Magnetization switching in small ferromagnetic ellipsoidal samples

Francois Alouges, Karine Beauchard, Mario Sigalotti

***Abstract*— This paper deals with the magnetization switching in small ferromagnetic particles. Particularly important for applications, this problem is tackled with two different controls, namely either a horizontal magnetic field or a current polarized in spin. We give in both cases necessary and sufficient conditions for controllability.**

I. INTRODUCTION

Ferromagnetic materials are nowadays used in numerous technological devices. Among these applications, magnetic storage is probably one of the most important areas and such materials are for instance found in hard-disks, or magnetic RAM. Both are composed of several small ferromagnetic particles capable of being magnetized in two opposite directions, allowing for the storage of one bit of information.

Being able to switch the magnetization in a quick and sure way into such a sample is therefore of prime interest. Not surprinsingly, the switching of the magnetization in small elongated particles has received a lot of attention (see for instance [10], [2] or [3] and references therein) after the pioneering work of Kikuchi [8] where an analytical solution is given in the case of a spherical particle uniformly magnetized.

We focus here on the so-called MRAM devices in which the information is stored in small ferromagnetic elements [7], and the switching is obtained via the application of a suitable magnetic field or the application through the sample of a suitable spin polarized electric current.

The aim of this paper is to provide a mathematical study of the phenomenon at the level of the PDE modelizing the behavior of the magnetization (Landau-Lifschitz-Gilbert equations) in the regime of small particles. We also generalize a recent work by two of the authors [1] to the case of more general controls by a magnetic field and to the case of the spin induced switching.

For more physical insight of the problem, we refer the reader to [5], [11], [12] and references therein.

Let us also quote [9] which treats a control problem in micromagnetics, but in the different context of moving a wall in a nano-wire.

This work was supported by the "Chair Mathematical modelling and numerical simulation, F-EADS ∼ Ecole Polytechnique ∼ INRIA ∼ F-X".

F. Alouges is with Centre de Mathématiques Appliquées, École Polytechnique Route de Saclay, 91128 Palaiseau Cedex, France, francois.alouges@polytechnique.edu

K. Beauchard is with CNRS, CMLA, ENS Cachan, CNRS, Universud, 61 Avenue du président Wilson, F-94230 Cachan, France, Karine.Beauchard@cmla.ens-cachan.fr

M. Sigalotti is with INRIA, Centre de Recherche Nancy - Grand Est, Équipe-projet CORIDA, and Institut Élie Cartan, BP 239, 54506 Vandœuvre-lès-Nancy, France Mario.Sigalotti@inria.fr

In the following, we explain the micromagnetic model and the Landau-Lifschitz-Gilbert equations which describe statics and dynamics of the magnetization inside a ferromagnetic material. We have dimensionalized the terms in such a way that most of the physical constants (*e.g.* saturation magnetization, gyromagnetic factor, exchange constant, etc.) are scaled to unity.

The magnetization m inside a ferromagnetic body, located in a space domain Ω, is a three dimensional vector field defined on Ω of magnitude 1 through the sample. The evolution of the magnetization is modelized by the Landau-Lifschitz equation,

$$\frac{\partial m}{\partial t} = \alpha[H(m) - \langle H(m), m\rangle m] - m \wedge H(m), x \in \Omega. \tag{1}$$

Here, $H(m)$ is the total (or effective) magnetic field whose expression is given below. It is induced by several physical phenomena (exchange, stray-field, anisotropy, exterior field), $\alpha > 0$ is a damping coefficient which depends on the material (we refer the reader to [4] or [6] for a more complete description of the physical model). In this equation, $\langle ., .\rangle$ denotes the Euclidean scalar product on $\mathbb{R}^3$ and $\wedge$ is the vectorial product. Equivalently, at least for smooth solutions, the equation (1) may be written under the form,

$$\alpha\frac{\partial m}{\partial t} + \left(m \wedge \frac{\partial m}{\partial t}\right) = (1+\alpha^2)[H(m) - \langle H(m), m\rangle m]. \tag{2}$$

For a ferromagnetic body without anisotropy, the magnetic field $H(m)$ can be expressed as

$$H(m) = -\frac{\partial \mathcal{E}}{\partial m} + H_{ext},$$

where H_{ext} is a magnetic field applied to the sample and $\mathcal{E}(m)$ is the micromagnetic energy associated to a given magnetization m,

$$\mathcal{E}(m) := \frac{A}{2}\int_\Omega |\nabla m|^2 - \frac{1}{2}\int_\Omega \langle H_d(m), m\rangle. \tag{3}$$

This leads to

$$H(m) = A\Delta m + H_d(m) + H_{ext}, \tag{4}$$

where A is the so-called exchange constant [4], that will be taken equal to 1 to simplify the presentation, and $H_d(m)$ is the stray field generated by the magnetization m itself via Maxwell equations. It is well known that $H_d(m)$ is the $L^2(\mathbb{R}^3)$-orthogonal projection of $-m1_\Omega$ on gradients, from which we deduce

$$\|H_d(m)\|_{L^2(\mathbb{R}^3)} \leqslant \|m\|_{L^2(\Omega)}, \forall m \in L^2(\Omega). \tag{5}$$

In the particular case of ellipsoidal domains, the stray field of constant magnetizations is constant on the domain [13] and can therefore be written as

$$H_d(m) = -Dm_\sharp + \widetilde{H}_d(m) \text{ on } \Omega$$

where

$$m_\sharp(t) := \frac{1}{|\Omega|}\int_\Omega m(t,x)dx,$$

$D = \text{diag}(\alpha_1, \alpha_2, \alpha_3)$, $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \alpha_3 \leqslant 1$ depend on the size of the 3 axes of the ellipsoid and $\widetilde{H}_d(m) := H_d(m - m_\sharp)$ satisfies, in view of (5),

$$\|\widetilde{H}_d(m)\|_{L^2(\mathbb{R}^3)} \leqslant \|m - m_\sharp\|_{L^2(\Omega)}. \quad (6)$$

In this article, we focus on the case

$$0 \leqslant \alpha_1 < \alpha_2 \leqslant \alpha_3 \leqslant 1, \quad (7)$$

ensuring that the global minimizers of the micromagnetic energy (for uniform magnetizations) are $\pm e_1$.

The natural boundary conditions associated to the problem (1) are of Neumann type,

$$\frac{\partial m}{\partial \nu}(t,x) = 0, \; x \in \partial\Omega, \; t \in (0,T). \quad (8)$$

The system (1)-(4)-(8) is a nonlinear control system in which

- the state is the magnetization m, with $m(t) : \Omega \to S^2$, for every t,
- the control is the time-dependent external magnetic field $H_{ext} : \mathbb{R}_+ \to \mathbb{R}^3$.

In the particular case of uniform (in space) magnetizations, the Landau-Lifschitz equation becomes the ordinary differential system

$$\alpha\frac{dm}{dt} + m \wedge \frac{dm}{dt} = (1+\alpha^2)[H_0(m) - \langle H_0(m), m\rangle m], \quad (9)$$

where

$$H_0(m) = -Dm + H_{ext}. \quad (10)$$

In [1], we were interested in the existence and the properties of a 3D uniform control $H_{ext}(t)$ that steers the solution m of (1) from $m(0) = e_1$ to $m(T) = -e_1$.

In this article, we are interested in the same question, but under different restrictions on the magnetic field, motivated by the applications:

- in a first step, we ask H_{ext} to be uniform in space and to take values in a vector subspace V of $\mathbb{R}^3$ of dimension 2. Indeed, in the device, the external magnetic field is horizontal (*i.e.* in $Span(e_1, e_2)$),
- in a second step, we study the spin induced switching. Although the control is no longer an external field, it can be modeled by assuming that H_{ext}, in the preceding model, is of the form

$$H_{ext}(t,x) = h(t)m(t,x) \wedge e, \quad (11)$$

where $h(t) \in \mathbb{R}$ is an amplitude and $e \in S^2$ is fixed.

In Section II, we study the feasibility of the switching from $m(0) = e_1$ to $m(T) = -e_1$ for the ODE (9)-(10) under the two previous restrictions on H_{ext}. In Section III, considering the full PDE model (1)-(4)-(8), we prove that non-uniform magnetizations m_0 that are close enough to e_1 can be steered asymptotically to $-e_1$. In both cases, we propose explicit controls realizing the switching.

All along this article we will use the following notations. The family (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$ (this is the basis in which D is actually diagonal), and if $x \in \mathbb{R}^3$, we denote its components by x_1, x_2 and x_3.

II. UNIFORM MAGNETIZATION SWITCHING

A. With 2D fields

In this section, we study the ODE (9)-(10) under the constraint

$$H_{ext}(t) \in V, \forall t \in [0, +\infty), \quad (12)$$

where V is a fixed 2D vector subspace of $\mathbb{R}^3$.

First, let us introduce the concept of *feasible trajectory* and some notations. A *feasible trajectory* of (9)-(10) is an absolutely continuous (a.c.) curve $M : [0,T] \to S^2$ such that there exists a measurable bounded control function $H_{ext} : [0,T] \to \mathbb{R}^3$ for which (9) is satisfied at almost every $t \in [0,T]$. We denote by β the positive constant $(1+\alpha^2)$. For $m \in S^2$, $S(m)$ is the 3×3 skew-symmetric matrix such that $S(m)v = m \wedge v$ for every $v \in \mathbb{R}^3$. Notice that, for every $m \in S^2$, $(\alpha Id + S(m))$ is invertible and preserves $m^\perp$; moreover its restriction to $m^\perp$ is a dilatation of coefficient $\sqrt{\beta}$ composed with a rotation of angle $\theta := \arctan(1/\alpha) \in (0, \pi/2)$. (See Figure 1.)

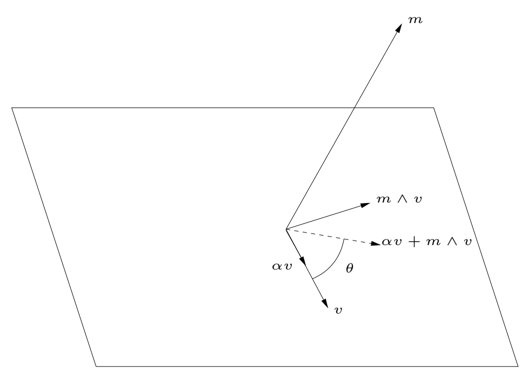

Fig. 1. Action of $\alpha + S(m)$ on $m^\perp$.

Hence $(\alpha Id + S(m))^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$ is itself an invertible transformation, which acts on $m^\perp$ as a dilatation of coefficient $1/\sqrt{\beta}$ composed with a rotation of angle $-\theta \in (-\pi/2, 0)$. Finally, we introduce the orthogonal projection Π_m from $\mathbb{R}^3$ to $m^\perp$,

$$\Pi_m(v) := v - \langle v, m\rangle m, \forall v \in \mathbb{R}^3, \forall m \in S^2,$$

and rewrite (9) as

$$\frac{dm}{dt} = \beta(\alpha Id + S(m))^{-1}(-\Pi_m(Dm) + \Pi_m(H_{ext})).$$

Proposition 1 *Let V be a 2D vector subspace of $\mathbb{R}^3$. Let $m : [0,T] \to S^2$ be a C^1 curve such that, for every $t^* \in$*

$[0,T]$ *for which* $m(t^*) \in V$*, then* $\frac{dm}{dt}(t^*)$ *belongs to the 1D affine subspace of* $\mathbb{R}^3$

$$\frac{dm}{dt}(t^*) \in (\alpha Id + S(m))^{-1}(-\Pi_m(Dm) + V \cap m^{\perp}).$$

Then m *is feasible for (9)-(10)-(12). Thus, (9)-(10)-(12) is controllable.*

Proof: If $m \in S^2 \setminus V$, then $\Pi_m(V) = m^{\perp}$, and the set of admissible velocities at m is equal to the entire tangent space $T_m S^2 = m^{\perp}$.

Now, let $m \in S^2 \cap V$, then $\Pi_m(V) = V \cap m^{\perp}$ is one dimensional. Notice that $(\alpha Id + S(m))^{-1}\Pi_m(V)$ is a 1D subspace of $m^{\perp}$ that is transversal to $V \cap m^{\perp}$. The set of admissible velocities at m is therefore an affine 1D subspace of $m^{\perp}$ that is transversal to $V \cap m^{\perp}$. In particular, if $J : \mathbb{R}^3 \to \mathbb{R}$ denotes a linear function such that $V = J^{-1}(0)$, then $m^{\perp}$ contains two admissible velocities $v_1(m), v_2(m)$ such that the directional derivatives at m of J in the directions $v_1(m)$ and $v_2(m)$, denoted respectively by $L_{v_1(m)}J(m)$ and $L_{v_2(m)}J(m)$, satisfy

$$L_{v_1(m)}J(m) > 0 > L_{v_2(m)}J(m),$$

that is, $v_1(m)$ and $v_2(m)$ point towards, respectively, the upper and the lower hemisphere of S^2 with respect to the equator $S^2 \cap V$.

In order to establish the controllability of (9)-(10), it suffices to notice the feasibility of every C^1 curve $m : [0,T] \to S^2$ such that $\dot{m}(t) \in \{v_1(m), v_2(m)\}$ for every $t \in [0,T]$ for which $m(t) \in V$. Indeed, every two points of S^2 can clearly be connected by such a curve. ■

Let us focus on the particular case $V = \text{Span}(e_1, e_2)$, which is the most interesting for the applications to MRAMs. Then, the magnetization switching may be done in arbitrarily small time (with large controls), or with quite small controls (in large time).

Proposition 2 *Let* $V := \textit{Span}(e_1, e_2)$*.*

(1) *Let* $T > 0$*. There exists* $H_{ext} \in C^0([0,T], V) \cap C^{\infty}((0,T), V)$ *such that the solution of (9) with* $m(0) = e_1$ *satisfies* $m(T) = -e_1$*.*

(2) *For every* $\epsilon > 0$*, there exists* $T > 0$ *and* $H_{ext} \in C^0([0,T], V) \cap C^{\infty}((0,T), V)$ *satisfying* $\|H_{ext}\|_{L^{\infty}(0,T)} \leqslant 2(\alpha_3 - \alpha_1) + \epsilon$ *such that the solution of (9) with* $m(0) = e_1$ *satisfies* $m(T) = -e_1$*.*

Proof: **(1)** Let $T > 0$ and $m \in C^{\infty}([0,T], S^2)$ be such that $m(0) = -m(T) = e_1$, $\dot{m}(0) = -\dot{m}(T) = v := -(\alpha Id + S(e_1))^{-1}e_2$, $m_3(t) > 0, \forall t \in (0,T)$. Then, m is a feasible trajectory associated to

$$H_{ext}(t) := \mu(t) - \frac{\langle \mu(t), e_3 \rangle}{\langle m(t), e_3 \rangle} m(t),$$

for every $t \in (0,T)$, where

$$\mu(t) := \frac{1}{\beta}[\alpha Id + S(m(t))]\frac{dm}{dt}(t) + \Pi_{m(t)}(Dm(t)).$$

Let us prove that H_{ext} is continuous at $t = 0$ and $t = T$. Since $\langle m, e_3 \rangle = t\langle v, e_3 \rangle + o(t)$ when $t \to 0$ and $\langle v, e_3 \rangle \neq 0$ (up to a dilatation, v is the rotation of e_2 around e_1 of angle $-\theta \in (-\pi/2, 0)$), it is sufficient to prove that the zero order term of $\langle \mu(t), e_3 \rangle$ when $t \to 0$ is zero. Indeed, we have

$$\begin{aligned} \mu(t) &= \tfrac{1}{\beta}[\alpha Id + S(e_1)]v + O(t) \\ &= -\tfrac{e_2}{\beta} + O(t) \text{ when } t \to 0. \end{aligned}$$

(2) For $\epsilon > 0$, the feasible trajectory $m_{\epsilon} : [0, T/\epsilon] \to S^2$, $m_{\epsilon}(t) := m(\epsilon t)$ is associated to a control $H_{ext,\epsilon}$ that satisfies

$$\|H_{ext,\epsilon}(t)\| \leqslant 2(\alpha_3 - \alpha_1) + C\epsilon,$$

where $C > 0$ is independent of ϵ, because

$$\begin{aligned} \|\Pi_m(Dm)\|^2 &= \sum_{k=1}^{3}[\alpha_k - (\alpha_1 m_1^2 + \alpha_2 m_2^2 + \alpha_3 m_3^2)]^2 m_k^2 \\ &\leqslant (\alpha_3 - \alpha_1)^2, \forall m \in S^2, \\ \tfrac{|\langle \Pi_m(Dm), e_3 \rangle|}{|\langle m, e_3 \rangle|} &= |\alpha_3 - (\alpha_1 m_1^2 + \alpha_2 m_2^2 + \alpha_3 m_3^2)| \\ &\leqslant \alpha_3 - \alpha_1, \forall m \in S^2. \end{aligned}$$

■

B. Spin induced switching

In this section, we study the ODE (9)-(10) under the constraint

$$H_{ext}(t) = h(t) m(t) \wedge e \tag{13}$$

where $h : [0,T] \to \mathbb{R}$ is an amplitude (the control) and $e \in S^2$ is a fixed vector.

Proposition 3 *The system (9)-(10)-(13) is controllable if and only if* e *is not an eigenvector of* D*.*

Proof: If e is an eigenvector of D, then, for every h the solution of (9)-(10)-(13) with $m(0) = e$ is $m(t) \equiv e$, thus, the system is not controllable.

Now, let us assume that e is not an eigenvector of D. We denote by X_0 and X_1 the vector fields on S^2 defined by

$$X_0(m) := \beta(\alpha Id + S(m))^{-1}(-\Pi_m(Dm)),$$

$$X_1(m) := \beta(\alpha Id + S(m))^{-1}(m \wedge e).$$

Although the trajectories of $\pm X_1$ are not themselves feasible, they can be approximated with arbitrary precision by feasible trajectories on bounded intervals of time, by taking for $h(t)$ a constant value $\pm A$, with $A > 0$ large enough.

Notice that

$$X_1(m) = \beta(\alpha Id + S(m))^{-1}(m \wedge \Pi_m(e))$$

and that the vector field $m \mapsto \Pi_m(e)$ is tangent at every point $m \in S^2$ to the meridian passing through e and m, and vanishes only at $m = \pm e$. Since $(\alpha Id + S(m))^{-1}$ acts, up to a dilatation, as a rotation of angle $-\theta \in (-\pi/2, 0)$ (independent of m) around m, it follows that the trajectories of X_1 are heteroclinic orbits connecting $-e$ to e such that the phase portrait around each of these two points is a focus.

Since e is not an eigenvector of D, then $X_0(e) \neq 0$, so the local phase portrait of X_0 around e and $-e$ is a foliation by parallel lines.

A possible control strategy is the following. Let $m_0, m_1 \in S^2$ and let us assume that $m_0, m_1 \notin \{e, -e\}$. Since integral curves of X_1 from points different from $\pm e$ turn infinitely many times around e, there exist p_0 and p_1 on the integral curve of X_0 passing through e such that X_1 connects m_i to p_i for $i = 0, 1$, and X_0 connects p_0 to p_1.

Then, approximating curves of X_1 by feasible ones, we get p'_0, p'_1 on the integral curve of X_0 passing through e such that the two feasible curves connect m_0 to p'_0 and p'_1 to m_1, and X_0 connects p'_0 to p'_1. Concatenating these three curves, we obtain a feasible curve connecting m_0 to m_1.

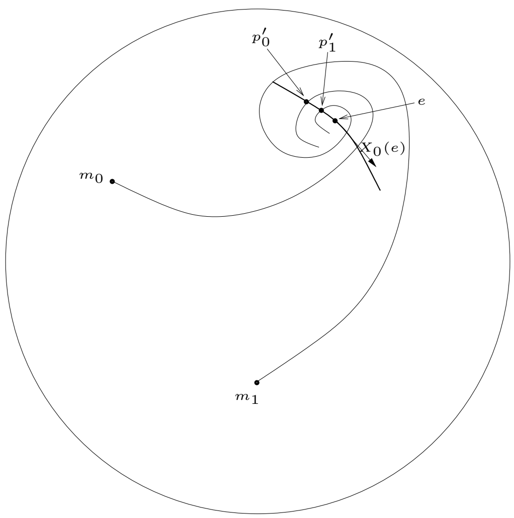

Fig. 2. In order to go from m_0 to m_1, first join m_0 to p'_0 following an approximated integral curve of X_1, then p'_0 to p'_1 using X_0, and finally p'_1 to m_1 by an approximated integral curve of $-X_1$.

A similar argument works when m_0 or m_1 belong to $\{-e, e\}$. ■

Remark 1 *The proof above can be easily adapted to show that when the external field appearing in Section II-A takes value in a one-dimensional vector space V, then (9)-(10)-(12) is controllable if and only if V is not contained in an eigenspace of D.*

When $e \in \{\pm e_2, \pm e_3\}$, the system (9)-(10)-(13) is not controllable, but the control design proposed in the previous proof may still be used to switch the magnetization from e_1 to $-e_1$. Moreover, under the hypotheses of the previous proposition, this motion may be done in arbitrarily small time.

Proposition 4 ***(1)*** *The magnetization switching from e_1 to $-e_1$ is possible for (9)-(10)-(13) if and only if $e \neq \pm e_1$.*

(2) *We assume that e does not belong to an eigenspace of D. For every $T > 0$ there exists $h \in L^\infty((0,T), \mathbb{R})$ piecewise constant taking at most 3 different values such that the solution of (9)-(10)-(13) with $m(0) = e_1$ satisfies $m(T) = -e_1$.*

Proof: **(1)** If $e \in \{e_1, -e_1\}$ then, for every h, the solution of (9)-(10)-(13) with $m(0) = e$ is $m \equiv e$, thus the switching is impossible.

If e does not belong to an eigenspace of D, then the switching is possible thanks to the previous proposition.

Let $\alpha_2 = \alpha_3$ and $e \in \text{span}\{e_1, e_2\}$. Then $-\Pi_m(Dm)$ is a vector tangent to the meridian connecting e_1 and m, and points towards the meridian orthogonal to e_1. Then the integral lines of X_0 converge towards such meridian and the argument seen in the proof of the previous proposition still imply the controllability from e_1 to $-e_1$.

Let us assume that $e \in \{\pm e_2\}$ and that $\alpha_2 < \alpha_3$. We have

$$dX_0(e_2) = \begin{pmatrix} -\alpha(\alpha_2 - \alpha_1) & -(\alpha_3 - \alpha_2) \\ -(\alpha_2 - \alpha_1) & \alpha(\alpha_3 - \alpha_2) \end{pmatrix}$$

on $e_2^\perp$ and so $dX_0(e_2)$ has two real eigenvalues with opposite signs. Thus, the phase portrait of X_0 around e_2 is a saddle point and the previous analysis may be adapted (the integral curves of a focus necessarily cross the ones of a saddle point).

Now, let us assume that $e \in \{\pm e_3\}$ and that $\alpha_2 < \alpha_3$. We have

$$dX_0(e_3) = \begin{pmatrix} -\alpha(\alpha_3 - \alpha_1) & -(\alpha_3 - \alpha_2) \\ \alpha_3 - \alpha_1 & -\alpha(\alpha_3 - \alpha_2) \end{pmatrix}$$

on $e_3^\perp$. When

$$\alpha^2(3\alpha_3 - \alpha_2 - \alpha_1)^2 - 2\beta(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2) \geqslant 0 \quad (14)$$

the phase portrait of X_0 around e_3 is a node, a line field or a degenerate node, thus the previous analysis may be adapted to this case. When (14) does not hold, then e_3 is a focus and the trajectories of X_0 turn around e_3 in the opposite sense than the ones of X_1, thus, the previous analysis may be adapted to this case.

(2) In the control process presented in the previous proof, the approximations of the curves of X_1 can be covered in arbitrarily small time by taking large enough constant controls (cf time rescaling). The segment of curve of X_0 is covered with a fixed velocity, but this segment may be chosen arbitrarily small, thus, this part of the control process may also be done in arbitrarily small time.

■

III. ASYMPTOTIC SWITCHING FOR THE PDE

Let Ω be a 3D ellipsoid with $|\Omega| = 1$. In this section, we consider the Landau-Lifschitz PDE on the domain $\Omega_\lambda := \sqrt{\lambda}\Omega$, when $\lambda \to 0$, $\lambda > 0$. A space change of variables shows that it is equivalent to study the following Landau-Lifschitz PDE on the fixed domain Ω,

$$\begin{cases} \dfrac{\partial m}{\partial t} = \alpha \Pi_m H_\lambda(m) - m \wedge H_\lambda(m), x \in \Omega, t \in (0,T) \\ \dfrac{\partial m}{\partial \nu}(t,x) = 0, x \in \partial\Omega, t \in (0,T), \\ m(0,x) = m_0(x), x \in \Omega, \end{cases} \quad (15)$$

with an effective magnetic field

$$H_\lambda(m) := \frac{\Delta m}{\lambda} + H_d(m) + H_{ext}(t) \quad (16)$$

associated to the micromagnetic energy

$$\mathcal{E}_\lambda(m) := \int_\Omega \frac{1}{2\lambda}|\nabla m|^2 + \frac{1}{2}\int_{\mathbb{R}^3} |H_d(m)|^2. \qquad (17)$$

Then, one has the following result, proved in [1, Proposition 7].

Proposition 5 *Let Ω be a 3D ellipsoid. There exists $\lambda^* = \lambda^*(\Omega) > 0$ such that, for every $\lambda \in (0, \lambda^*)$, the micromagnetic energy $\mathcal{E}_\lambda$ has exactly two global minimizers : $m \equiv e_1$ and $m \equiv -e_1$.*

The goal of this section it to prove that, on a small enough ellipsoid (λ small), the magnetizations m_0 that are close enough to e_1 in $H^2(\Omega, S^2)$ can be asymptotically steered to $-e_1$ in $H^s(\Omega, S^2)$, $\forall s < 2$. In Section III-A, we prove that, on a small enough ellipsoid, with an initial condition m_0 close enough to e_1 (resp. $-e_1$) in $H^2(\Omega, S^2)$, there exists a global solution of (15)-(16) with $H_{ext} = 0$ and this solution converges to e_1 (resp. $-e_1$) in H^s, $\forall s < 2$. Then, we get the conclusion in the case of 2D external fields and in the case of the spin induced switching model in Section III-B thanks to the results of Section II.

A. Exponential convergence without control

The goal of this section is to prove the following result.

Theorem 1 *Let Ω be a 3D ellipsoid and $\alpha > 0$. There exists $\lambda^* = \lambda^*(\Omega, \alpha) > 0$ such that, for every $\lambda \in (0, \lambda^*)$, there exists $\eta = \eta(\Omega, \alpha, \lambda) > 0$ such that, for every $m_0 \in H^2(\Omega, S^2)$ with*

$$\frac{\partial m_0}{\partial \nu} \equiv 0 \text{ on } \partial\Omega \qquad (18)$$

and

$$\|m_0 - e_1\|_{H^2} < \eta, \qquad (19)$$

there exists a unique global smooth solution of (15)-(16) with $H_{ext} \equiv 0$ and it satisfies

$$\|m(t) - e_1\|_{H^s} \to 0 \text{ when } t \to +\infty, \forall s < 2.$$

The proof of Theorem 1 uses the following result, which is proved in [1, Theorems 2, 4 and Proposition 9].

Theorem 2 *Let Ω be a 3D ellipsoid, $\alpha > 0$ and $H_{ext} \in L^\infty(\mathbb{R}_+, \mathbb{R}^3)$. There exist $\lambda_0^* = \lambda_0^*(\Omega, \alpha, \|H_{ext}\|_{L^\infty}) > 0$, $\eta_0 = \eta_0(\Omega, \alpha) > 0$, such that, for every $\lambda \in (0, \lambda_0^*)$, for every $m_0 \in H^2(\Omega, S^2)$ with (18) and $\|\Delta m_0\|_{L^2} < \eta_0$, there exists a global smooth solution of (15)-(16) and it satisfies*

$$\|\Delta m(t)\|_{L^2} \leqslant \|\Delta m_0\|_{L^2}, \forall t > 0, \qquad (20)$$

$$\|\nabla m(t)\|_{L^2} \leqslant \|\nabla m_0\|_{L^2} e^{-ct}, \forall t > 0, \qquad (21)$$

where $c = c(\Omega, \alpha, \lambda) > 0$. Moreover, such smooth solutions depend continuously on m_0 for the topology $C^0([0,T], H^2(\Omega, S^2)), \forall T > 0$.

Remark 2 *The same proof as in [1] gives the previous result for (15)-(16)-(11) replacing the assumption on H_{ext} by $h \in L^\infty(\mathbb{R}_+, \mathbb{R})$.*

Proof of Theorem 1: First, let us recall that, along the trajectories of (15), (16), the energy $\mathcal{E}_\lambda(m)$ is non-increasing (see, for instance [1, Proposition 8]).

Let Ω be a 3D ellipsoid and $\alpha > 0$. One may assume that $\alpha_1 = 0$. Let λ_0^* and η_0^* be as in Theorem 2 (with $H_{ext} \equiv 0$) and $\mathcal{C}_P$ be the Poincaré constant

$$\|m - m_\sharp\|_{L^2} \leqslant \mathcal{C}_P \|\nabla m\|_{L^2}, \forall m \in H^1(\Omega, \mathbb{R}^3). \qquad (22)$$

(Recall that $m_\sharp = (m_{\sharp 1}, m_{\sharp 2}, m_{\sharp 3})$ denotes the average value of m on Ω.) We take $\lambda^* = \lambda_0^*$. Let $\lambda \in (0, \lambda^*)$ and $c > 0$ be as in Theorem 2 (cf. (21)). Let $\eta \in (0, \eta_0)$ be small enough so that, for every $\tilde{m} \in H^2(\Omega, S^2)$ with $\|\tilde{m} - e_1\|_{H^2} < \eta$, one has

$$\tilde{m}_{\sharp 1} > 0, \qquad (23)$$

$$\mathcal{E}_\lambda(\tilde{m}) < \frac{\alpha_2}{4}, \qquad (24)$$

$$\|\tilde{m}_\sharp\|^2 - \frac{2}{c}\Big(\frac{1}{\lambda} + (2 + \alpha_3)\mathcal{C}_P\Big)\|\nabla \tilde{m}\|_{L^2} \geqslant \frac{1}{2} \qquad (25)$$

(notice that (24) is possible because $\mathcal{E}_\lambda(e_1) = 0$). Let $m_0 \in H^2(\Omega, S^2)$ with (18) and (19). Let $(t_n)_{n\in\mathbb{N}}$ be an increasing diverging sequence in $[0, +\infty)$. Thanks to (20), the sequence $(m(t_n))_{n\in\mathbb{N}}$ is bounded in $H^2(\Omega, S^2)$. Therefore, there exists $m_\infty \in H^2(\Omega, S^2)$ such that, up to extracting a subsequence, $m(t_n) \to m_\infty$ weakly in H^2 and strongly in H^1 (m_∞ takes values in S^2 because the embedding $H^2(\Omega) \to L^\infty(\Omega)$ is compact). Because of (21), m_∞ is constant on Ω. Since $t \mapsto \mathcal{E}_\lambda[m(t)]$ is not increasing, then the solution of (15)-(16) (or equivalently of (9)-(10)) with initial condition $m(0) = m_\infty$ and $H_{ext} = 0$ is necessarily stationary. Thus, $Dm_\infty = \langle Dm_\infty, m_\infty\rangle m_\infty$, i.e. $m_\infty \in \{\pm e_1, \pm e_2, \pm e_3\}$. We have $\mathcal{E}_\lambda(m_\infty) \leqslant \mathcal{E}_\lambda(m_0) < \alpha_2/4$ thus $m_\infty = \pm e_1$.

Working by contradiction, we assume that $m_\infty = -e_1$. Then the function $m_\sharp : [0, +\infty) \to \mathbb{R}$ is continuous and satisfies $m_{\sharp 1}(0) > 0$ (because of (23)) and $m_{\sharp 1}(+\infty) = -1$. Thus, there exists $T \in (0, +\infty)$ such that $m_{\sharp 1}(T) = 0$. Then, we have

$$\begin{aligned} \mathcal{E}_\lambda[m(T)] &\geqslant \tfrac{1}{2\lambda}\|\nabla m(T)\|_{L^2}^2 + \tfrac{\langle Dm_\sharp(T), m_\sharp(T)\rangle}{2} \\ &\geqslant \tfrac{\alpha_2}{2}\|m_\sharp(T)\|^2 \end{aligned} \qquad (26)$$

where the first inequality is proved in [1, proof of Proposition 7]. Integrating the first equation of (15) over Ω, using (21), (6) and (22), we get

$$\frac{dm_\sharp}{dt} = F(m_\sharp) + G(t),$$

$$F(m) := \alpha[-Dm + \langle Dm, m\rangle m] + m \wedge Dm,$$

$$\|G(t)\| \leqslant \left(\frac{1}{\lambda} + (2 + \alpha_3)\mathcal{C}_P\right)\|\nabla m_0\|_{L^2} e^{-ct}, \forall t > 0.$$

We deduce from

$$\frac{d}{dt}\|m_\sharp\|^2 = 2\langle G(t), m_\sharp\rangle,$$

the previous inequality and (25) that

$$\begin{aligned} \|m_\sharp(T)\|^2 &\geqslant \|m_{0\sharp}\|^2 - \frac{2}{c}\left(\frac{1}{\lambda} + (2 + \alpha_3)\mathcal{C}_P\right)\|\nabla m_0\|_{L^2} \\ &\geqslant \frac{1}{2}. \end{aligned}$$

Thus (26) leads to $\mathcal{E}_\lambda[m(T)] \geqslant \alpha_2/4$. However, (24) gives $\mathcal{E}_\lambda[m(T)] \leqslant \mathcal{E}_\lambda[m_0] < \alpha_2/4$. This is a contradiction. Therefore $m_\infty = e_1$. ■

B. With 2D fields or the spintronic model

Thanks to Theorem 1 and the study made in Sections II-A and II-B one easily gets the following theorem.

Theorem 3 *Let Ω be a 3D ellipsoid, $\alpha > 0$, $V := Span(e_1, e_2)$. Let $T > 0$ and $\widetilde{H}_{ext} \in C^0([0,T], V)$ be such that the solution $\widetilde{m}$ of the ODE (9)-(10)-(12) with $\widetilde{m}(0) = -e_1$ satisfies $\widetilde{m}(T) = e_1$. We define $H_{ext} \in L^\infty(\mathbb{R}_+, V)$ by*

$$H_{ext}(t) := \begin{cases} \widetilde{H}_{ext}(t), \forall t \in [0,T], \\ 0, \forall t > T. \end{cases}$$

There exists $\lambda^* > 0$ such that, for every $\lambda \in (0, \lambda^*)$, there exists $\eta > 0$ such that, for every $m_0 \in H^2(\Omega, S^2)$ with (18) and (19), there exists a unique global solution of (15)-(16) and it converges to e_1 in $H^s(\Omega)$, $\forall s < 2$.

Theorem 4 *Let Ω be a 3D ellipsoid, $\alpha > 0$, $e \in S^2 \backslash \{\pm e_1\}$. Let $T > 0$ and $\widetilde{h} \in L^\infty(0,T)$ be such that the solution $\widetilde{m}$ of the ODE (9)-(10)-(13) with $\widetilde{m}(0) = -e_1$ satisfies $\widetilde{m}(T) = e_1$. We define $h \in L^\infty(\mathbb{R}_+, \mathbb{R})$ by*

$$h(t) := \begin{cases} \widetilde{h}(t), \forall t \in [0,T], \\ 0, \forall t > T. \end{cases}$$

There exists $\lambda^* > 0$ such that, for every $\lambda \in (0, \lambda^*)$, there exists $\eta > 0$ such that, for every $m_0 \in H^2(\Omega, S^2)$ with (18) and (19), there exists a unique global solution of (15)-(16)-(11) and it converges to e_1 in $H^s(\Omega)$, $\forall s < 2$.

IV. CONCLUSIONS AND FUTURE WORKS

In this paper, we have tackled the problem of magnetization switching with either a bidimensional external magnetic field or a spin polarized current. In both cases, we have studied the controllability of the system for infinitely small ellipsoidal particles, giving a complete answer to the problem. We have then extended the results to the full PDE model (Landau-Lifschitz-Gilbert equations) in the case of small enough particles. Although only very few results are available for the full PDE in all generality, we have been able to use the fact that in small enough particles the magnetization remains almost constant in space for all time. This is however not the case for the sizes of particles used in current devices [11], and a more complicated space structure occurs, although minimizers are still almost constant in space. We intend to explore this situation in the near future.

REFERENCES

[1] F. Alouges and K. Beauchard, Magnetization switching on small ferromagnetic ellipsoidal samples. *ESAIM:COCV (to appear)*, 2008.

[2] M. Bauer, J. Fassbender, B. Hillebrands and R. L. Stamps, Switching behavior of a Stoner particle beyond the relaxation time limit, *Phys. Rev. B*, **61**, 3410–3416, 2000

[3] G. Bertotti and I. Mayergoyz, *The Science of Hysteresis*, Academic Press, 2006

[4] W. F. Brown, Micromagnetics. *Interscience Publishers*, 1963.

[5] A. Fert, V. Cros, J.-M. George, J. Grollier, H. Jaffres, A. Hamzic, A. Vaures, G. Faini, J. Ben Youssef and H. Le Gall, Magnetization reversal by injection and transfer of spin: experiments and theory, *Journal of Magnetism and Magnetic Materials*, **272-276** 1706–11, 2004

[6] A. Hubert and R. Schäfer, Magnetic Domains: The Analysis of Magnetic Microstructures. *Springer*, 1998, Berlin.

[7] W.C. Jeong, J.H. Park, J.H. Oh, G.T. Jeong, H.S. Jeong and Kinam Kim, Highly scalable MRAM using field assisted current induced switching, *Symposium on VLSI Technology Digest of Technical Papers*, 2005

[8] R. Kikuchi, On the minimum of magnetization reversal time, *J. Appl. Phys.*, **27**, 1352–1357, 1956

[9] G. Carbou, S. Labbé and E. Trélat, Control of travelling walls in a ferromagnetic nanowire. *Discrete Contin. Dyn. Syst. Ser. S*, 1, 2008, pp.51-59.

[10] J. C. Mallinson, Damped gyromagnetic switching, *IEEE Trans. Magn.*, **36**, 1976–1981, 2000

[11] M. D. Stiles and J. Miltat, Spin transfer torque and dynamics, *Spin dynamics in confined magnetic structures III*, Topics Appl. Phys., 225–308, B. Hillebrands and A. Thiaville Eds, Springer, 2006

[12] D. V. Berkov and J. Miltat, Spin-torque driven magnetization dynamics: Micromagnetic modeling, *Journal of Magnetism and Magnetic Materials*, **320**, 1238–1259, 2008

[13] J.L A. Osborn, Demagnetizing factors of the general ellipsoid, *Phys. Rev.* **67**, 351–357, 1945