) - T_2 f(x,u)| = \mathcal{O}_0\big(|x|^3 + |u|^3\big). \tag{6.54}$$

From (6.54), (6.51) and Hölder's inequality, one has:

$$f(x,u) - T_2 f(x,u) = \mathcal{O}_0\Big(\|u\|_{L^1}^3 + |u|^3\Big) = \mathcal{O}_0\Big(\|u\|_{L^3}^3 + |u|^3\Big). \tag{6.55}$$

Plugging estimates (6.52), (6.53) and (6.55) into (6.48) and integrating yields:

$$x(t) - \hat{x}(t) = \mathcal{O}_0\left(\|u\|_{L^3}^3 + \int_0^t |x(s) - \hat{x}(s)| \mathrm{d}s\right). \tag{6.56}$$

Applying Grönwall's lemma to (6.56) provides the estimate:

$$x(t) - \hat{x}(t) = \mathcal{O}_0\Big(\|u\|_{L^3}^3\Big). \tag{6.57}$$

Let $\alpha \in \mathbb{R}^n$. We consider the transformed control $v := u - (\alpha, \hat{x})$. With this new control, the evolution equation (6.47) becomes $\dot{\hat{x}} = g(\hat{x}, v)$ where $g(\hat{x}, v) := T_2 f(\hat{x}, v + (\alpha, \hat{x}))$. Hence, applying Lemma 8 to this system yields:

$$\hat{x}(t) = \mathcal{O}_0(\|v\|_{L^1}) = \mathcal{O}_0(\|u - (\alpha, \hat{x})\|_{L^1}). \tag{6.58}$$

Combining (6.57) with (6.58), using the triangular inequality and Hölder's inequality gives:

$$x(t) - \hat{x}(t) = \mathcal{O}_0\Big(\|u - (\alpha, \hat{x})\|_{L^3}^3\Big). \tag{6.59}$$

Let $G : \mathcal{S}_1 \to \mathcal{S}_1^\perp$ be the smooth function associated with system (6.47) by Theorem 2. Since G' is bounded in a vicinity of zero, estimate (6.59) yields:

$$\mathbb{P}^\perp x(t) - G(\mathbb{P}x(t)) = \mathbb{P}^\perp \hat{x}(t) - G(\mathbb{P}\hat{x}(t)) + \mathcal{O}_0\Big(\|u - (\alpha, \hat{x})\|_{L^3}^3\Big). \tag{6.60}$$

Estimate (6.60) allows to transpose all the conclusions on the state $\hat{x}$ to conclusions on the state x since the remainder is exactly of the same size as those that have been managed in Section 5, provided that one chooses the α corresponding to the Brunovský transform. $\square$

Corollary 3. *Let $f_0 \in C^3(\mathbb{R}^n, \mathbb{R}^n)$ and $f_1 \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ with $f_0(0) = 0$. There exists $G \in C^2(\mathcal{S}_1, \mathcal{S}_1^\perp)$ with $G(0) = 0$ and $G'(0) = 0$ such that the conclusions of Theorem 3 hold.*

Proof. We consider a regularized system:

$$\dot{\hat{x}} = \hat{f}_0(\hat{x}) + u\hat{f}_1(\hat{x}), \tag{6.61}$$

where $\hat{f}_0 := T_2 f_0$ is the second-order Taylor expansion of f_0 at zero and $\hat{f}_1 := T_1 f_1$ is the first-order Taylor expansion of f_1 at zero. Using the notations introduced in Section 5, we consider the respective auxiliary systems ξ_1 and $\hat{\xi}_1$ associated with systems (1.3) and (6.61). One has:

$$\dot{\xi}_1 = f_0(\xi_1) + u_1 f_2(\xi_1) + u_1^2 G_1(u_1, \xi_1), \tag{6.62}$$

$$\dot{\hat{\xi}}_1 = \hat{f}_0(\hat{\xi}_1) + u_1 \hat{f}_2(\hat{\xi}_1) + u_1^2 \hat{G}_1(u_1, \hat{\xi}_1). \tag{6.63}$$

From Lemma 9 and estimate (4.24), we have $\xi_1 = \mathcal{O}_1(1)$ and $\hat{\xi}_1 = \mathcal{O}_1(1)$. Since $f_0 \in C^3(\mathbb{R}^n, \mathbb{R}^n)$ and $\hat{f}_0 = T_2 f_0$,

$$\begin{aligned} f_0(\xi_1) - \hat{f}_0(\hat{\xi}_1) &= \Big(f_0(\xi_1) - f_0(\hat{\xi}_1)\Big) + \Big(f_0(\hat{\xi}_1) - \hat{f}_0(\hat{\xi}_1)\Big) \\ &= \mathcal{O}_1\Big(|\xi_1 - \hat{\xi}_1|\Big) + \mathcal{O}_1\Big(|\hat{\xi}_1|^3\Big). \end{aligned} \tag{6.64}$$

Similarly, since $f_2 = -[f_0, f_1] \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and $\hat{f}_2 = -[\hat{f}_0, \hat{f}_1] = f_2 + O(x^2)$ one has:

$$\begin{aligned} f_2(\xi_1) - \hat{f}_2(\hat{\xi}_1) &= \Big(f_2(\xi_1) - f_2(\hat{\xi}_1)\Big) + \Big(f_2(\hat{\xi}_1) - \hat{f}_2(\hat{\xi}_1)\Big) \\ &= \mathcal{O}_1\Big(|\xi_1 - \hat{\xi}_1|\Big) + \mathcal{O}_1\Big(|\hat{\xi}_1|^2\Big). \end{aligned} \tag{6.65}$$

Last, one checks that $\hat{G}_1(0,0) = \frac{1}{2}[\hat{f}_1, \hat{f}_2](0) = \frac{1}{2}[f_1, f_2](0) = G_1(0,0)$. Moreover, we have $G_1 \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Indeed, since f_1 is C^2, ϕ_1 is C^3 with respect to τ and C^2 with respect to p, thus F_1 is C^3 with respect to τ and C^1 with respect to p. Thus G_1, which is obtained from F_1 through a Taylor formula with integral remainder, is at least C^1 in (τ, p). Hence:

$$G_1(u_1, \xi_1) - \hat{G}_1(u_1, \hat{\xi}_1) = \mathcal{O}_1\Big(|u_1| + |\xi_1| + |\hat{\xi}_1|\Big). \tag{6.66}$$

From Lemma 9 and estimate (4.24), we also have $\xi_1 = \mathcal{O}_1(\|u_1\|_{L^1})$ and $\hat{\xi}_1 = \mathcal{O}_1(\|u_1\|_{L^1})$. Therefore, plugging estimates (6.64), (6.65) and (6.66) into the difference of (6.63) with (6.62), integrating in time and using Hölder's inequality yields:

$$\xi_1(t) - \hat{\xi}_1(t) = \mathcal{O}_1\left(\|u_1\|_{L^3}^3\right) + \int_0^t \mathcal{O}_1\left(|\xi_1(s) - \hat{\xi}_1(s)|\right)\mathrm{d}s. \tag{6.67}$$

Applying Grönwall's lemma to (6.67) gives:

$$r(t) := \xi_1(t) - \hat{\xi}_1(t) = \mathcal{O}_1\left(\|u_1\|_{L^3}^3\right). \tag{6.68}$$

We denote by ϕ_1 and $\hat{\phi}_1$, the flows associated with f_1 and $\hat{f}_1$. Hence:

$$x = \phi_1(u_1, \xi_1) = \phi_1(u_1, \hat{\xi}_1 + r) = \phi_1(u_1, \hat{\phi}_1(-u_1, \hat{x}) + r) = \Phi(u_1, \hat{x}) + \mathcal{O}_1(|r|), \tag{6.69}$$

thanks to the C^1 regularity of ϕ_1, where we introduce the map:

$$\Phi(\tau, p) := \phi_1\left(\tau, \hat{\phi}_1(-\tau, p)\right). \tag{6.70}$$

Differentiating (6.70) and using the shorthand notation $\hat{p}_1 := \hat{\phi}_1(-\tau, p)$ yields:

$$\begin{aligned} \partial_\tau \Phi(\tau, p) &= \partial_\tau \phi_1\left(\tau, \hat{p}_1\right) - \partial_p \phi_1\left(\tau, \hat{p}_1\right) \cdot \partial_\tau \hat{\phi}_1(-\tau, p) \\ &= f_1(\phi_1(\tau, \hat{p}_1)) - \partial_p \phi_1\left(\tau, \hat{p}_1\right) \cdot \hat{f}_1(\hat{p}_1) \\ &= \Psi(\tau, \hat{p}_1) - \partial_p \phi_1(\tau, \hat{p}_1) \cdot \left(\hat{f}_1(\hat{p}_1) - f_1(\hat{p}_1)\right), \end{aligned} \tag{6.71}$$

where we introduced:

$$\Psi(\tau, p) := f_1(\phi_1(\tau, p)) - \partial_p \phi_1\left(\tau, p\right) \cdot f_1(p). \tag{6.72}$$

One checks that $\Psi(0, p) = 0$. Moreover, using Schwarz's theorem, we obtain:

$$\partial_\tau \Psi(\tau, p) = f_1'(\phi_1(\tau, p)) \cdot f_1(\phi_1(\tau, p)) - \partial_{\tau p} \phi_1(\tau, p) \cdot f_1(p) = f_1'(\phi_1(\tau, p)) \cdot \Psi(\tau, p). \tag{6.73}$$

From (6.73), we deduce that $\Psi(\tau, p) = 0$. Hence, (6.71) yields:

$$\partial_\tau \Phi(u_1, \hat{x}) = \mathcal{O}_1\left(|\hat{\xi}_1|^2\right). \tag{6.74}$$

For $p \in \mathbb{R}^n$, let us denote by $\mathbb{P}_0(p)$ the component of $\mathbb{P}p$ along $b_0 = b = f_1(0)$ in the basis of $\mathcal{S}_1$ made up of $(b_0, \dots b_{d-1})$ (see (3.22)). Considering the function $\tau \mapsto \mathbb{P}_0(\hat{\phi}_1(\tau, 0))$, thanks to the local inversion theorem, there exists a smooth function $\beta_0 : \mathbb{R} \to \mathbb{R}$ such that, if $p = \hat{\phi}_1(\tau, 0)$ then $\tau = \beta_0(\mathbb{P}_0(p))$ for τ and p small enough. Hence, since $\hat{x} = \hat{\phi}_1(u_1, \hat{\xi}_1)$, the C^1 regularity of $\hat{\phi}_1$ yields:

$$u_1 = \beta_0(\mathbb{P}_0 \hat{x}) + \mathcal{O}_1\left(|\hat{\xi}_1|\right). \tag{6.75}$$

Gathering (6.68), (6.69), (6.74) and (6.75) yields:

$$x = \Phi(\beta_0(\mathbb{P}_0 \hat{x}), \hat{x}) + \mathcal{O}_1\left(\|u_1\|_{L^3}^3\right). \tag{6.76}$$

The cubic remainder is of the same size as those that are absorbed during the usual proof (in fact, during the proof, one absorbs such remainders on a transformed control $v = u - (\alpha, \hat{x})$ but we have already seen that the involved norms on u_1 and v_1 can be interchanged, as in (6.32) and (6.33)). Thus, we only need to study the first part of this equation and prove that it defines a manifold. We look at:

$$x = \Phi(\beta_0(\mathbb{P}_0\hat{x}), \mathbb{P}\hat{x} + \hat{G}(\mathbb{P}\hat{x})). \tag{6.77}$$

Once again, the local inversion theorem tells us that we can express $\mathbb{P}\hat{x}$ as a C^2 function $\rho(\mathbb{P}x)$. Thus, we set:

$$G(p_\parallel) := \mathbb{P}^\perp \Phi(\beta_0(\mathbb{P}_0\rho(p_\parallel)), \rho(p_\parallel) + \hat{G}(\rho(p_\parallel)). \tag{6.78}$$

This concludes the proof of the theorem for less regular vector fields. □

7 Explicit approximation of the invariant manifold

We construct an explicit quadratic approximation of the invariant manifold involved in our quadratic alternative theorems. We also exhibit a second-order approximation of the differential systems that stays exactly within this manifold when $\mathcal{S}_2 = \mathcal{S}_1$.

We continue to denote by $1 \leqslant d < n$ the dimension of $\mathcal{S}_1$. Using the notations introduced in Subsection 3.2, (3.22), (3.26) and (3.27), we define:

$$\mathcal{M}_2 := \left\{ \sum_{0\leqslant i<d} \alpha_i b_i + \mathbb{P}^\perp \left(\frac{1}{2} \sum_{0\leqslant i<d} \alpha_i^2 L_i b_i + \sum_{0\leqslant i<j<d} \alpha_i \alpha_j L_i b_j \right), \ \alpha \in \mathbb{R}^d \right\}. \tag{7.1}$$

Since $b_0, \dots b_{d-1}$ are d independent vectors which span $\mathcal{S}_1$, equation (7.1) defines a global smooth manifold of $\mathbb{R}^n$ of dimension d. Indeed, it is defined as the image of a smooth injective function, with smooth inverse. We recover from (7.1) that the tangent subspace to $\mathcal{M}_2$ at the origin is $\mathcal{S}_1$. For $x \in \mathbb{R}^n$, and $0 \leqslant k \leqslant d-1$, we introduce the notation $\mathbb{P}_k(x)$ to denote the component of $\mathbb{P}x$ along b_k in the basis $(b_0, \dots b_{d-1})$. Formally, one could also define $\mathcal{M}_2$ as the set of $x \in \mathbb{R}^n$ such that:

$$\mathbb{P}^\perp x = \frac{1}{2} \sum_{0\leqslant i<d} \mathbb{P}_i(x)^2 \mathbb{P}^\perp(L_i b_i) + \sum_{0\leqslant i<j<d} \mathbb{P}_i(x)\mathbb{P}_j(x)\mathbb{P}^\perp(L_i b_j). \tag{7.2}$$

7.1 Local expansion of the invariant manifold

Proposition 7. *The manifold $\mathcal{M}_2$ defined by (7.1) is a local approximation of second-order of the manifold $\mathcal{M}$ involved in the main quadratic alternative theorems around the origin.*

Proof. The manifold $\mathcal{M}$ is defined in (2.4). First, both manifolds contain the origin and admit $\mathcal{S}_1$ as their tangent subspace at the origin. Here, we check that the second-order terms contained in $\mathcal{S}_1^\perp$ are the same. Since both definitions are given as graphs, we need to check that the second-order Taylor expansions match. From (5.36), we have $G(p_\parallel) = \Theta(p_\parallel, 0)$. From (5.32) and (5.34), we have:

$$\forall p_\parallel \in \mathcal{S}_1, \quad F(p_\parallel, G(p_\parallel)) = 0. \tag{7.3}$$

Differentiating (7.3) twice with respect to $p_\parallel$ yields:

$$\partial_\parallel^2 F + 2\partial_\perp \partial_\parallel F \cdot G' + \partial_\perp^2 F \cdot G'G' + \partial_\perp F \cdot G'' = 0. \tag{7.4}$$

At the origin, $G = 0$, $G' = 0$ and $\partial_\perp F = \mathrm{Id}$ from (5.35). Hence, (7.4) yields:

$$G''(0) = -\partial_\parallel^2 F(0,0). \tag{7.5}$$

We must compute a Taylor approximation of F with respect to $p_\parallel$. Let ϕ_j be defined by (4.3). Using a Taylor expansion with respect to time, one has:

$$\phi_j(\tau, p) =_{[3]} \phi_j(0,p) + \tau \partial_\tau \phi_j(0,p) + \frac{1}{2}\tau^2 \partial_\tau^2 \phi_j(0,p). \tag{7.6}$$

Using (4.3) and (7.6), one has:

$$\phi_j(\tau, p) =_{[3]} p + \tau \left(f_j(0) + f_j'(0)p\right) + \frac{1}{2}\tau^2 f_j'(0) f_j(0). \tag{7.7}$$

Let $\alpha_1, \dots \alpha_d \in \mathbb{R}^d$. Iterated application of (7.7) yields:

$$\begin{aligned} \phi_d^{-\alpha_d} \circ \dots \circ \phi_1^{-\alpha_1}(p) =_{[3]} p - \sum_{i=1}^d \alpha_i f_i(0) + \frac{1}{2}\sum_{i=1}^d \alpha_i^2 f_i'(0) f_i(0) \\ - \sum_{i=1}^d \alpha_i f_i'(0) \left(p - \sum_{j=1}^{i-1} \alpha_j f_j(0)\right). \end{aligned} \tag{7.8}$$

Using (5.18) and (5.20), one obtains, for $p_\parallel \in \mathcal{S}_1$, since $\alpha_i(0) = 0$:

$$p_\parallel =_{[2]} \alpha_1(p_\parallel) f_1(0) + \ldots + \alpha_d(p_\parallel) f_d(0). \tag{7.9}$$

Plugging (7.8) and (7.9) into definition (5.21) yields:

$$\begin{aligned} F(p_\parallel, 0) &=_{[3]} \frac{1}{2}\sum_{i=1}^d \alpha_i^2(p_\parallel) \mathbb{P}^\perp(f_i'(0) f_i(0)) - \sum_{i=1}^d \alpha_i(p_\parallel) \sum_{j=i}^d \alpha_j(p_\parallel) \mathbb{P}^\perp(f_i'(0) f_j(0)) \\ &=_{[3]} -\frac{1}{2}\sum_{i=1}^d \alpha_i^2(p_\parallel) \mathbb{P}^\perp(f_i'(0) f_i(0)) - \sum_{1 \leqslant i < j \leqslant d} \alpha_i(p_\parallel)\alpha_j(p_\parallel) \mathbb{P}^\perp(f_i'(0) f_j(0)). \end{aligned} \tag{7.10}$$

For $1 \leqslant k \leqslant d$, recalling the definition of f_k (see (4.2)) and using Lemma 4, one has $f_k(0) = (-1)^{k-1} b_{k-1}$ and $f_k'(0) = (-1)^{k-1} L_{k-1}$. From (7.5) and (7.10), we have:

$$G(p_\parallel) =_{[3]} \frac{1}{2}\sum_{i=0}^{d-1} \mathbb{P}_i(p_\parallel)^2 \mathbb{P}^\perp(L_i b_i) + \sum_{0 \leqslant i < j < d} \mathbb{P}_i(p_\parallel) \mathbb{P}_j(p_\parallel) \mathbb{P}^\perp(L_i b_j). \tag{7.11}$$

Since (7.11) matches (7.2), it concludes the proof of Proposition 7. ☐

7.2 Construction of an homogeneous second-order system

We construct an homogeneous second-order system that provides a good approximation of any differential system and stays exactly within $\mathcal{M}_2$, under the assumption that $\mathcal{S}_2 = \mathcal{S}_1$. Approximating the behavior of nonlinear systems using homogeneous (with respect to amplitude dilatations) approximations has already been used in various contexts: for small-time local controllability (around an equilibrium in [38] and around a trajectory in [8]), for large-time local controllability in [20], for stabilization in [19] and the construction of Lyapunov functions in [32].

As proposed in Section 5, given some control-affine or nonlinear differential system, we decompose the state x as $y + z + r$, where y denotes the linear part, z the quadratic part and r a remainder (which is thus at least cubic in the control). Thanks to the linear theory, we know that y lives in $\mathcal{S}_1$. Hence, an homogeneous approximation of x up to the second order is the quantity:

$$\zeta := \mathbb{P}y + \mathbb{P}^\perp z. \tag{7.12}$$

In (7.12), we write $\mathbb{P}y$ instead of y to highlight the orthogonality with respect to the second term $\mathbb{P}^\perp z$. Using the notations introduced in (2.3), (3.20), (3.24) and (3.25), we recall that:

$$\dot{y} = H_0 y + ub, \tag{7.13}$$

$$\dot{z} = H_0 z + uH_1 y + Q_0(y,y) + \frac{1}{2}u^2 d_0. \tag{7.14}$$

Since $\mathcal{S}_1$ is stable under multiplication by H_0, we have the relations:

$$\mathbb{P}H_0\mathbb{P} = H_0\mathbb{P}, \tag{7.15}$$

$$\mathbb{P}^\perp H_0 \mathbb{P}^\perp = \mathbb{P}^\perp H_0. \tag{7.16}$$

Applying $\mathbb{P}$ to (7.13) and using (7.15), then $\mathbb{P}^\perp$ to (7.14) and using (7.16), one obtains:

$$\mathbb{P}\dot{\zeta} = H_0\mathbb{P}\zeta + u\mathbb{P}b, \tag{7.17}$$

$$\mathbb{P}^\perp\dot{\zeta} = \mathbb{P}^\perp H_0\zeta + u\mathbb{P}^\perp H_1\mathbb{P}\zeta + \mathbb{P}^\perp Q_0(\mathbb{P}\zeta,\mathbb{P}\zeta) + \frac{1}{2}u^2\mathbb{P}^\perp d_0. \tag{7.18}$$

Combining (7.17) and (7.18) leads to the following ODE for ζ:

$$\dot{\zeta} = g_0(\zeta) + ug_1(\zeta) + \frac{1}{2}u^2\mathbb{P}^\perp d_0, \tag{7.19}$$

where:

$$g_0(\zeta) := (H_0\mathbb{P} + \mathbb{P}^\perp H_0)\zeta + \mathbb{P}^\perp Q_0(\mathbb{P}\zeta,\mathbb{P}\zeta), \tag{7.20}$$

$$g_1(\zeta) := b + \mathbb{P}^\perp H_1 \mathbb{P}\zeta. \tag{7.21}$$

We prove in the following lemma that system (7.19) exhibits nice properties concerning the Lie brackets of g_0 and g_1 since they are fully explicit.

Lemma 17. *Let $j,k \in \mathbb{N}$. For any $\zeta \in \mathbb{R}^n$, we have:*

$$\mathrm{ad}^k_{g_0}(g_1)(\zeta) = b_k + \mathbb{P}^\perp L_k \mathbb{P}\zeta, \tag{7.22}$$

$$\left[\mathrm{ad}^k_{g_0}(g_1), \mathrm{ad}^j_{g_0}(g_1)\right](\zeta) = \mathbb{P}^\perp(L_j b_k - L_k b_j). \tag{7.23}$$

Proof. We proceed by induction on $k \in \mathbb{N}$. From (3.22), (3.26) and (7.21), (7.22) holds for $k = 0$. Let $k \in \mathbb{N}$ be such that (7.22) holds. Using (7.20), we compute the next bracket:

$$\begin{aligned} \mathrm{ad}^{k+1}_{g_0}(g_1)(\zeta) &= \left[g_0, \mathrm{ad}^k_{g_0}(g_1)\right](\zeta) \\ &= (\mathbb{P}^\perp L_k \mathbb{P})\left(H_0\mathbb{P}\zeta + \mathbb{P}^\perp H_0\zeta + \mathbb{P}^\perp Q_0(\mathbb{P}\zeta,\mathbb{P}\zeta)\right) \\ &\quad - H_0\mathbb{P}\left(b_k + \mathbb{P}^\perp L_k\mathbb{P}\zeta\right) - \mathbb{P}^\perp H_0\left(b_k + \mathbb{P}^\perp L_k \mathbb{P}\zeta\right) \\ &\quad - 2\mathbb{P}^\perp Q_0(b_k,\mathbb{P}\zeta) - 2\mathbb{P}^\perp Q_0(\mathbb{P}\mathbb{P}^\perp L_k\mathbb{P}\zeta,\mathbb{P}\zeta). \end{aligned} \tag{7.24}$$

Thanks to (7.15), (7.16) and the relation $\mathbb{P}\mathbb{P}^\perp = 0$, we deduce from (7.24) that:

$$\mathrm{ad}^{k+1}_{g_0}(g_1)(\zeta) = b_{k+1} + \mathbb{P}^\perp\left(L_k H_0 - H_0 L_k - 2Q_0(b_k,\cdot)\right)\mathbb{P}\zeta. \tag{7.25}$$

The conclusion follows from (7.25) because we obtain the same recursion relation as in (3.27). Thus (7.22) holds for any $k \in \mathbb{N}$. Using (7.22) and $\mathbb{P}\mathbb{P}^\perp = 0$, we compute:

$$\begin{aligned} \left[\mathrm{ad}^k_{g_0}(g_1), \mathrm{ad}^j_{g_0}(g_1)\right](\zeta) &= \mathbb{P}^\perp L_j \mathbb{P}\left(b_k + \mathbb{P}^\perp L_k \mathbb{P}\zeta\right) - \mathbb{P}^\perp L_k\mathbb{P}\left(b_j + \mathbb{P}^\perp L_j \mathbb{P}\zeta\right) \\ &= \mathbb{P}^\perp(L_j b_k - L_k b_j). \end{aligned} \tag{7.26}$$

Hence, (7.26) concludes the proof of (7.23). ☐

Lemma 18. *Assume that $\mathcal{S}_2 + \mathbb{R}d_0 = \mathcal{S}_1$. Then, there exists a smooth manifold $\mathfrak{M} \subset \mathbb{R}^n$ of dimension $d := \dim \mathcal{S}_1$, such that any trajectory of system (7.19) with $\zeta(0) = 0$ satisfies $\zeta(t) \in \mathfrak{M}$ for any $t \geqslant 0$.*

Proof. We start by proving that, for any $\zeta \in \mathbb{R}^n$:

$$\text{Lie } \{g_0, g_1\}(\zeta) = \text{Span } \left\{ b_k + \mathbb{P}^\perp L_k \mathbb{P}\zeta, \;\; 0 \leqslant k < d \right\}. \tag{7.27}$$

We use Lemma 17. From (3.32) and (7.23), we obtain:

$$\begin{aligned} \left[\text{ad}^k_{g_0}(g_1), \text{ad}^j_{g_0}(g_1)\right](\zeta) &= \mathbb{P}^\perp (L_j b_k - L_k b_j) \\ &= \mathbb{P}^\perp \left[\text{ad}^k_{f_0}(f_1), \text{ad}^j_{f_0}(f_1)\right](0) \\ &= 0. \end{aligned} \tag{7.28}$$

Hence, from (7.28), all brackets containing g_1 at least two times vanish identically. Thus, the space Lie $\{g_0, g_1\}(\zeta)$ is spanned by brackets containing g_1 exactly once. Using (7.22) and $\mathbb{P}^\perp L_j b_k = \mathbb{P}^\perp L_k b_j$, we compute:

$$\begin{aligned} \text{ad}^j_{g_0}(g_1)(\zeta) &= b_j + \mathbb{P}^\perp L_j \mathbb{P}\zeta \\ &= b_j + \mathbb{P}^\perp L_j \left(\sum_{l=0}^{d-1} \mathbb{P}_l(\zeta) b_l \right) \\ &= b_j + \sum_{l=0}^{d-1} \mathbb{P}_l(\zeta) \mathbb{P}^\perp L_l b_j \\ &= b_j + \sum_{k=0}^{d-1} \mathbb{P}_k(b_j) \sum_{l=0}^{d-1} \mathbb{P}_l(\zeta) \mathbb{P}^\perp L_l b_k \\ &= \sum_{k=0}^{d-1} \mathbb{P}_k(b_j) b_k + \sum_{k=0}^{d-1} \mathbb{P}_k(b_j) \mathbb{P}^\perp L_k \left(\sum_{l=0}^{d-1} \mathbb{P}_l(\zeta) b_l \right) \\ &= \sum_{k=0}^{d-1} \mathbb{P}_k(b_j) \left(b_k + \mathbb{P}^\perp L_k \mathbb{P}\zeta \right). \end{aligned} \tag{7.29}$$

Equation (7.29) proves that the d first brackets span Lie $\{g_0, g_1\}(\zeta)$ and thus (7.27) holds. From the definition of $\mathcal{S}_1$ and $d = \dim \mathcal{S}_1$, the family $b_0, \dots b_{d-1}$ is free. Thus, from (7.27), we have that, for any $\zeta \in \mathbb{R}^n$, the dimension of Lie $\{g_0, g_1\}(\zeta)$ is exactly d. Hence, from the Frobenius theorem (as stated in [13, Corollary 3.26]), the state $\zeta(t)$ must evolve within a manifold of $\mathbb{R}^n$ of dimension d (see also [21, Theorem 4, page 45] or [41, Theorem 2.20, page 48] for proofs of this geometric result). □

7.3 Exact evolution within the quadratic manifold

We prove that Lemma 18 actually holds with $\mathfrak{M} = \mathcal{M}_2$ defined in (7.1). Using the assumption $\mathcal{S}_2 = \mathcal{S}_1$ and (3.32), one has $\mathbb{P}^\perp L_j b_k = \mathbb{P}^\perp L_k b_j$. Hence (7.1) can be rewritten in a more symmetric way as:

$$\mathcal{M}_2 = \left\{ \sum_{k=0}^{d-1} \alpha_k b_k + \frac{1}{2} \sum_{j=0}^{d-1} \sum_{k=0}^{d-1} \alpha_j \alpha_k \mathbb{P}^\perp L_j b_k, \;\; (\alpha_0, \dots \alpha_{d-1}) \in \mathbb{R}^d \right\}. \tag{7.30}$$

Equivalently, it corresponds to the set of $x \in \mathbb{R}^n$ for which the following vector-valued second-order polynomial vanishes:

$$Q(x) := \mathbb{P}^\perp x - \frac{1}{2}\sum_{j=0}^{d-1}\sum_{k=0}^{d-1}\mathbb{P}_j(x)\mathbb{P}_k(x)\mathbb{P}^\perp L_j b_k. \tag{7.31}$$

We compute the evolution of Q along trajectories $t \mapsto \zeta(t)$ by differentiating (7.31):

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(\zeta(t)) = \mathbb{P}^\perp\dot{\zeta} - \frac{1}{2}\sum_{j=0}^{d-1}\sum_{k=0}^{d-1}\mathbb{P}_j(\dot{\zeta})\mathbb{P}_k(\zeta)\mathbb{P}^\perp L_j b_k - \frac{1}{2}\sum_{j=0}^{d-1}\sum_{k=0}^{d-1}\mathbb{P}_j(\zeta)\mathbb{P}_k(\dot{\zeta})\mathbb{P}^\perp L_j b_k. \tag{7.32}$$

Recalling that $\mathbb{P}^\perp L_j b_k = \mathbb{P}^\perp L_k b_j$ and using (7.17) and (7.18), we have from (7.32):

$$\begin{aligned}\frac{\mathrm{d}Q}{\mathrm{d}t} &= \mathbb{P}^\perp H_0\mathbb{P}^\perp\zeta + u\mathbb{P}^\perp H_1\mathbb{P}\zeta + \mathbb{P}^\perp Q_0(\mathbb{P}\zeta,\mathbb{P}\zeta) \\ &\quad - \sum_{j=0}^{d-1}\sum_{k=0}^{d-1}\mathbb{P}_j(H_0\mathbb{P}\zeta + ub)\mathbb{P}_k(\zeta)\mathbb{P}^\perp L_j b_k.\end{aligned} \tag{7.33}$$

Since $b = b_0$ (see (3.22)) and $L_0 = H_1$ (see (3.26)), we have:

$$\sum_{j=0}^{d-1}\sum_{k=0}^{d-1}\mathbb{P}_j(ub)\mathbb{P}_k(\zeta)\mathbb{P}^\perp L_j b_k = u\sum_{k=0}^{d-1}\mathbb{P}_k(\zeta)\mathbb{P}^\perp L_0 b_k = u\mathbb{P}^\perp H_1\mathbb{P}\zeta. \tag{7.34}$$

From (7.33) and (7.34), one has:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \mathbb{P}^\perp H_0\mathbb{P}^\perp\zeta + \mathbb{P}^\perp Q_0(\mathbb{P}\zeta,\mathbb{P}\zeta) - \sum_{j=0}^{d-1}\sum_{k=0}^{d-1}\mathbb{P}_j(H_0\mathbb{P}\zeta)\mathbb{P}_k(\zeta)\mathbb{P}^\perp L_j b_k. \tag{7.35}$$

From (3.27), (7.16) and (7.31), one has:

$$\begin{aligned}&\mathbb{P}^\perp H_0\mathbb{P}^\perp\zeta + \mathbb{P}^\perp Q_0(\mathbb{P}\zeta,\mathbb{P}\zeta) \\ &\qquad = \mathbb{P}^\perp H_0 Q + \frac{1}{2}\sum_{j=0}^{d-1}\mathbb{P}_j(\zeta)\mathbb{P}^\perp H_0\mathbb{P}^\perp L_j\mathbb{P}\zeta + \mathbb{P}^\perp Q_0(\mathbb{P}\zeta,\mathbb{P}\zeta) \\ &\qquad = \mathbb{P}^\perp H_0 Q + \frac{1}{2}\sum_{j=0}^{d-1}\mathbb{P}_j(\zeta)\mathbb{P}^\perp\left(H_0 L_j\mathbb{P}\zeta + 2Q_0(b_j,\mathbb{P}\zeta)\right) \\ &\qquad = \mathbb{P}^\perp H_0 Q - \frac{1}{2}\sum_{j=0}^{d-1}\mathbb{P}_j(\zeta)\mathbb{P}^\perp L_{j+1}\mathbb{P}\zeta + \frac{1}{2}\sum_{j=0}^{d-1}\mathbb{P}_j(\zeta)\mathbb{P}^\perp L_j H_0\mathbb{P}\zeta.\end{aligned} \tag{7.36}$$

The last term in (7.36) can be further simplified using (3.22). Indeed:

$$\begin{aligned}\frac{1}{2}\sum_{j=0}^{d-1}\mathbb{P}_j(\zeta)\mathbb{P}^\perp L_j H_0\mathbb{P}\zeta &= \frac{1}{2}\sum_{j=0}^{d-1}\sum_{k=0}^{d-1}\mathbb{P}_j(\zeta)\mathbb{P}_k(\zeta)\mathbb{P}^\perp L_j H_0 b_k \\ &= -\frac{1}{2}\sum_{j=0}^{d-1}\sum_{k=0}^{d-1}\mathbb{P}_j(\zeta)\mathbb{P}_k(\zeta)\mathbb{P}^\perp L_j b_{k+1} \\ &= -\frac{1}{2}\sum_{j=0}^{d-1}\sum_{k=0}^{d-1}\mathbb{P}_j(\zeta)\mathbb{P}_k(\zeta)\mathbb{P}^\perp L_{k+1} b_j \\ &= -\frac{1}{2}\sum_{k=0}^{d-1}\mathbb{P}_k(\zeta)\mathbb{P}^\perp L_{k+1}\mathbb{P}\zeta.\end{aligned} \tag{7.37}$$

Similarly, the last term of (7.35) can be rewritten:

$$
\begin{aligned}
-\sum_{j=0}^{d-1}\sum_{k=0}^{d-1}\mathbb{P}_j(H_0\mathbb{P}\zeta)\mathbb{P}_k(\zeta)\mathbb{P}^\perp L_j b_k &= -\sum_{j=0}^{d-1}\sum_{k=0}^{d-1}\sum_{l=0}^{d-1}\mathbb{P}_l(\zeta)\mathbb{P}_j(H_0 b_l)\mathbb{P}_k(\zeta)\mathbb{P}^\perp L_k b_j \\
&= -\sum_{k=0}^{d-1}\sum_{l=0}^{d-1}\mathbb{P}_l(\zeta)\mathbb{P}_k(\zeta)\mathbb{P}^\perp L_k H_0 b_l \\
&= \sum_{k=0}^{d-1}\sum_{l=0}^{d-1}\mathbb{P}_l(\zeta)\mathbb{P}_k(\zeta)\mathbb{P}^\perp L_k b_{l+1} \\
&= \sum_{k=0}^{d-1}\sum_{l=0}^{d-1}\mathbb{P}_l(\zeta)\mathbb{P}_k(\zeta)\mathbb{P}^\perp L_{l+1} b_k \\
&= \sum_{l=0}^{d-1}\mathbb{P}_l(\zeta)\mathbb{P}^\perp L_{l+1}\mathbb{P}\zeta.
\end{aligned} \tag{7.38}
$$

Hence, grouping (7.35), (7.36), (7.37) and (7.38), we conclude that:

$$
\frac{\mathrm{d}}{\mathrm{d}t} Q\left(\zeta(t)\right) = \mathbb{P}^\perp H_0 Q\left(\zeta(t)\right). \tag{7.39}
$$

Straight-forward integration of (7.39) yields, for any $t \geqslant 0$:

$$
Q\left(\zeta(t)\right) = e^{t\mathbb{P}^\perp H_0} Q(\zeta(0)). \tag{7.40}
$$

In particular, we deduce from (7.40) that the evolution of Q along trajectories of $\zeta(t)$ does not depend on the control. Moreover, when $\zeta(0) = 0$, $Q(\zeta(0)) = Q(0) = 0$ and this remains true for any positive time: the quadratic homogeneous model cannot leave the manifold. Hence Lemma 18 holds with $\mathfrak{M} = \mathcal{M}_2$.

Remark 3. *In this paper, we introduced two different "quadratic" approximations for a nonlinear system: $y + z$ and ζ. The decomposition $y + z$ used in Section 5 is quite classical but it does not behave as well as ζ. Indeed ζ provides an approximation which is homogeneous with respect to dilatations of order one in $\mathcal{S}_1$ and order two in $\mathcal{S}_1^\perp$ (while $y + z$ mixes first and second-order terms in $\mathcal{S}_1$). A clear indication that ζ behaves more nicely than $y + z$ is Lemma 18, since ζ lives exactly within a given manifold. This remark might hint towards introducing well-prepared homogeneous approximations instead of standard approximations to study local properties. Of course, the approximation $y + z$ is also relevant. It lives within $\mathcal{M}_2$ up to a cubic residual. Indeed, since $\zeta = y + \mathbb{P}^\perp z$ and using (7.31):*

$$
Q(y+z) = Q(\zeta) + \mathcal{O}_\gamma(|\zeta||Pz|) = \mathcal{O}_\gamma\left(\|u_\gamma\|_{L^\infty}^3\right). \tag{7.41}
$$

7.4 Examples of approximate invariant manifolds

We consider variations around the toy system exposed in Example 4 to illustrate the difference between $\mathcal{M}$ and its approximation $\mathcal{M}_2$. When the system is already second-order homogeneous, we have $\mathcal{M}_2 = \mathcal{M}$. However, this is not always the case.

Example 15 (Higher-order terms)**.** *We consider the following variation:*

$$
\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = 2ux_1 + 3ux_1^2. \end{cases} \tag{7.42}
$$

System (7.42) *also satisfies* $\mathcal{S}_1 = \mathbb{R}e_1$ *and* $\mathcal{S}_2 = \mathcal{S}_1$. *However, the invariant manifold* $\mathcal{M}$ *defined in* (2.4) *takes into account the cubic term and is given by:*

$$\mathcal{M} = \left\{(x_1, x_2) \in \mathbb{R}^2,\ \ x_2 = x_1^2 + x_1^3\right\}. \tag{7.43}$$

Here, $\mathcal{M}_2$ *is only a local approximation of* (7.43) *of second-order (see Figure 3, left plot).*

In Example 15, the difference between $\mathcal{M}$ and $\mathcal{M}_2$ is due to cubic terms. It is of also possible to build systems for which $f(x,u)$ is a polynomial of degree two but $\mathcal{M} \neq \mathcal{M}_2$. Indeed, the homogeneity with respect to dilatations for a system is not constrained by the degree of the polynomials defining the dynamics.

Example 16 (Non-polynomial invariant manifold)**.** *We consider the following variation*

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = 2ux_1 + ux_2. \end{cases} \tag{7.44}$$

System (7.44) *also satisfies* $\mathcal{S}_1 = \mathbb{R}e_1$ *and* $\mathcal{S}_2 = \mathcal{S}_1$. *The invariant manifold is given by:*

$$\mathcal{M} = \left\{(x_1, x_2) \in \mathbb{R}^n,\ \ x_2 = 2\left(e^{x_1} - 1 - x_1\right)\right\}. \tag{7.45}$$

One checks that $\mathcal{M}_2$ *is indeed the local approximation of* (7.45) *because of the Taylor expansion* $2(e^{x_1} - 1 - x_1) = x_1^2 + O(x_1^3)$ *for* $|x_1| \leqslant 1$ *(see Figure 3, right plot).*

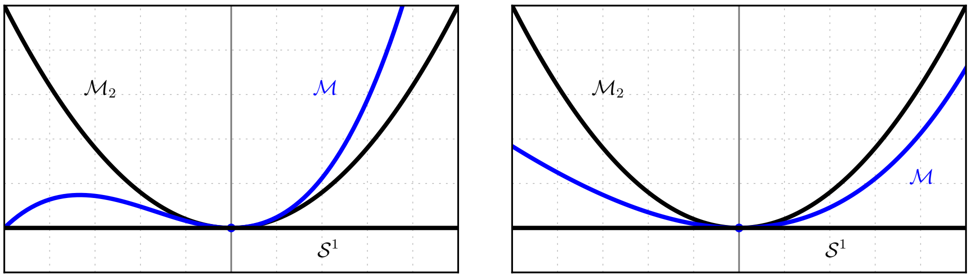

Figure 3: Influence of left-out terms on the invariant manifold.

8 On other notions of small-time local controllability

As sketched in the introduction, multiple definitions of small-time local controllability can be found in the literature. In this work, we chose to put the focus on the smallness assumption concerning the control, because we think that it is the easiest way to highlight the links between the functional setting and the geometric properties of the Lie brackets. However, other choices are possible; we explore one and explain how it relates to our definition.

8.1 Small-state small-time local controllability

Definition 11. *We say that a differential system is* small-state small-time locally controllable *when, for any* $T > 0$, *for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that, for any* $x^*, x^\dagger \in \mathbb{R}^n$ *with* $|x^*| + |x^\dagger| \leqslant \delta$, *there exists a trajectory* $(x,u) \in C^0([0,T],\mathbb{R}^n) \times L^q((0,T),\mathbb{R})$ *such that* $x(0) = x^*$, $x(T) = x^\dagger$ *and:*

$$\forall t \in [0,T], \quad |x(t)| \leqslant \varepsilon. \tag{8.1}$$

In particular, the control is a priori *allowed to be large in* $L^q((0,T),\mathbb{R})$ *(with* $q=\infty$ *for nonlinear systems and* $q=1$ *for control-affine systems).*

Here again, other choices would be possible: Definition 11 is linked to the L^∞-norm of the state along the trajectory, but one could also consider stronger norms.

8.2 Relations between state-smallness and control-smallness

Small-state small-time local controllability can be linked to the notions studied in this work.

Lemma 19. *If a differential system is* L^∞ *small-time locally controllable (for Definition 4), then it is small-state small-time locally controllable (for Definition 11).*

Proof. Let $f\in C^\infty(\mathbb{R}^n\times\mathbb{R},\mathbb{R}^n)$ with $f(0,0)=0$. Let $T>0$ and $\varepsilon>0$. We define $M:=\sup\{|f'(p,\tau)|,\ \ p\in\bar{B}(0,\varepsilon),\tau\in[-\varepsilon,\varepsilon]\}$, $C:=(1+MT)Te^{MT}$ and $\sigma:=\varepsilon/(2C)$. Moreover, considering $(x,u)\in C^0([0,T],\mathbb{R}^n)\times L^\infty((0,T),\mathbb{R})$ with $|x(0)|\leqslant\sigma$ and $\|u\|_{L^\infty}\leqslant\sigma$, we have, as long as $x(t)\in\bar{B}(0,\varepsilon)$:

$$\begin{aligned}|x(t)|&\leqslant|x(0)|+\left|\int_0^t f(x(s),u(s))\mathrm{d}s\right|\\&\leqslant|x(0)|+Mt\,\|u\|_{L^\infty}+M\int_0^t|x(s)|\mathrm{d}s.\end{aligned}\tag{8.2}$$

Applying Grönwall's lemma to (8.2) yields:

$$|x(t)|\leqslant e^{MT}(T|x(0)|+MT^2\,\|u\|_{L^\infty}).\tag{8.3}$$

Using (8.3) and the definition of C leads to:

$$|x(t)|\leqslant C(|x(0)|+\|u\|_{L^\infty})\leqslant\varepsilon.\tag{8.4}$$

Since we assumed that the system is L^∞ small-time locally controllable, from Definition 4, there exists $\delta_\sigma>0$ such that, for any $x^*,x^\dagger\in\mathbb{R}$ with $|x^*|+|x^\dagger|\leqslant\delta_\sigma$, there exists a trajectory from x^* to $x^\dagger$ with a control smaller than σ. Therefore, thanks to (8.4), small-state small-time locally controllable if we set $\delta:=\min(\sigma,\delta_\sigma)$. ☐

For control-affine systems, a more precise result can be obtained.

Lemma 20. *A control-affine system is* $W^{-1,\infty}$ *small-time locally controllable (for Definition 4) if and only if it is small-state small-time locally controllable (for Definition 11).*

Proof. Let $f_0,f_1\in C^\infty(\mathbb{R}^n,\mathbb{R}^n)$ with $f_0(0)=0$. The forward implication is proved as above, using an enhanced estimate as in Lemma 9, obtained via the use of the first auxiliary system. We turn to the reverse implication.

Heuristic. The key argument is that $\|u_1\|_{L^\infty}$ can be estimated from $\|x\|_{L^\infty}$. Indeed, one has:

$$x=x(0)+u_1f_1(0)+\text{lower order terms}.\tag{8.5}$$

Thus we can hope to recover u_1 from the knowledge of the state. The lower order terms in (8.5) are easily estimated when it is known that u_1 is small. Here, this is not *a priori* the case. Hence, we need to invert relation (8.5) so that the lower order terms can be estimated from the state and not from the control.

Construction of an input-to-state map. Let $T > 0$ be fixed. For $p \in \mathbb{R}^n$, we denote as previously by $\mathbb{P}_0 p$ the component of $\mathbb{P}p$ along b_0 in the basis $(b_0, \dots b_{d-1})$ of $\mathcal{S}_1$. We introduce the spaces:

$$A := \{(x^*, v) \in \mathbb{R}^n \times C^0([0,T],\mathbb{R});\ \ v(0) = 0\}, \tag{8.6}$$

$$B := \{(x^*, \pi) \in \mathbb{R}^n \times C^0([0,T],\mathbb{R});\ \ \pi(0) = \mathbb{P}_0 x^*\} \tag{8.7}$$

and the following input-to-state map:

$$\mathcal{F} : \begin{cases} A \to B, \\ (x^*, v) \mapsto (x^*, \mathbb{P}_0 \phi_1(v, \xi_1)), \end{cases} \tag{8.8}$$

where ϕ_1 is defined by (4.3) and ξ_1 is the solution to:

$$\dot{\xi}_1 = f_0(\xi_1) + v f_2(\xi_1) + v^2 G_1(v, \xi_1), \tag{8.9}$$

with initial data $\xi_1(0) = x^*$. One has $\mathcal{F}(0,0) = (0,0)$. Straightforward Grönwall estimates prove that $\mathcal{F}$ is well-defined and C^1 on a small neighborhood of $(0,0)$ in A.

Local inversion at zero. For $(x^*, v) \in A$, we compute:

$$\mathcal{F}'(0,0) \cdot (x^*, v) = (x^*, \mathbb{P}_0 (\partial_\tau \phi_1(0,0) v + \partial_p \phi_1(0,0) y_1)), \tag{8.10}$$

where y_1 is the solution to $y_1(0) = x^*$ and:

$$\dot{y}_1 = H_0 y_1 + v H_0 b. \tag{8.11}$$

Since $\mathbb{P}_0 \partial_\tau \phi_1(0,0) = \mathbb{P}_0 f_1(0) = 1$ and $\partial_p \phi_1(0,0) = \mathrm{Id}$, equation (8.10) yields:

$$\mathcal{F}'(0,0) \cdot (x^*, v) = (x^*, v + \mathbb{P}_0 y_1), \tag{8.12}$$

From (8.11), one obtains:

$$\mathbb{P}_0 y_1(t) = \mathbb{P}_0(e^{tH_0} x^*) + \int_0^t v(s) \mathbb{P}_0 \left(e^{(t-s)H_0} H_0 b \right) \mathrm{d}s. \tag{8.13}$$

Let $(x^*, \pi) \in B$. Solving $\mathcal{F}'(0,0) \cdot (x^*, v) = (x^*, \pi)$ amounts to finding a $v \in C^0([0,T],\mathbb{R})$ with $v(0) = 0$ such that:

$$v(t) + \int_0^t v(s) \mathbb{P}_0 \left(e^{(t-s)H_0} H_0 b \right) \mathrm{d}s = \pi(t) - \mathbb{P}_0(e^{tH_0} x^*) =: h(t), \tag{8.14}$$

where $h(0) = 0$. We are faced with a linear Volterra integral equation of second-kind, with a smooth kernel. We refer to [10, Section 1.2] for an introduction on this topic. Classical theory for such problems (see e.g. [10, Theorem 1.2.3]) yields the existence of a continuous resolvent kernel K such that (8.14) is equivalent to:

$$v(t) = h(t) + \int_0^t K(t,s) h(s) \mathrm{d}s. \tag{8.15}$$

Hence $\mathcal{F}'(0,0)$ is invertible. The inverse function theorem then allows us to conclude that there exists $C, \delta_A, \delta_B > 0$ such that, for any $(x^*, v) \in A$, if $|x^*| \leqslant \delta_A$ and $\|v\|_{C^0} \leqslant \delta_A$ and $\mathcal{F}(x^*, v) = (x^*, \pi)$ with $\|\pi\|_{C^0} \leqslant \delta_B$, then:

$$\|v\|_{L^\infty([0,T])} \leqslant \frac{C}{2} \left(|x^*| + \|\pi\|_{L^\infty([0,T])} \right) \leqslant C \|\phi_1(v, \xi_1)\|_{L^\infty([0,T])}. \tag{8.16}$$

Moreover, since the map $\mathcal{F}$ is causal (in the sense that the value of $\mathbb{P}_0\phi_1(v,\xi_1)$ at time $t \in [0,T]$ only depends on the values of v on the past time interval $[0,t]$), the same property holds for its inverse. Thus, estimate (8.16) yields:

$$\forall T' \in (0,T], \quad \|v\|_{L^\infty([0,T'])} \leqslant C \, \|\phi_1(v,\xi_1)\|_{L^\infty([0,T'])} . \tag{8.17}$$

Progressive estimation. We assume that the system is small-state small-time locally controllable. Let $\eta > 0$. We define $\varepsilon := \min\{\delta_A, \delta_B, \delta_A/(2C), \eta/C\}$. Let $\delta > 0$ be given by the application of Definition 11. For any states $x^*, x^\dagger \in \bar{B}(0,\delta/2)$, there exists a trajectory $(x,u) \in C^0([0,T],\mathbb{R}^n) \times L^1((0,T),\mathbb{R})$ with $x(0) = x^*$, $x(T) = x^\dagger$ and $\|x\|_{L^\infty} \leqslant \varepsilon$. Moreover, $u_1 \in C^0([0,T])$ with $u_1(0) = 0$. Let $\bar{T} := \sup\{T' \in [0,T]; \ \|u_1\|_{L^\infty(0,T')} \leqslant \delta_A/2\}$. Since u_1 is continuous and vanishes at the initial time, one has $\bar{T} > 0$.

By contradiction, let us assume that $\bar{T} < T$. Then, by continuity of u_1, there exists $T' \in (\bar{T}, T]$ such that $\|u_1\|_{L^\infty(0,T')} \leqslant \delta_A$. Thus, we can apply (8.17) and obtain that $\|u_1\|_{L^\infty(0,T')} \leqslant C\varepsilon \leqslant \delta_A/2$. Hence $\bar{T} = T$.

Eventually, we can apply (8.16) and obtain that $\|u_1\|_{L^\infty(0,T)} \leqslant C\varepsilon \leqslant \eta$. Therefore, the system is also $W^{-1,\infty}$ small-time locally controllable. $\square$

Conclusion and perspectives

We proved that quadratic approximations for differential systems can lead either to drifts quantified by Sobolev norms of the control or to the existence of an invariant manifold at the second-order. Thus, when a nonlinear system does not satisfy the linear Kalman condition, one needs to go at least up to the third order expansion to hope for positive results concerning small-time local controllability.

Our work highlights the importance of the norm hypothesis in the definition of small-time local controllability, even for differential systems. Indeed, although the state lives in $\mathbb{R}^n$, we have proved that the controllability properties depend strongly on the norm of the control chosen in the definition of the notion. We expect that other geometric results might be improved by exploring the link between Lie brackets and functional settings.

For systems governed by partial differential equations, we expect that the behaviors proved in finite dimension can also be observed. For example, the first author and Morancey obtain in [7] a drift quantified by the H^{-1}-norm of the control, which prevents small-time local controllability, under an assumption corresponding to $[f_1,[f_0,f_1]](0) \notin \mathcal{S}_1$. In [14], Coron and Crépeau observe that the behavior of the second-order expansion of a Korteweg-de-Vries system is fully determined by the position of the linear approximation (thus recovering a kind of invariant manifold up to the second order).

It is also known that new phenomenons can occur. For example, in [27], the second author obtains a drift quantified by the $H^{-5/4}$-norm of the control for a Burgers system, which thus does not seem to be linked with an integer order Lie bracket and is specific to the infinite dimensional setting.

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