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Local controllability and non-controllability for a 1D wave equation with bilinear control

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ABSTRACT

We consider a linear wave equation, on the interval $(0,1)$, with bilinear control and Neumann boundary conditions. We study the controllability of this nonlinear control system, locally around a constant reference trajectory. We prove that the following results hold generically.

- For every $T>2$, this system is locally controllable in $H^3\times H^2$, in time T, with controls in $L^2((0,T),\mathbb{R})$.
- For $T=2$, this system is locally controllable up to codimension one in $H^3\times H^2$, in time T, with controls in $L^2((0,T),\mathbb{R})$: the reachable set is (locally) a non-flat submanifold of $H^3\times H^2$ with codimension one.
- For every $T<2$, this system is not locally controllable, more precisely, the reachable set, with controls in $L^2((0,T),\mathbb{R})$, is contained in a non-flat submanifold of $H^3\times H^2$, with infinite codimension.

The proof of these results relies on the inverse mapping theorem and second order expansions.

1. Introduction

1.1. Main result

The goal of this article is to investigate the exact controllability of the wave equation with bilinear controls. We consider the following 1D wave equation

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$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2}(t,x) = \dfrac{\partial^2 w}{\partial x^2}(t,x) + u(t)\mu(x)w(t,x), \quad x \in (0,1),\ t \in (0,T), \\ \dfrac{\partial w}{\partial x}(t,0) = \dfrac{\partial w}{\partial x}(t,1) = 0, \end{cases} \tag{1}$$

where $\mu \in H^2((0,1),\mathbb{R})$. The system (1) is a bilinear control system, in which

- the state is $(w, \frac{\partial w}{\partial t})$,
- the control is the real valued function $u : [0,T] \to \mathbb{R}$.

Let us introduce some conventions and notations. Unless otherwise specified, the functions are real valued. The operator A is defined by

$$D(A) := \{\varphi \in H^2(0,1);\ \varphi'(0) = \varphi'(1) = 0\}, \qquad A\varphi := -\varphi''. \tag{2}$$

Its eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ and eigenvectors $(\varphi_k)_{k\in\mathbb{N}}$ are

$$\begin{aligned} \lambda_0 &:= 0, \qquad \varphi_0(x) := 1, \\ \lambda_k &:= (k\pi)^2, \qquad \varphi_k(x) := \sqrt{2}\cos(k\pi x), \quad \forall k \in \mathbb{N}^*. \end{aligned} \tag{3}$$

We define the spaces

$$H^s_{(0)}(0,1) := D\big(A^{s/2}\big), \quad \forall s > 0, \tag{4}$$

equipped with the norm

$$\|\varphi\|_{H^s_{(0)}} := \left(\sum_{k=0}^{\infty} \big|k_*^s \langle \varphi, \varphi_k\rangle\big|^2\right)^{1/2},$$

where $k_* := \max\{k,1\}$, $\forall k \in \mathbb{N}$ and $\langle .,.\rangle$ is the $L^2(0,1)$-scalar product. Notice that

$$\begin{aligned} &H^1_{(0)}(0,1) = H^1(0,1), \\ &H^2_{(0)}(0,1) = \{\varphi \in H^2(0,1);\ \varphi'(0) = \varphi'(1) = 0\}, \\ &H^3_{(0)}(0,1) = \{\varphi \in H^3(0,1);\ \varphi'(0) = \varphi'(1) = 0\}. \end{aligned}$$

The goal of this article is to prove that, under generic assumptions on μ, the system (1) is locally controllable around the reference trajectory $(w(t,x) = 1, u(t) = 0)$, if and only if $T > 2$. The restriction $T > 2$ is not surprising because this wave equation has a propagation speed equal to 1, but, in this article, a particular attention is given to the case $T \leqslant 2$. Precisely, we prove the following results.

- When $T > 2$, the system (1) is locally controllable in $H^3_{(0)} \times H^2_{(0)}(0,1)$ with $L^2(0,T)$-controls.
- When $T = 2$, the system (1) is not locally controllable in $H^3_{(0)} \times H^2_{(0)}(0,1)$ with $L^2(0,T)$-controls because the reachable set is (locally) a non-flat submanifold of $H^3_{(0)} \times H^2_{(0)}(0,1)$ with codimension one. However, the system (1) is locally controllable up to codimension one: one can control the couple $(w - \int_0^1 w(x)\,dx, \partial w/\partial t)$. Moreover, for any reachable (local) target, there exists a unique (small) control allowing to reach this target.
- When $T < 2$, the system (1) is strongly not controllable: the reachable set, with $L^2(0,T)$-controls, is (locally) contained in a non-flat submanifold of $H^3_{(0)} \times H^2_{(0)}(0,1)$ with infinite codimension.

The goal of this article is the proof of the following theorem.

Theorem 1. *Let $\mu \in H^2(0,1)$. We assume*

$$\exists c > 0 \quad \text{such that} \quad \frac{c}{k_*^2} \leqslant \big|\langle \mu, \varphi_k \rangle\big|, \quad \forall k \in \mathbb{N}. \tag{5}$$

(1) *Let $T > 2$. There exist $\delta > 0$ and a C^1-map*

$$\Gamma_T : \mathcal{V}_T \to L^2(0,T)$$
$$(w_f, \dot{w}_f) \mapsto \Gamma_T(w_f, \dot{w}_f)$$

where

$$\mathcal{V}_T := \big\{(w_f, \dot{w}_f) \in H^3_{(0)} \times H^2_{(0)}(0,1);\ \|w_f - 1\|_{H^3_{(0)}} + \|\dot{w}_f\|_{H^2_{(0)}} < \delta\big\},$$

such that $\Gamma_T(1,0) = 0$ and for every $(w_f, \dot{w}_f) \in \mathcal{V}_T$, the solution of (1) *with initial condition*

$$\left(w, \frac{\partial w}{\partial t}\right)(0,x) = (1,0), \quad \forall x \in (0,1), \tag{6}$$

and control $u = \Gamma_T(w_f, \dot{w}_f)$ satisfies $(w, \frac{\partial w}{\partial t})(T) = (w_f, \dot{w}_f)$.

(2) *Let $T = 2$. There exist $\delta, r > 0$ and a C^1-map*

$$\Gamma_T : \mathcal{V}_T \to B_r\big[L^2(0,T)\big]$$
$$(\tilde{w}_f, \dot{w}_f) \mapsto \Gamma_T(\tilde{w}_f, \dot{w}_f)$$

where

$$\mathcal{V}_T := \left\{(\tilde{w}_f, \dot{w}_f) \in H^3_{(0)} \times H^2_{(0)}(0,1);\ \int_0^1 \tilde{w}_f(x)\,dx = 0,\ \|\tilde{w}_f\|_{H^3_{(0)}} + \|\dot{w}_f\|_{H^2_{(0)}} < \delta\right\},$$

$$B_r\big[L^2(0,T)\big] := \big\{u \in L^2\big((0,T),\mathbb{R}\big);\ \|u\|_{L^2} < r\big\},$$

such that $\Gamma_T(0,0) = 0$ and for every $(\tilde{w}_f, \dot{w}_f) \in \mathcal{V}_T$, $u \in B_r[L^2(0,T)]$, the solution of (1), (6) *satisfies*

$$w(T) - \int_0^1 w(T,x)\,dx = \tilde{w}_f \quad \text{and} \quad \frac{\partial w}{\partial t}(T) = \dot{w}_f,$$

if and only if $u = \Gamma_T(\tilde{w}_f, \dot{w}_f)$.

The reachable set from (6) *is, locally, a C^1-submanifold with codimension one. More precisely, there exist $r' > 0$ and a locally surjective nonlinear C^1-map $G_T : H^3_{(0)} \times H^2_{(0)}(0,1) \to \mathbb{R}$ such that, for every $u \in B_{r'}[L^2(0,T)]$, the solution of* (1), (6) *satisfies*

$$G_T\left[\left(w, \frac{\partial w}{\partial t}\right)(T)\right] = 0.$$

(3) *We assume*

$$\frac{(\mu^2)'(1) \pm (\mu^2)'(0)}{\mu'(1) \pm \mu'(0)} \neq \frac{\int_0^1 \mu(x)^2\,dx}{\int_0^1 \mu(x)\,dx}. \tag{7}$$

Let $T < 2$. The reachable set from (6) is, locally, contained in a C^1-submanifold of $H^3_{(0)} \times H^2_{(0)}(0,1)$, with infinite codimension, that does not coincide with its tangent space at $(1,0)$. More precisely, there exist $r > 0$, a strict vector subspace R_T of $H^3_{(0)} \times H^2_{(0)}(0,1)$ with infinite dimension and a locally surjective C^1-map

$$G_T : H^3_{(0)} \times H^2_{(0)}(0,1) \to R_T$$

such that, for every $u \in B_r[L^2(0,T)]$, the solution of (1), (6) satisfies

$$G_T\left[\left(w, \frac{\partial w}{\partial t}\right)(T)\right] = 0.$$

Remark 1. Notice that, when (5) holds, then $\int_0^1 \mu = \langle \mu, \varphi_0 \rangle \neq 0$ and $\mu'(1) \pm \mu'(0) \neq 0$. Indeed, thanks to the Riemann–Lebesgue Lemma, we have

$$\begin{aligned}\langle \mu, \varphi_k \rangle &= \frac{\sqrt{2}}{(k\pi)^2}\big((-1)^k \mu'(1) - \mu'(0)\big) - \frac{\sqrt{2}}{(k\pi)^2} \int_0^1 \mu''(x)\cos(k\pi x)\,dx \\ &= \frac{\sqrt{2}}{(k\pi)^2}\big((-1)^k \mu'(1) - \mu'(0)\big) + o\left(\frac{1}{k^2}\right) \quad \text{when } k \to +\infty. \end{aligned} \tag{8}$$

Therefore each term in (7) is well defined.

Remark 2. The assumptions (5) and (7) hold simultaneously, for example, with $\mu(x) = x^2$, because

$$\langle x^2, \varphi_0 \rangle = \int_0^1 x^2\,dx = \frac{1}{3},$$

$$\langle x^2, \varphi_k \rangle = \int_0^1 x^2 \sqrt{2}\cos(k\pi x)\,dx = \frac{(-1)^k 2\sqrt{2}}{(k\pi)^2}, \quad \forall k \in \mathbb{N}^*,$$

$$\frac{(\mu^2)'(1) \pm (\mu^2)'(0)}{\mu'(1) \pm \mu'(0)} = 2, \quad \text{and} \quad \frac{\int_0^1 \mu(x)^2\,dx}{\int_0^1 \mu(x)\,dx} = \frac{3}{5}. \tag{9}$$

But (5) and (7) are not always satisfied. For example, (5) does not hold when $\langle \mu, \varphi_k \rangle = 0$ for some $k \in \mathbb{N}$, or when μ has a symmetry with respect to $x = 1/2$. However, the assumptions (5) and (7) are generic in $H^2(0,1)$ (see Appendix A for a proof), thus, Theorem 1 is very general.

Remark 3. In Theorem 1, the spaces are optimal. Indeed, we will see in this article that, for every control $u \in L^2(0,T)$, there exists a unique solution of (1), (6) and it satisfies

$$\left(w, \frac{\partial w}{\partial t}\right)(T) \in H^3_{(0)} \times H^2_{(0)}(0,1).$$

Remark 4. Let us mention Ref. [14] by Coron, Rouchon and the author, in which a negative result, similar to the statement (3) of Theorem 1 is proved. In this reference, we consider the Bloch equation

$$\frac{\partial M}{\partial t}(t,\omega)=\begin{pmatrix}0 & -\omega & v(t)\\ \omega & 0 & u(t)\\ -v(t) & -u(t) & 0\end{pmatrix}M(t,\omega),\quad t\in[0,+\infty),\ \omega\in\big(\omega_*,\omega^*\big),$$

where $-\infty\leqslant\omega_*<\omega^*\leqslant+\infty$, $u,v:[0,+\infty)\to\mathbb{R}$. It is a control system where the state is the function $M=M(t,\omega)$ and the control is $(u,v):[0,+\infty)\to\mathbb{R}^2$. This system is a prototype for infinite-dimensional bilinear control systems, with continuous spectrum. In [14, Theorem 2], we prove that, when $\omega_*=-\infty$ and $\omega^*=+\infty$, then, this system is not exactly controllable, locally around the reference trajectory $(M_{ref}=e_3, u_{ref}=0, v_{ref}=0)$, with small $L^2(0,T)$-controls. The proof consists in proving that the reachable set from $M(0,\omega)=e_3$, in time T, with small $L^2(0,T)$-controls, is locally a non-flat submanifold of some functional space, with infinite codimension. The proof of this result relies on the inverse mapping theorem, and second order expansions, as in the present article.

In this article, the same letter C denotes a positive constant that can change from one line to another one.

1.2. Sketch of the proof

The proof of Theorem 1 relies on the inverse mapping theorem, applied to the end point map

$$\Theta_T: u\mapsto\left(w,\frac{\partial w}{\partial t}\right)(T),\tag{10}$$

where w solves (1), (6).

First, we prove that, for every $T>0$, the map Θ_T is C^1 between the following spaces

$$\Theta_T: L^2(0,T)\to H^3_{(0)}\times H^2_{(0)}(0,1).$$

Then, the local controllability of the nonlinear system when $T>2$ (i.e. the local surjectivity of Θ_T) is a consequence of the surjectivity of $d\Theta_T(0)$. And the non-controllability of the nonlinear system when $T\leqslant 2$ is a consequence of the injectivity and non-surjectivity of $d\Theta_T(0)$. More precisely, we prove the following results.

- When $T>2$, the continuous linear map $d\Theta_T(0): L^2(0,T)\to H^3_{(0)}\times H^2_{(0)}(0,1)$ has a continuous right inverse. This means that the linearized system around the reference trajectory $(w(t,x)=1, u(t)=0)$ is controllable, in time T, in $H^3_{(0)}\times H^2_{(0)}(0,1)$, with controls in $L^2(0,T)$.
- When $T=2$, the continuous linear map $d\Theta_T(0): L^2(0,T)\to H^3_{(0)}\times H^2_{(0)}(0,1)$ is injective, its image R_T is a vector subspace of $H^3_{(0)}\times H^2_{(0)}(0,1)$ with codimension one, and the map $d\Theta_T(0): L^2(0,T)\to R_T$ has a continuous (left and right) inverse. This means that the linearized system around the reference trajectory $(w(t,x)=1, u(t)=0)$ is controllable up to codimension one, in time T, in $H^3_{(0)}\times H^2_{(0)}(0,1)$, with controls in $L^2(0,T)$: it misses exactly one direction. Moreover, for every reachable target, there exists a unique control allowing this motion in time $T=2$.
- When $T<2$, the continuous linear map $d\Theta_T(0): L^2(0,T)\to H^3_{(0)}\times H^2_{(0)}(0,1)$ is injective, its image R_T is a vector subspace of $H^3_{(0)}\times H^2_{(0)}(0,1)$ with infinite codimension and the map $d\Theta_T(0): L^2(0,T)\to R_T$ has a continuous (left and right) inverse. This means that the linearized system around the reference trajectory $(w(t,x)=1, u(t)=0)$ is strongly not controllable: it misses an infinite number of directions. Moreover, for every reachable target, there exists a unique control allowing this motion.

Thus, by applying the inverse mapping theorem, we prove that the reachable set in time $T \leqslant 2$ is a strict submanifold of $H^3_{(0)} \times H^2_{(0)}(0,1)$. Now, let us explain how we prove this submanifold is not flat. First, we prove that the image of the quadratic form $d^2\Theta_T(0)$ is not contained in the image of the linear map $d\Theta_T(0)$. Then, thanks to a second order expansion of Θ_T around 0, we see that the (local) submanifold (i.e. the image of Θ_T) does not coincide with its tangent space at $(1,0)$ (i.e. the image of $d\Theta_T(0)$).

Remark 5. The first (local) exact controllability result, for an infinite-dimensional bilinear system, has been proved in [10], for a Schrödinger equation. In [10], the strategy is the same as in this article: first, we prove the controllability of the linearized system and then, we conclude by applying an inverse mapping theorem. However, because of an a priori loss of regularity, we use the Nash–Moser implicit function theorem, instead of the classical inverse mapping theorem. Thus, the analysis is quite complicated.

One of the interests of the present article is to provide an example of infinite-dimensional bilinear control system (i.e. Eq. (1)), for which the proof of the (local) exact controllability relies only on the classical inverse mapping theorem, and is rather simple. In order to avoid the use of the Nash–Moser theorem, we emphasize a 'hidden' regularization effect for Eq. (1).

1.3. A review of previous results

1.3.1. A previous negative result for this equation

The following result is due to Ball, Marsden and Slemrod [5, Theorem 3.6].

Theorem 2. *Let X be a Banach space with infinite dimension. Let $\mathcal{A}$ be the generator of a C^0-group of bounded operators of X and $\mathcal{B}$ be a bounded operator of X. For $w_0 \in X$ and $p \in L^1_{loc}([0,+\infty),\mathbb{R})$, $U[T;p,w_0]$ denotes the value at time T of the unique weak solution of*

$$\begin{cases} \dfrac{dw}{dt} = \mathcal{A}w + p(t)\mathcal{B}w(t), \\ w(0) = w_0. \end{cases} \tag{11}$$

For every $w_0 \in X$, the reachable set from w_0,

$$\mathcal{R}(w_0) := \left\{U[T;p,w_0];\ T \geqslant 0,\ p \in L^r_{loc}\big([0,+\infty),\mathbb{R}\big),\ r > 1\right\}$$

has an empty interior in X.

A consequence of this theorem is the non-controllability of the system (11), in X, with controls $p \in L^r_{loc}([0,+\infty),\mathbb{R})$, $r > 1$.

Theorem 2 applies to the system (1), written in first order form, with

$$\begin{aligned} &X := H^2_{(0)} \times H^1(0,1), \\ &D(\mathcal{A}) := H^2_{(0)} \times H^1(0,1), \qquad \mathcal{A} := \begin{pmatrix} 0 & I \\ \partial_x^2 & 0 \end{pmatrix}, \\ &D(\mathcal{B}) := L^2 \times L^2(0,1), \qquad \mathcal{B} := \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix}. \end{aligned} \tag{12}$$

Indeed, for every $(w_0, \dot{w}_0) \in H^2_{(0)} \times H^1(0,1)$, we have

$$e^{\mathcal{A}t}\begin{pmatrix} w_0 \\ \dot{w}_0 \end{pmatrix} = \begin{pmatrix} w(t) \\ \dot{w}(t) \end{pmatrix},$$

where

$$w(t) = \big(\langle w_0, \varphi_0\rangle + \langle \dot w_0, \varphi_0\rangle t\big)\varphi_0 + \sum_{k=1}^{\infty}\left(\langle w_0, \varphi_k\rangle \cos(\sqrt{\lambda_k}t) + \frac{1}{\sqrt{\lambda_k}}\langle \dot w_0, \varphi_k\rangle \sin(\sqrt{\lambda_k}t)\right)\varphi_k,$$

$$\dot w(t) = \langle \dot w_0, \varphi_0\rangle\varphi_0 + \sum_{k=1}^{\infty}\big(-\sqrt{\lambda_k}\langle w_0, \varphi_k\rangle \sin(\sqrt{\lambda_k}t) + \langle \dot w_0, \varphi_k\rangle \cos(\sqrt{\lambda_k}t)\big)\varphi_k.$$

Thus $\mathcal{A}$ generates a C^0-group of bounded operators of X. Moreover, when $\mu \in H^1(0,1)$, the restriction of $\mathcal{B}$ to X defines a bounded operator of X. For a precise definition of weak solutions of (1), we refer to Proposition 2. Thanks to Theorem 2, we have the following non-controllability result for (1).

Proposition 1. *Let $\mu \in H^1(0,1)$, $T > 0$ and $(w_0, \dot w_0) \in H^2_{(0)} \times H^1(0,1)$. For $u \in L^1_{loc}[0,+\infty)$, $U[T; u, w_0, \dot w_0]$ denotes the value at time T of the weak solution of (1) with initial condition*

$$\left(w, \frac{\partial w}{\partial t}\right)(0) = (w_0, \dot w_0).$$

The reachable set from $(w_0, \dot w_0)$,

$$\mathcal{R}(w_0, \dot w_0) := \big\{U[T; u, w_0, \dot w_0];\ T > 0,\ u \in L^r_{loc}[0,+\infty),\ r > 1\big\}$$

has an empty interior in $H^2_{(0)} \times H^1(0,1)$.

Thus, the system (1) is not controllable in $H^2_{(0)} \times H^1(0,1)$ with controls in $L^r_{loc}[0,+\infty)$, $r > 1$.

Remark 6. Notice that Theorem 2 does not apply with

$$\tilde X := H^3_{(0)} \times H^2_{(0)}(0,1).$$

Indeed, $(e^{t\mathcal{A}})_{t\in\mathbb{R}}$ defines a C^0-group of bounded operators of $\tilde X$, but $\mathcal{B}$ does not map $\tilde X$ into $\tilde X$: for $\varphi \in H^3_{(0)}(0,1)$ (i.e. $\varphi \in H^3(0,1)$ and $\varphi'(0) = \varphi'(1) = 0$), we have $(\mu\varphi)'(0) = \mu'(0)\varphi(0)$ and $(\mu\varphi)'(1) = \mu'(1)\varphi(1)$ that may not vanish.

Such a negative controllability result may be rather weak, because it does not prevent from positive controllability results, in different functional spaces. For example, the reachable set $\mathcal{R}(w_0, \dot w_0)$ may be the whole space $H^3_{(0)} \times H^2_{(0)}(0,1)$ (which has an empty interior in $H^2_{(0)} \times H^1_{(0)}(0,1)$) and then the system would be controllable in $H^3_{(0)} \times H^2_{(0)}(0,1)$. In this article, we prove that this is indeed the case, at least locally, when $T > 2$. On the contrary, when $T < 2$, the system (1) is not controllable in a very strong sense (stronger that Ball, Marsden and Slemrod's one): the reachable set $\mathcal{R}(1,0)$ is locally a non-flat submanifold of $H^3_{(0)} \times H^2_{(0)}(0,1)$, with infinite codimension. In particular, when $T < 2$, no positive exact controllability result can be expected in smoother spaces (because the manifold is not flat). Thus, the results of this article complete the ones of [5].

The same kind of situation arises with bilinear Schrödinger or beam equations (see [15,10–13]).

1.3.2. Iterated Lie brackets for general bilinear systems

Now, let us discuss the exact controllability of general bilinear systems.

First, the controllability of *finite-dimensional* bilinear control systems (i.e. modeled by an ordinary differential equation) is well understood. Let us consider the control system

$$\frac{dX}{dt} = AX + u(t)BX, \tag{13}$$

where $X(t) \in \mathbb{R}^n$ is the state, A, B are $n * n$ matrices, and $t \mapsto u(t) \in \mathbb{R}$ is the control. The controllability of (13) is linked to the rank of the Lie algebra spanned by A and B (see for example [2] by Agrachev and Sachkov, [21, Chapter 3] by Coron or [22] by D'Alessandro).

In *infinite dimension*, there are cases where formal computations on iterated Lie brackets provide the right intuition. For instance, it holds for the non-controllability of the harmonic quantum oscillator with bilinear control (see [37] by Mirrahimi and Rouchon). However, such formal computations on Lie brackets are sometimes less powerful in infinite dimension than in finite dimension. It is precisely the case for our system. Let us compute formally the iterated Lie brackets of the operators $\mathcal{A}$ and $\mathcal{B}$ defined by (12), at the point

$$\mathcal{W}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We have

$$[\mathcal{A}, \mathcal{B}]\mathcal{W}_0 = (\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A})\mathcal{W}_0 = \mathcal{A}\mathcal{B}\mathcal{W}_0 = \begin{pmatrix} \mu \\ 0 \end{pmatrix}$$

(μ is assumed to belong to $H^2(0,1)$). Notice that $[\mathcal{A}, \mathcal{B}]\mathcal{W}_0$ does not belong to $D(\mathcal{A})$ because μ' may not vanish at 0 and 1. Thus, in order to compute the iterated Lie bracket $[\mathcal{A}, [\mathcal{A}, \mathcal{B}]]\mathcal{W}_0$, one needs to extend the definition of $\mathcal{A}$ to couples $(w_0, w_1) \in H^2 \times H^1(0,1)$ such that w_0' does not vanish at 0 and 1. A natural choice is

$$\mathcal{A}\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} := \begin{pmatrix} w_1 \\ w_0'' - w_0'(1)\delta_1 + w_0'(0)\delta_0 \end{pmatrix}, \quad \forall (w_0, w_1) \in H^2 \times H^1(0,1), \tag{14}$$

where δ_0 and δ_1 are Dirac masses at the points $x = 0$ and $x = 1$. With this definition, we get formally

$$\big[\mathcal{A}, [\mathcal{A}, \mathcal{B}]\big]\mathcal{W}_0 = \begin{pmatrix} 0 \\ \mu'' - \mu'(1)\delta_1 + \mu'(0)\delta_0 \end{pmatrix},$$
$$\big[\mathcal{A}, \big[\mathcal{A}, [\mathcal{A}, \mathcal{B}]\big]\big]\mathcal{W}_0 = \begin{pmatrix} \mu'' - \mu'(1)\delta_1 + \mu'(0)\delta_0 \\ 0 \end{pmatrix}.$$

But again, $[\mathcal{A}, [\mathcal{A}, [\mathcal{A}, \mathcal{B}]]]\mathcal{W}_0$ does not belong to $H^2 \times H^1(0,1)$, thus the definition (14) cannot be used to compute $[\mathcal{A}, [\mathcal{A}, [\mathcal{A}, [\mathcal{A}, \mathcal{B}]]]]\mathcal{W}_0$. Moreover, even if we could give a sense to any iterated Lie bracket, because of the presence of Dirac masses, it would not be clear which space the Lie brackets should generate in case of local controllability around the reference trajectory $(w(t,x) = 1, u(t) = 0)$. Therefore, the way the Lie algebra rank condition could be used directly in infinite dimension is not clear.

Finally, let us cite important articles about the controllability of PDEs, in which positive results are proved by applying such geometric control methods but to the (finite-dimensional) Galerkin approximations of the equation. In [3] by Agrachev and Sarychev and [41] by Shirikyan, the authors prove exact controllability results for dissipative equations. In [19] by Chambrion, Mason, Sigalotti and Boscain, the authors prove approximate controllability results for Schrödinger equations. At present, no exact controllability result has been proved, with such geometric control methods, for non-dissipative PDEs.

1.3.3. Wave equation with bilinear control

Now, let us cite few articles about the controllability of wave equations with bilinear control. In [31], Khapalov considers the following control system

$$\begin{cases} \frac{\partial^2 y}{\partial t^2}(t,x) = \frac{\partial^2 y}{\partial x^2}(t,x) + v(t,x)y(t,x) - \gamma(t)\frac{\partial y}{\partial t}(t,x) - F(t,x,y), \\ x \in (0,1),\ t \in (0,+\infty), \\ y(t,0) = y(t,1) = 0, \end{cases} \tag{15}$$

in which the controls are $v \in L^\infty((0,+\infty)\times(0,1))$ and $\gamma \in L^\infty(0,+\infty)$. This equation represents a semilinear vibrating string, with clamped ends, with a variable axial load $v(t,x)$ and a variable damping gain $\gamma(t)$. The nonlinearity F is fixed. Such controllability problems may arise in the context of 'smart materials', whose properties can be altered by applying various factors (temperature, electric current, magnetic field). In [31], the author proves the global approximate controllability to nonnegative equilibrium states: $\forall (y_0,y_1) \in H^1_0 \times L^2(0,1)$ with $(y_0,y_1) \neq 0$, $\forall y_d \in L^1(0,1)$ with $y_d \geqslant 0$ a.e. on $(0,1)$, $\forall \epsilon > 0$ there exist $T = T(\epsilon, y_0, y_1, y_d) > 0$ and piecewise-constant-in-time controls (v,γ) such that the solution of (15) with initial condition

$$\left(y, \frac{\partial y}{\partial t}\right)(0) = (y_0, y_1),$$

satisfies

$$\|y(T) - y_d\|_{L^2} + \left\|\frac{\partial y}{\partial t}\right\|_{L^2} < \epsilon.$$

The proof consists in, first, finding a control (v,γ) that realizes the approximate controllability for the homogeneous truncated system (i.e. with $F=0$), and then, proving that, the nonlinear system with the same control follows closely the linear one. We also refer to [30] and [28] by Khapalov for similar results on similar equations (with $\gamma = 0$ or $F = 0$) and to [33] for a general survey.

1.3.4. Wave equation with linear controls

Now, let us cite few articles about the controllability of wave equations with distributed or boundary controls acting linearly on the state. There is a huge literature on this subject. One of the best result has been obtained by Bardos, Lebeau and Rauch in [6]. See also the paper [18] by Burq and Gérard, the paper [17] by Burq for improvements or simpler proofs, and the paper [43] by Zuazua for semilinear equations. Let us also mention the survey paper [40] by Russell and the books [21] by Coron, [25] by Fursikov and Imanuvilov, [35] by Jacques Louis Lions and [34] by Komornik, where one can find plenty of results and useful references.

1.3.5. Other results about infinite-dimensional bilinear systems

In recent years, important progress have been made about the controllability of Schrödinger equations with bilinear control.

The first results were negative: in [42], Turinici adapted Theorem 2 to linear Schrödinger equations; in [27], Illner, Lange and Teismann adapted it to nonlinear equations; in [37], Mirrahimi and Rouchon proved a stronger negative result for the quantum harmonic oscillator.

Concerning exact controllability issues, local results for 1D models have been proved in [10,11] by the author, who proposed a simplified proof in a joint work with Laurent [15]; almost global results have been proved in [13], by Coron and the author. In [20], Coron proved the existence of a positive minimal time required for the local controllability of the 1D model studied in [10].

Now, let us cite some approximate controllability results. In [16] Mirrahimi and the author proved the global approximate controllability, in infinite time, for a 1D model and in [36] Mirrahimi proved a similar result for equations involving a continuous spectrum. Approximate controllability, in finite time, has been proved for particular models by Adami and Boscain in [1], by using adiabatic theory and intersection of the eigenvalues in the space of controls. Approximate controllability, in finite time, for more general models, have been studied by 3 teams, with different tools: by Chambrion, Mason, Sigalotti, Boscain in [19], with geometric control methods; by Nersesyan in [38,39] with feedback controls and variational methods; and by Ervedoza and Puel in [24] thanks to a simplified model.

Let us emphasize that the local exact controllability of [15] and the global approximate controllability of [38,39] can be put together in order to get the global exact controllability of 1D models in large time (see [39]).

Optimal control techniques have also been investigated for Schrödinger equations with a nonlinearity of Hartree type in [7,8] by Baudouin, Kavian, Puel and in [23] by Cancés, Le Bris, Pilot. An algorithm for the computation of such optimal controls is studied in [9] by Baudouin and Salomon.

Finally, let us also cite [29,32] by Khapalov for approximate controllability results about the heat equation, [12] by the author for an exact controllability result about a 1D beam equation, and [14] for a negative exact controllability result and positive approximate controllability results for the Bloch equation.

1.4. A toy model for 2D quantum systems

Finally, let us emphasize that the system (1) may be considered as a toy model for 2D (i.e. $n=2$) Schrödinger bilinear control systems,

$$\begin{cases} i\dfrac{\partial\psi}{\partial t} = -\Delta\psi - u(t)\mu(x)\psi, & x\in\Omega,\ t\in[0,T], \\ \psi(t,x)=0, & x\in\partial\Omega, \end{cases} \tag{16}$$

where Ω is a bounded regular open subset of $\mathbb{R}^n$, and $\mu:\Omega\to\mathbb{R}$ is a smooth function.

The system (16) represents a quantum particle in an infinite square potential well Ω, subjected to a $1D$ uniform (in space) time dependent electric field with amplitude $u(t)$. The function μ is the dipolar moment of the particle. The exact controllability in finite time of such systems is a challenging problem, when $n\geqslant 2$.

In the references above, the approximate controllability results [19,38,39] hold in any space dimension ($\forall n\in\mathbb{N}^*$), but the local exact controllability results [15] hold only in 1D ($n=1$). Thus, the global exact controllability is proved only in 1D (see [39]). It would be interesting to know if the same program works in any dimension, i.e. if the local exact controllability result also holds in 2D and 3D.

A key point in the proof of [15] is the following property: the eigenvalues of the Laplacian on a 1D domain (take, for instance $\lambda_k=(k\pi)^2$, $k\in\mathbb{N}^*$ with $\Omega=(0,1)$) satisfy a gap condition:

$$\exists\delta>0 \quad \text{such that} \quad \lambda_{k+1}-\lambda_k\geqslant\delta, \quad \forall k\in\mathbb{N}^*.$$

Such a property does not hold on 2D and 3D domains, for which we only know the Weyl formula,

$$\exists d>0,\ \alpha\in(0,n/2) \quad \text{such that} \quad \operatorname{Card}\big\{k\in\mathbb{N};\ \mu_k\in[0,t]\big\} = dt^{n/2}+O\big(t^\alpha\big) \quad \text{when } t\to+\infty. \tag{17}$$

The system (1) may be considered as a toy model for (16) with $n=2$. Indeed, the spectrum of the underlying operator $\mathcal{A}$ defined by (12) satisfies the Weyl formula (17) with $n=2$, its eigenvalues are $(ik\pi)_{k\in\mathbb{N}}$ with the associated eigenvectors $(X_k)_{k\in\mathbb{N}}$,

$$X_k := \begin{pmatrix} \varphi_k \\ ik\pi\varphi_k \end{pmatrix}, \quad \forall k \in \mathbb{N}^*$$

(see (3) for a definition of φ_k). The control system (1) is easier to deal with than (16) because the spectrum of the underlying operator has more structure.

1.5. Structure of this article

This article is organized as follows.

Section 2 is dedicated to the well posedness of the Cauchy problem (1), (6).

In Section 2.1, we state classical results about existence, uniqueness, regularity, and bounds for the solutions of a more general Cauchy problem.

In Section 2.2, improving these classical results, we prove that the end point map Θ_T, defined by (10), is C^1 from $L^2(0,T)$ to $H^3_{(0)} \times H^2_{(0)}(0,1)$.

In Section 3 we consider the linearized system of (1) around the reference trajectory $(w(t,x)=1, u(t)=0)$. We study its controllability in $H^3_{(0)} \times H^2_{(0)}(0,1)$ with $L^2(0,T)$-controls.

In Section 4, we study the second order term around $(w(t,x)=1, u(t)=0)$. We prove that, for every $T \geqslant 2$, the image of the quadratic form $d^2\Theta_T(0)$ is not contained in the image of the linear map $d\Theta_T(0)$.

In Section 5 we prove Theorem 1, by applying the inverse mapping theorem.

Finally, Section 6 is dedicated to conclusions, open problems and perspectives.

2. Well posedness and C^1 regularity of the end point map

This section is dedicated to the statement of existence, uniqueness, regularity results, and bounds for the solutions of the Cauchy problem

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2}(t,x) = \dfrac{\partial^2 w}{\partial x^2}(t,x) + u(t)\mu(x)w(t,x) + f(t,x), & x \in (0,1),\ t \in \mathbb{R}_+, \\ \dfrac{\partial w}{\partial x}(t,0) = \dfrac{\partial w}{\partial x}(t,1) = 0, \\ w(0,x) = w_0(x), \\ \dfrac{\partial w}{\partial t}(0,x) = \dot{w}_0(x). \end{cases} \tag{18}$$

These results are presented in Section 2.1. Then, in Section 2.2, improving the results of Section 2.1, we prove that the map Θ_T, defined by (10), is of class C^1 from $L^2(0,T)$ to $H^3_{(0)} \times H^2_{(0)}(0,1)$.

2.1. Existence, uniqueness, regularity and bounds

In order to study the well posedness of (18), it is convenient to write it in first order form. With the notations

$$\mathcal{W} := \begin{pmatrix} w \\ \frac{\partial w}{\partial t} \end{pmatrix}, \qquad \mathcal{W}_0 := \begin{pmatrix} w_0 \\ \dot{w}_0 \end{pmatrix}, \qquad \mathcal{F}(t,x) := \begin{pmatrix} 0 \\ f(t,x) \end{pmatrix},$$

and $\mathcal{A}$, $\mathcal{B}$ defined by (12), Eq. (18) may be written

$$\begin{cases} \dfrac{\partial \mathcal{W}}{\partial t}(t,x) = \mathcal{A}\mathcal{W}(t,x) + u(t)\mathcal{B}\mathcal{W}(t,x) + \mathcal{F}(t,x), \\ \mathcal{W}(0) = \mathcal{W}_0. \end{cases} \tag{19}$$

The operator $\mathcal{A}$ generates a C^0-group of bounded operators of $H^{s+1}_{(0)} \times H^s_{(0)}(0,1)$, for every $s \geqslant 0$ (see (4) for a definition) and the operator $\mathcal{B}$ is bounded on $H^2_{(0)} \times H^1(0,1)$ when $\mu \in W^{1,\infty}(0,1)$. These two facts allow to prove the following classical existence result of weak solutions for (18).

Proposition 2. *Let $\mu \in H^1(0,1)$ and $T > 0$. There exists $C = C(\mu, T) > 0$ such that, for every $u \in L^1(0,T)$, $(w_0, \dot{w}_0) \in H^2_{(0)} \times H^1(0,1)$, and $f \in L^1((0,T), H^1(0,1))$, there exists a unique weak solution of* (18)*, i.e. a function*

$$\left(w, \frac{\partial w}{\partial t}\right) \in C^0\big([0,T], H^2_{(0)} \times H^1(0,1)\big)$$

such that the following equality holds in $H^2_{(0)} \times H^1(0,1)$, for every $t \in [0,T]$,

$$\mathcal{W}(t) = e^{\mathcal{A}t}\mathcal{W}_0 + \int_0^t e^{\mathcal{A}(t-\tau)}\big(u(\tau)\mathcal{B}\mathcal{W}(\tau) + \mathcal{F}(\tau)\big)\, d\tau, \tag{20}$$

and this weak solution satisfies

$$\left\| \left(w, \frac{\partial w}{\partial t}\right) \right\|_{C^0([0,T], H^2_{(0)} \times H^1)} \leqslant C\big(\|(w_0, \dot{w}_0)\|_{H^2_{(0)} \times H^1} + \|f\|_{L^1((0,T),H^1)}\big) e^{C\|u\|_{L^1}}. \tag{21}$$

Proof. The existence and uniqueness come from a fixed point argument on the map F defined on $C^0([0,T], H^2_{(0)} \times H^1(0,1))$ by $F(\mathcal{W}) := \xi$ where

$$\xi(t) = e^{\mathcal{A}t}\mathcal{W}_0 + \int_0^t e^{\mathcal{A}(t-\tau)}\big(u(\tau)\mathcal{B}\mathcal{W}(\tau) + \mathcal{F}(\tau)\big)\, d\tau, \quad \forall t \in [0,T].$$

F maps $C^0([0,T], H^2_{(0)} \times H^1(0,1))$ into itself because $\mathcal{B}$ and $e^{\mathcal{A}t}$ preserve $H^2_{(0)} \times H^1(0,1)$. When $\|u\|_{L^1((0,T),\mathbb{R})}$ is small enough, then F is a contraction, because

$$\begin{aligned}
\big\|F(\mathcal{W}_1)(t) - F(\mathcal{W}_2)(t)\big\|_{H^2_{(0)} \times H^1} &= \left\| \int_0^t e^{\mathcal{A}(t-\tau)} u(\tau)\mathcal{B}\big(\mathcal{W}_1(\tau) - \mathcal{W}_2(\tau)\big)\, d\tau \right\|_{H^2_{(0)} \times H^1} \\
&\leqslant \int_0^t \big|u(\tau)\big| \big\|e^{\mathcal{A}(t-\tau)}\mathcal{B}\big(\mathcal{W}_1(\tau) - \mathcal{W}_2(\tau)\big)\big\|_{H^2_{(0)} \times H^1}\, d\tau \\
&\leqslant C_1 \int_0^t \big|u(\tau)\big| \big\|\mathcal{B}\big(\mathcal{W}_1(\tau) - \mathcal{W}_2(\tau)\big)\big\|_{H^2_{(0)} \times H^1}\, d\tau \\
&\leqslant C_1 C_2 \|u\|_{L^1(0,T)} \|\mathcal{W}_1 - \mathcal{W}_2\|_{C^0([0,T], H^2_{(0)} \times H^1)},
\end{aligned}$$

where $C_1 = C_1(\mathcal{A}, T)$, $C_2 = C_2(\mathcal{B}) > 0$. Thus, F has a unique fixed point $\mathcal{W} \in C^0([0,T], H^2_{(0)} \times H^1)$ that satisfies (20). If $\|u\|_{L^1((0,T),\mathbb{R})}$ is not small, one may use $0 = T_0 < T_1 < \cdots < T_n = T$ where, for $i = 0, \ldots, n-1$, $\|u\|_{L^1(T_i, T_{i+1})}$ is small enough so that the previous result holds on $[T_i, T_{i+1}]$, for

$i = 0, \dots, n-1$. Then we glue the solutions defined on $[T_0, T_1]$, $[T_1, T_2], \dots, [T_{n-1}, T_n]$. We deduce from the equality (20) that

$$\big\|\mathcal{W}(t)\big\|_{H^2_{(0)}\times H^1} \leqslant C_1\left(\|\mathcal{W}_0\|_{H^2_{(0)}\times H^1} + \|F\|_{L^1((0,T),H^2_{(0)}\times H^1)} + \int_0^t \big|u(\tau)\big| C_2 \big\|\mathcal{W}(\tau)\big\|_{H^2_{(0)}\times H^1}\,d\tau\right),$$

and Gronwall's Lemma gives (21). □

Remark 7. This proof does not work with $H^3_{(0)} \times H^2_{(0)}(0,1)$ instead of $H^2_{(0)} \times H^1(0,1)$ because $\mathcal{B}$ does not conserve $H^3_{(0)} \times H^2_{(0)}(0,1)$. Indeed, for $\varphi \in H^3_{(0)}(0,1)$ (i.e. $\varphi \in H^3(0,1)$ and $\varphi'(0) = \varphi'(1) = 0$), we have $(\mu\varphi)' = \mu'\varphi$ at $x = 0, 1$ that may not vanish. Thus it is not obvious that the map Θ_T defined by (10) maps $L^2(0,T)$ into $H^3_{(0)} \times H^2_{(0)}$.

2.2. C^1 regularity of the end point map

Thanks to Proposition 2, we can consider the map Θ_T defined by (10), and we know that it is continuous from $L^2(0,T)$ to $H^2_{(0)} \times H^1(0,1)$. The goal of this section is the proof of the following hidden regularization effect.

Theorem 3. *Let $T > 0$ and $\mu \in H^2(0,1)$. The map Θ_T defined by (10) is C^1 between the following spaces*

$$\Theta_T : L^2(0,T) \to H^3_{(0)} \times H^2_{(0)}(0,1).$$

Moreover, for every $u, v \in L^2(0,T)$, we have

$$d\Theta_T(u).v = \left(W, \frac{\partial W}{\partial t}\right)(T) \tag{22}$$

where W is the weak solution of

$$\begin{cases} \dfrac{\partial^2 W}{\partial t^2} = \dfrac{\partial^2 W}{\partial x^2} + u(t)\mu(x)W(t,x) + v(t)\mu(x)w(t,x), \quad x \in (0,1),\ t \in (0,T), \\ \dfrac{\partial W}{\partial x}(t,0) = \dfrac{\partial W}{\partial x}(t,1) = 0, \\ W(0,x) = 0, \\ \dfrac{\partial W}{\partial t}(0,x) = 0, \end{cases} \tag{23}$$

and w is the weak solution of (1), (6)*.*

In Section 2.2.1, we state preliminary results useful for the proof of Theorem 3, which is detailed in Section 2.2.2.

2.2.1. Preliminaries

Lemma 1. *Let $T > 0$. There exists $C = C(T) > 0$ such that, for every $g \in L^2(0,T)$,*

$$\left(\sum_{k\in\mathbb{N}}\left|\int_0^T g(t)e^{ik\pi t}\,dt\right|^2\right)^{1/2} \leqslant C\|g\|_{L^2(0,T)}.$$

Proof. Let $n \in \mathbb{N}^*$ be such that $2(n-1) < T \leqslant 2n$. Extending g by zero on $[T, 2n]$ and using the Bessel–Parseval inequality, we get

$$\sum_{k\in\mathbb{N}} \left| \frac{1}{2n} \int_0^T g(t) e^{ik\pi t}\, dt \right|^2 \leqslant \frac{1}{2n} \int_0^T \left|g(t)\right|^2 dt.$$

Thus, Lemma 1 holds with $C(T) := \sqrt{2n}$. □

For $s \geqslant 0$, we use the spaces

$$h^s\left(\mathbb{N}^*, \mathbb{C}\right) := \left\{ a = (a_k)_{k\in\mathbb{N}^*} \in \mathbb{C}^{\mathbb{N}};\ \sum_{k=1}^{\infty} \left|k^s a_k\right|^2 < +\infty \right\}$$

equipped with the norm

$$\|a\|_{h^s} := \left(\sum_{k=1}^{\infty} \left|k^s a_k\right|^2 \right)^{1/2}.$$

Let us recall that for a function $g \in L^2(0,1)$, the following equivalence holds: $g \in H^s_{(0)}((0,1), \mathbb{C}) \Leftrightarrow (\langle g, \varphi_k \rangle)_{k\in\mathbb{N}} \in h^s(\mathbb{N}^*, \mathbb{C})$. Thanks to Lemma 1, we have the following result.

Lemma 2. *Let $T > 0$. There exists $C = C(T) > 0$ such that, for every $w \in L^2(0,T)$, $f \in C^0([0,T], H^2(0,1))$, the sequence $S_0 = (S_{0,k})_{k\in\mathbb{N}^*}$ defined by*

$$S_{0,k} := \int_0^T w(t) \left\langle f(t), \varphi_k \right\rangle e^{i\sqrt{\lambda_k} t}\, dt, \quad \forall k \in \mathbb{N}^*,$$

belongs to $h^2(\mathbb{N}^*, \mathbb{C})$ and

$$\|S_0\|_{h^2} \leqslant C \|w\|_{L^2} \|f\|_{C^0([0,T],H^2)}.$$

Proof. Thanks to the equation $A\varphi_k = \lambda_k \varphi_k$, two integrations by part and the equalities $\varphi_k(1) = (-1)^k\sqrt{2}$, $\varphi_k(0) = \sqrt{2}$ (see (3)), we get the decomposition

$$S_{0,k} = \frac{1}{\lambda_k} \int_0^T w(t) \left\langle -\partial_x^2 f(t), \varphi_k \right\rangle e^{i\sqrt{\lambda_k} t}\, dt + \frac{(-1)^k \sqrt{2}}{\lambda_k} \int_0^T w(t) \partial_x f(t,1) e^{i\sqrt{\lambda_k} t}\, dt$$
$$- \frac{\sqrt{2}}{\lambda_k} \int_0^T w(t) \partial_x f(t,0) e^{i\sqrt{\lambda_k} t}\, dt,$$

called $S_0 = S_0^a + S_0^b + S_0^c$. Thanks to (3) and Cauchy–Schwarz inequality, we have

$$\|S_0^a\|_{h^2} = \left(\sum_{k=1}^{\infty} \left| k^2 \frac{1}{\lambda_k} \int_0^T w(t) \langle -\partial_x^2 f(t), \varphi_k \rangle e^{i\sqrt{\lambda_k} t} \, dt \right|^2 \right)^{1/2}$$
$$\leqslant \frac{1}{\pi^2} \left(\sum_{k=1}^{\infty} \left(\int_0^T \left| w(t) \langle \partial_x^2 f(t), \varphi_k \rangle \right| dt \right)^2 \right)^{1/2}$$
$$\leqslant \frac{1}{\pi^2} \left(\sum_{k=1}^{\infty} \|w\|_{L^2}^2 \int_0^T \left| \langle \partial_x^2 f(t), \varphi_k \rangle \right|^2 dt \right)^{1/2}$$
$$\leqslant \frac{\sqrt{T}}{\pi^2} \|w\|_{L^2} \left\| \partial_x^2 f \right\|_{C^0([0,T],L^2)}$$
$$\leqslant \frac{\sqrt{T}}{\pi^2} \|w\|_{L^2} \|f\|_{C^0([0,T],H^2)}.$$

Thanks to Lemma 1, there exists $C = C(T) > 0$ such that

$$\left\| S_0^b \right\|_{h^2} \leqslant C \left\| w(t) \partial_x f(t,1) \right\|_{L^2} \leqslant C \|w\|_{L^2} \|f\|_{C^0([0,T],H^2)},$$
$$\left\| S_0^c \right\|_{h^2} \leqslant C \left\| w(t) \partial_x f(t,0) \right\|_{L^2} \leqslant C \|w\|_{L^2} \|f\|_{C^0([0,T],H^2)}. \qquad \square$$

2.2.2. Proof of Theorem 3

Proof of Theorem 3. Let $T > 0$ and $\mu \in H^2(0,1)$.

First step: We prove that Θ_T indeed maps $L^2(0,T)$ into $H^3_{(0)} \times H^2_{(0)}(0,1)$.

Let $u \in L^2(0,T)$ and w be the weak solution of (1), (6). Let

$$z_k := \langle w(T), \varphi_k \rangle + \frac{1}{i\sqrt{\lambda_k}} \left\langle \frac{\partial w}{\partial t}(T), \varphi_k \right\rangle, \quad \forall k \in \mathbb{N}^*. \tag{24}$$

It is sufficient to prove that $(z_k)_{k\in\mathbb{N}^*}$ belongs to $h^3(\mathbb{N}^*, \mathbb{C})$. From the formulation of a weak solution, we get

$$z_k = \frac{1}{i\sqrt{\lambda_k}} \int_0^T u(t) \langle \mu w(t), \varphi_k \rangle e^{i\sqrt{\lambda_k}(T-t)} \, dt, \quad \forall k \in \mathbb{N}^*.$$

From Proposition 2, we know that

$$\left(w, \frac{\partial w}{\partial t} \right) \in C^0\big([0,T], H^2_{(0)} \times H^1(0,1)\big).$$

Thus $\mu w \in C^0([0,T], H^2)$, and Lemma 2 proves that $(z_k)_{k\in\mathbb{N}^*}$ belongs to $h^3(\mathbb{N}^*, \mathbb{C})$.

Second step: We prove that the linear map $v \mapsto W$ is continuous from $L^2(0,T)$ to $H^3_{(0)} \times H^2_{(0)}(0,1)$. Let $u, v \in L^2(0,T)$ and w, W be the solutions of (1), (6) and (23). Let

$$Z_k := \langle W(T), \varphi_k \rangle + \frac{1}{i\sqrt{\lambda_k}} \left\langle \frac{\partial W}{\partial t}(T), \varphi_k \right\rangle, \quad \forall k \in \mathbb{N}^*.$$

It is sufficient to prove that $Z := (Z_k)_{k\in\mathbb{N}^*}$ belongs to $h^3(\mathbb{N}^*, \mathbb{C})$ and

$$\|Z\|_{h^3} \leqslant C\|v\|_{L^2},$$

for some constant $C = C(T, \mu, \|u\|_{L^2})$. From the formulation of a weak solution, we get

$$Z_k = \frac{1}{i\sqrt{\lambda_k}} \int_0^T \big(u(t)\langle \mu W(t), \varphi_k\rangle + v(t)\langle \mu w(t), \varphi_k\rangle\big) e^{i\sqrt{\lambda_k}(T-t)}\, dt, \quad \forall k \in \mathbb{N}^*.$$

From Proposition 2, we know that

$$\left\| \left(W, \frac{\partial W}{\partial t}\right)\right\|_{C^0([0,T], H^2_{(0)}\times H^1)} \leqslant C\|v\|_{L^2},$$

where $C = C(T, \mu, \|v\|_{L^2})$. Thus, applying Lemma 2, we get

$$\|Z\|_{h^3} \leqslant C\big[\|u\|_{L^2}\|\mu W\|_{C^0([0,T],H^2)} + \|v\|_{L^2}\|\mu w\|_{C^0([0,T],H^2)}\big] \leqslant C\|v\|_{L^2}$$

where $C = C(T, \mu, \|u\|_{L^2})$.

Third step: *We prove that $\Theta_T : L^2(0, T) \to H^3_{(0)} \times H^2_{(0)}(0, 1)$ is differentiable and* (22) *holds*. Let $u, v \in L^2(0, T)$, w, W, $\tilde{w}$ be the weak solutions of (1), (6), (23) and

$$\begin{cases} \dfrac{\partial^2 \tilde{w}}{\partial t^2} = \dfrac{\partial^2 \tilde{w}}{\partial x^2} + (u+v)(t)\mu\tilde{w}, \quad x \in (0,1),\ t \in (0,T), \\ \dfrac{\partial \tilde{w}}{\partial x}(t,0) = \dfrac{\partial \tilde{w}}{\partial x}(t,1) = 0, \\ \tilde{w}(0,x) = 1, \\ \dfrac{\partial \tilde{w}}{\partial t}(0,x) = 0. \end{cases} \tag{25}$$

Then, $\xi := \tilde{w} - w - W$ is the weak solution of

$$\begin{cases} \dfrac{\partial^2 \xi}{\partial t^2} = \dfrac{\partial^2 \xi}{\partial x^2} + (u+v)\mu\xi + v\mu W, \\ \dfrac{\partial \xi}{\partial x}(t,0) = \dfrac{\partial \xi}{\partial x}(t,1) = 0, \\ \xi(0,x) = 0, \\ \dfrac{\partial \xi}{\partial t}(0,x) = 0. \end{cases} \tag{26}$$

We want to prove that

$$\left\| \left(\xi, \frac{\partial \xi}{\partial t}\right)(T)\right\|_{H^3_{(0)}\times H^2_{(0)}} = o\big(\|v\|_{L^2}\big) \quad \text{when } \|v\|_{L^2} \to 0.$$

Let

$$y_k := \langle \xi(T), \varphi_k \rangle + \frac{1}{i\sqrt{\lambda_k}} \left\langle \frac{\partial \xi}{\partial t}(T), \varphi_k \right\rangle, \quad \forall k \in \mathbb{N}^*.$$

It is sufficient to prove that $y := (y_k)_{k\in\mathbb{N}^*}$ satisfies $\|y\|_{h^3} = O(\|v\|_{L^2}^2)$ when $\|v\|_{L^2} \to 0$. From the formulation of a weak solution, we get

$$y_k = \frac{1}{i\sqrt{\lambda_k}} \int_0^T \big((u+v)(t)\langle \mu\xi(t), \varphi_k\rangle + v(t)\langle \mu W(t), \varphi_k\rangle\big) e^{i\sqrt{\lambda_k}(T-t)}\, dt, \quad \forall k \in \mathbb{N}^*.$$

From Proposition 2, we know that, when $\|v\|_{L^2} \leqslant 1$, we have

$$\begin{aligned}
\left\| \left(W, \frac{\partial W}{\partial t}\right) \right\|_{C^0([0,T],H^2_{(0)}\times H^1)} &\leqslant C\|v\mu w\|_{L^1((0,T),H^1)} \\
&\leqslant C\|v\|_{L^2}\|w\|_{C^0([0,T],H^1)} \\
&\leqslant C\|v\|_{L^2}, \\
\left\| \left(\xi, \frac{\partial \xi}{\partial t}\right) \right\|_{C^0([0,T],H^2_{(0)}\times H^1)} &\leqslant C\|v\mu W\|_{L^1((0,T),H^1)} \\
&\leqslant C\|v\|_{L^2}\|W\|_{C^0([0,T],H^1)} \\
&\leqslant C\|v\|_{L^2}^2,
\end{aligned}$$

where $C = C(\mu, T, \|u\|_{L^2}) > 0$. Thus, applying Lemma 2, we deduce that

$$\|y\|_{h^3} \leqslant \|u+v\|_{L^2}\|\mu\xi\|_{C^0([0,T],H^2)} + \|v\|_{L^2}\|\mu W\|_{C^0([0,T],H^2)} \leqslant C\|v\|_{L^2}^2.$$

Fourth step: *We prove the continuity of the map*

$$\begin{aligned}
d\Theta_T : L^2(0,T) &\to \mathcal{L}_c\big(L^2(0,T), H^3_{(0)} \times H^2_{(0)}(0,1)\big) \\
u &\mapsto d\Theta_T(u).
\end{aligned}$$

Actually, we prove this map is locally Lipschitz. Let $u, \tilde{u}, v \in L^2(0,T)$ with $\|u-\tilde{u}\|_{L^2} < 1$ and w, W, $\tilde{w}$, $\tilde{W}$ be the weak solutions of (1), (6), (23) and

$$\begin{cases}
\dfrac{\partial^2 \tilde{w}}{\partial t^2} = \dfrac{\partial^2 \tilde{w}}{\partial x^2} + \tilde{u}\mu\tilde{w}, \\
\dfrac{\partial \tilde{w}}{\partial x}(t,0) = \dfrac{\partial \tilde{w}}{\partial x}(t,1) = 0, \\
\tilde{w}(0,x) = 1, \\
\dfrac{\partial \tilde{w}}{\partial t}(0,x) = 0,
\end{cases}
\qquad
\begin{cases}
\dfrac{\partial^2 \tilde{W}}{\partial t^2} = \dfrac{\partial^2 \tilde{W}}{\partial x^2} + \tilde{u}\mu\tilde{W} + v\mu\tilde{w}, \\
\dfrac{\partial \tilde{W}}{\partial x}(t,0) = \dfrac{\partial \tilde{W}}{\partial x}(t,1) = 0, \\
\tilde{W}(0,x) = 0, \\
\dfrac{\partial \tilde{W}}{\partial t}(0,x) = 0.
\end{cases}$$

We have

$$\big[d\Theta_T(u) - d\Theta_T(\tilde{u})\big].v = \left(\Xi, \frac{\partial \Xi}{\partial t}\right)(T)$$

where $\Xi := W - \tilde{W}$ is the weak solution of

$$\begin{cases} \dfrac{\partial^2 \Xi}{\partial t^2} = \dfrac{\partial^2 \Xi}{\partial x^2} + u\mu\Xi + (u - \tilde{u})\mu\tilde{W} + v\mu(w - \tilde{w}), \\ \dfrac{\partial \Xi}{\partial x}(t, 0) = \dfrac{\partial \Xi}{\partial x}(t, 1) = 0, \\ \Xi(0, x) = 0, \\ \dfrac{\partial \Xi}{\partial t}(0, x) = 0. \end{cases}$$

Let

$$\omega_k := \langle \Xi(T), \varphi_k \rangle + \frac{1}{i\sqrt{\lambda_k}} \left\langle \frac{\partial \Xi}{\partial t}(T), \varphi_k \right\rangle, \quad \forall k \in \mathbb{N}^*.$$

It is sufficient to prove that $\omega := (\omega_k)_{k\in\mathbb{N}^*}$ satisfies

$$\|\omega\|_{h^3} \leqslant C\|u - \tilde{u}\|_{L^2}\|v\|_{L^2}, \tag{27}$$

where $C = C(\mu, T, \|u\|_{L^2}) > 0$. We have, for every $k \in \mathbb{N}^*$,

$$\omega_k = \frac{1}{i\sqrt{\lambda_k}} \int_0^T \big(u(t)\langle \mu\Xi(t), \varphi_k\rangle + (u - \tilde{u})(t)\langle \mu\tilde{W}(t), \varphi_k\rangle + v(t)\langle \mu(w - \tilde{w})(t), \varphi_k\rangle\big) e^{i\sqrt{\lambda_k}(T-t)}\, dt.$$

Thus, applying Lemma 2, we get

$$\|\omega\|_{h^3} \leqslant C\big[\|u\|_{L^2}\|\Xi\|_{C^0([0,T],H^2)} + \|u - \tilde{u}\|_{L^2}\|\tilde{W}\|_{C^0([0,T],H^2)} + \|v\|_{L^2}\|w - \tilde{w}\|_{C^0([0,T],H^2)}\big],$$

where $C = C(\mu, T, \|u\|_{L^2}) > 0$. Thanks to Proposition 2, we have

$$\begin{aligned} \|w - \tilde{w}\|_{C^0([0,T],H^2_{(0)})} &\leqslant C\big\|(u - \tilde{u})\mu w\big\|_{L^1((0,T),H^1)} \leqslant C\|u - \tilde{u}\|_{L^2}, \\ \|\tilde{W}\|_{C^0([0,T],H^2_{(0)})} &\leqslant C\|v\mu\tilde{w}\|_{L^1((0,T),H^1)} \leqslant C\|v\|_{L^2}, \\ \|\Xi\|_{C^0([0,T],H^2_{(0)})} &\leqslant C\big\|(u - \tilde{u})\mu\tilde{W} + v\mu(w - \tilde{w})\big\|_{L^1((0,T),H^1)} \\ &\leqslant C\big[\|u - \tilde{u}\|_{L^2}\|\tilde{W}\|_{C^0([0,T],H^1)} + \|v\|_{L^2}\|w - \tilde{w}\|_{C^0([0,T],H^1)}\big] \\ &\leqslant C\|u - \tilde{u}\|_{L^2}\|v\|_{L^2}, \end{aligned}$$

where $C = C(\mu, T, \|u\|_{L^2}) > 0$. Therefore, we have (27). □

3. Controllability of the linearized system

The goal of this section is the proof of the following results.

Theorem 4. *Let $\mu \in H^2(0, 1)$ be such that* (5) *holds.*

(1) *Let $T > 2$. The linear map $d\Theta_T(0) : L^2(0, T) \to H^3_{(0)} \times H^2_{(0)}(0, 1)$ has a continuous right inverse $d\Theta_T(0)^{-1} : H^3_{(0)} \times H^2_{(0)}(0, 1) \to L^2(0, T)$.*

(2) *Let $T = 2$. The image of the linear map $d\Theta_T(0) : L^2(0,T) \to H^3_{(0)} \times H^2_{(0)}(0,1)$ is a vector subspace R_T of $H^3_{(0)} \times H^2_{(0)}(0,1)$ with codimension one, and there exists a continuous (left and right) inverse $d\Theta_T(0)^{-1} : R_T \to L^2(0,T)$.*
(3) *Let $T < 2$. The image of the linear map $d\Theta_T(0) : L^2(0,T) \to H^3_{(0)} \times H^2_{(0)}(0,1)$ is a vector subspace R_T of $H^3_{(0)} \times H^2_{(0)}(0,1)$ with infinite codimension, and there exists a continuous (left and right) inverse $d\Theta_T(0)^{-1} : R_T \to L^2(0,T)$.*

This section is organized as follows. In Section 3.1, we state preliminary results, useful for the proof of Theorem 4, which is detailed in Section 3.2.

3.1. Preliminaries: trigonometric moment problems

Let us introduce the space

$$l^2_r\big([-1,+\infty),\mathbb{C}\big) := \big\{(d_k)_{k\geqslant -1};\ d_{-1}, d_0 \in \mathbb{R}\big\}, \tag{28}$$

equipped with the norm

$$\|d\|_{l^2_r} := \left(\sum_{k=-1}^{\infty} |d_k|^2\right)^{1/2}.$$

For the case $T > 2$, the result of the following Proposition 3 is needed.

Proposition 3. *Let $T > 2$. There exists a continuous linear map*

$$L_T : l^2_r\big([-1,+\infty),\mathbb{C}\big) \to L^2(0,T)$$
$$d = (d_k)_{k\geqslant -1} \mapsto L_T(d)$$

such that, for every sequence $d = (d_k)_{k\geqslant -1} \in l^2_r([-1,+\infty),\mathbb{C})$ the function $u := L_T(d)$ solves the moment problem

$$\begin{cases} \displaystyle\int_0^T (T-t)u(t)\,dt = d_{-1}, \\ \displaystyle\int_0^T u(t)e^{ik\pi t}\,dt = d_k, \quad \forall k \in \mathbb{N}. \end{cases} \tag{29}$$

Proof. Let $T > 2$. The set

$$Z := \mathrm{Cl}_{L^2((0,T),\mathbb{C})}\big(\mathrm{Span}\big\{e^{ik\pi t};\ k \in \mathbb{Z}\big\}\big)$$

(i.e. Z is the closure in $L^2((0,T),\mathbb{C})$ of the vector space generated by the set $\{e^{ik\pi t};\ k \in \mathbb{Z}\}$) is a closed vector subspace of $L^2((0,T),\mathbb{C})$ with infinite codimension. Let us prove that $t \notin Z$. Working by contradiction, we assume that $t \in Z$. After successive integrations, we get

$$t^j \in \mathrm{Cl}_{C^0([0,T],\mathbb{C})}\big(\mathrm{Span}\big\{t, e^{ik\pi t};\ k \in \mathbb{Z}\big\}\big), \quad \forall j \in \mathbb{N} \text{ with } j \geqslant 2.$$

The Stone–Weierstrass theorem ensures that $\{1, t^j;\ \ j \in \mathbb{N},\ \ j \geqslant 2\}$ is dense in $C^0([0,T],\mathbb{C})$, thus, it is also dense in $L^2((0,T),\mathbb{C})$. Since $t \in Z$, we deduce that Z is dense in $L^2((0,T),\mathbb{C})$, which is impossible. Therefore, $t \notin Z$, and we have the following orthogonal decomposition

$$L^2\big((0,T),\mathbb{C}\big) = Z \oplus Z^{\perp},$$
$$T - t = z + z^{\perp}$$

where $z^{\perp} \neq 0$. For $d = (d_k)_{k \geqslant -1} \in l^2_r([-1,+\infty),\mathbb{C})$, we define

$$L_T(d) := v + \left(d_{-1} - \int_0^T (T-t)v(t)\,dt\right)\frac{z^{\perp}}{\|z^{\perp}\|^2_{L^2}}$$

where

$$v := \left(\sum_{k\in\mathbb{Z}} d_k e^{-ik\pi t}\right)1_{[0,2]}(t)$$

and $d_{-k} := \overline{d_k}, \forall k \in \mathbb{N}^*$. The function $L_T(d)$ is real valued because v and $z^{\perp}$ are. From Bessel–Parseval identity, we have

$$\|v\|^2_{L^2(0,T)} = \frac{1}{2}\left[|d_0|^2 + 2\sum_{k=1}^{\infty}|d_k|^2\right],$$

thus there exists $C = C(T)$ such that

$$\big\|L_T(d)\big\|_{L^2(0,T)} \leqslant C(T)\|d\|_{l^2_r}. \qquad \Box$$

For the case $T < 2$, the results of the following Propositions 4 and 5 are needed.

Proposition 4. *For every $T \in (0,2\pi)$, there exists an extraction $\xi : \mathbb{Z} \to \mathbb{Z}$ such that $(e^{i\xi(k)t})_{k\in\mathbb{Z}}$ is a Riesz basis of $L^2(0,T)$.*

This proposition is a consequence of a more general result due to Horvath and Joo in [26]. For particular values of T, we also have the following stronger result.

Proposition 5. *Let $T \in (0,2\pi)$ be of the form*

$$T = \frac{(2r-1)\pi}{p} \quad \textit{with}\ r, p \in \mathbb{N}^*.$$

There exists an extraction $\xi : \mathbb{Z} \to \mathbb{Z}$ such that $\xi(-k) = -\xi(k)$, $\forall k \in \mathbb{Z}$ and $(e^{i\xi(k)t})_{k\in\mathbb{Z}}$ is a Riesz basis of $L^2(0,T)$.

Proof. First, let us recall that the Kadec 1/4 Theorem says that, if the real valued sequence $(\delta_n)_{n\in\mathbb{Z}}$ satisfies

$$\sup_{n\in\mathbb{Z}} |\delta_n| < 1/4,$$

then $(e^{i(n+\delta_n)t})_{n\in\mathbb{Z}}$ is a Riesz basis of $L^2(0,2\pi)$. Avdonin made the important remark that here, the 1/4 bound is sufficient to hold only for an average of the perturbations δ_n. Namely, if

- $(\delta_n)_{n\in\mathbb{Z}}$ is bounded,
- $(n+\delta_n)_{n\in\mathbb{Z}}$ is separated, i.e.

$$\inf\big\{(n+\delta_n)-(m+\delta_m);\ n,m\in\mathbb{Z},\ n\neq m\big\}>0,$$

- and we have

$$\lim_{K\to+\infty}\sup_{x\in\mathbb{R}}\frac{1}{K}\left|\sum_{x<n<x+K}\delta_n\right|<\frac{1}{4},$$

then $(e^{i(n+\delta_n)t})_{n\in\mathbb{Z}}$ is a Riesz basis of $L^2(0,2\pi)$ (see [4]).

Now, let us prove Proposition 5. Let $\xi:\mathbb{Z}\to\mathbb{Z}$ be the extraction such that $\xi(0)=0$ and the image of ξ is

$$R[\xi]=\bigcup_{n\in\mathbb{Z}}\{2np-r+1,2np-r+2,\dots,2np+r-1\}.$$

This means that we keep $(2r-1)$ frequencies over $2p$, in chains centered at the frequencies $2np$, $n\in\mathbb{Z}$. For this extraction, the average shift (with respect to $\{2pn/(2r-1);\ n\in\mathbb{Z}\}$) is equal to zero. Indeed, on any chain, the global shift is equal to zero. Thus, $(e^{i\xi(k)t})_{k\in\mathbb{Z}}$ is a Riesz basis of $L^2(0,T)$. $\Box$

3.2. Study of the linearized system

The goal of this subsection is the proof of Theorem 4.

Proof of Theorem 4. Let $\mu\in H^2(0,1)$ be such that (5) holds. Let $v\in L^2(0,T)$. We have

$$d\Theta_T(0).v=\left(W,\frac{\partial W}{\partial t}\right)(T)$$

where W is the weak solution of

$$\begin{cases}\dfrac{\partial^2 W}{\partial t^2}=\dfrac{\partial^2 W}{\partial x^2}+v(t)\mu(x), & x\in(0,1),\ t\in(0,T),\\ \dfrac{\partial W}{\partial x}(t,0)=\dfrac{\partial W}{\partial x}(t,1)=0,\\ W(0,x)=0,\\ \dfrac{\partial W}{\partial t}(0,x)=0.\end{cases}\tag{30}$$

We have

$$W(T)=\left(\langle\mu,\varphi_0\rangle\int_0^T(T-t)v(t)\,dt\right)\varphi_0+\sum_{k=1}^{\infty}\left(\frac{\langle\mu,\varphi_k\rangle}{\sqrt{\lambda_k}}\int_0^T v(t)\sin\big[\sqrt{\lambda_k}(T-t)\big]\,dt\right)\varphi_k,$$

$$\frac{\partial W}{\partial t}(T)=\left(\langle\mu,\varphi_0\rangle\int_0^T v(t)\,dt\right)\varphi_0+\sum_{k=1}^{\infty}\left(\langle\mu,\varphi_k\rangle\int_0^T v(t)\cos\left[\sqrt{\lambda_k}(T-t)\right]dt\right)\varphi_k.$$

Thus, for $(W_f,\dot W_f)\in H^3_{(0)}\times H^2_{(0)}(0,1)$, the equality $d\Theta_T(0).v=(W_f,\dot W_f)$ is equivalent to the moment problem

$$\begin{cases}\displaystyle\int_0^T (T-t)v(t)\,dt=d_{-1}(W_f,\dot W_f),\\ \displaystyle\int_0^T v(t)\,dt=d_0(W_f,\dot W_f),\\ \displaystyle\int_0^T v(t)e^{-i\sqrt{\lambda_k}t}\,dt=d_k(W_f,\dot W_f),\quad \forall k\in\mathbb{N}^*,\end{cases}$$

where $d(W_f,\dot W_f)=(d_k(W_f,\dot W_f))_{k\geqslant -1}$ is the sequence defined by

$$d_{-1}(W_k,\dot W_f):=\frac{\langle W_f,\varphi_0\rangle}{\langle\mu,\varphi_0\rangle},$$
$$d_0(W_k,\dot W_f):=\frac{\langle \dot W_f,\varphi_0\rangle}{\langle\mu,\varphi_0\rangle},$$
$$d_k(W_k,\dot W_f):=\frac{e^{-i\sqrt{\lambda_k}T}}{\langle\mu,\varphi_k\rangle}\left(\langle\dot W_f,\varphi_k\rangle+i\sqrt{\lambda_k}\langle W_f,\varphi_k\rangle\right),\quad\forall k\in\mathbb{N}^*. \tag{31}$$

Thanks to (5), the map

$$d: H^3_{(0)}\times H^2_{(0)}(0,1)\to l^2_r\left([-1,+\infty),\mathbb{C}\right)$$
$$(W_f,\dot W_f)\mapsto d(W_f,\dot W_f)$$

is continuous (see (28) for a definition of $l^2_r([-1,+\infty),\mathbb{C})$).

(1) We assume $T>2$. Thanks to Proposition 3, the expression

$$d\Theta_T(0)^{-1}(W_f,\dot W_f):=L_T\left[d(W_f,\dot W_f)\right]$$

gives a suitable right inverse.

(2) We assume $T=2$. Then the family $(e^{ik\pi t})_{k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(0,T)$ and we have

$$(T-t)=\sum_{k\in\mathbb{Z}}\alpha_k e^{-ik\pi t}\quad\text{in } L^2(0,T),$$

where $(\alpha_k)_{k\in\mathbb{Z}}\in l^2(\mathbb{Z},\mathbb{C})$. Then, the image of $d\Theta_T(0)$ is the vector space

$$R_T:=\left\{(W_f,\dot W_f)\in H^3_{(0)}\times H^2_{(0)}(0,1);\ d_{-1}(W_f,\dot W_f)=\sum_{k\in\mathbb{Z}}\alpha_k\tilde d_k(W_f,\dot W_f)\right\},$$

where

$$\tilde{d}_k(W_f, \dot{W}_f) := d_k(W_f, \dot{W}_f), \quad \forall k \in \mathbb{N},$$
$$\tilde{d}_{-k}(W_f, \dot{W}_f) := \overline{d_k(W_f, \dot{W}_f)}, \quad \forall k \in \mathbb{N}^*. \tag{32}$$

The map $d\Theta_T(0) : L^2(0,T) \to R_T$ has an inverse defined by

$$d\Theta_T(0)^{-1}(W_f, \dot{W}_f) = t \mapsto \sum_{k\in\mathbb{Z}} \tilde{d}_k(W_f, \dot{W}_f) e^{ik\pi t},$$

which is continuous from R_T (equipped with the $H^3_{(0)} \times H^2_{(0)}(0,1)$-norm) to $L^2(0,T)$, thanks to the Bessel–Parseval identity.

(3) We assume $T < 2$. Let $\xi : \mathbb{Z} \to \mathbb{Z}$ be an extraction such that $(e^{-i\xi(k)\pi t})_{k\in\mathbb{Z}}$ is a Riesz basis of $L^2(0,T)$ (see Proposition 4). Then, there exists $(\beta_k)_{k\in\mathbb{Z}} \in l^2(\mathbb{Z},\mathbb{C})$ such that

$$T - t = \sum_{k\in\mathbb{Z}} \beta_k e^{-i\xi(k)\pi t} \quad \text{in } L^2(0,T)$$

and for every $n \in \mathbb{N}$ that do not belong to the image of ξ, there exists $(\gamma^n_k)_{k\in\mathbb{Z}} \in l^2(\mathbb{Z},\mathbb{C})$ such that

$$e^{-in\pi t} = \sum_{k\in\mathbb{Z}} \gamma^n_k e^{-i\xi(k)\pi t} \quad \text{in } L^2(0,T).$$

Then, the image of $d\Theta_T(0)$ is the vector space

$$R_T := \Bigg\{ (W_f, \dot{W}_f) \in H^3_{(0)} \times H^2_{(0)}(0,1);\ d_{-1}(W_f, \dot{W}_f) = \sum_{k\in\mathbb{Z}} \beta_k \tilde{d}_{\xi(k)}(W_f, \dot{W}_f) \text{ and}$$
$$\forall n \in \mathbb{N} - R(\xi),\ d_n(W_f, \dot{W}_f) = \sum_{k\in\mathbb{Z}} \gamma^n_k \tilde{d}_{\xi(k)}(W_f, \dot{W}_f) \Bigg\}. \tag{33}$$

The set R_T is a vector subspace of $H^3_{(0)} \times H^2_{(0)}(0,1)$ with infinite codimension because it is defined by an infinite number of linearly independent relations. Let $(\zeta_k)_{k\in\mathbb{Z}}$ be the biorthogonal family to $(e^{-i\xi(k)\pi t})_{k\in\mathbb{Z}}$ in $L^2(0,T)$. Then, the map $d\Theta_T(0) : L^2(0,T) \to R_T$ has a continuous inverse $d\Theta_T(0)^{-1} : R_T \to L^2(0,T)$ defined by

$$d\Theta_T(0)^{-1}(W_f, \dot{W}_f) = \sum_{k\in\mathbb{Z}} \tilde{d}_{\xi(k)}(W_f, \dot{W}_f)\zeta_k. \qquad \square$$

4. Second order term

Using the same kind of arguments as in the proof of Theorem 3, one may prove the following result.

Proposition 6. *Let $\mu \in H^2(0,1)$ and $T > 0$. The map Θ_T defined by* (10) *is twice differentiable at* 0 *and*

$$d^2\Theta_T(0).(v,v) = \left(v, \frac{\partial v}{\partial t} \right)(T)$$

where ν is the weak solution of

$$\begin{cases} \frac{\partial \nu}{\partial t} = \frac{\partial^2 \nu}{\partial x^2} + v(t)\mu(x)W, \\ \frac{\partial \nu}{\partial x}(t,0) = \frac{\partial \nu}{\partial x}(t,1) = 0, \\ \nu(0,x) = 0, \\ \frac{\partial \nu}{\partial t}(0,x) = 0, \end{cases} \tag{34}$$

and W is the weak solution of (30)*.*

The main result of this section is the following one.

Proposition 7. *Let $\mu \in H^2(0,1)$ be such that* (5) *and* (7) *hold and $T \in (0,2]$. We assume that, either $T = 2$, or $T = (2r-1)/p$ with $p, r \in \mathbb{N}^*$. The image of the quadratic form $d^2\Theta_T(0)$ is not contained in the image of the linear map $d\Theta_T(0)$.*

The following lemma is useful for the proof of Proposition 7.

Lemma 3. *Let $T > 0$, $D := \{(t,\tau) \in \mathbb{R}^2;\ 0 < \tau < t < T\}$ and $h \in L^2(D,\mathbb{R})$. If*

$$\int_0^T v(t) \int_0^t v(\tau) h(t,\tau)\, d\tau\, dt = 0, \quad \forall v \in L^2(0,T),$$

then $h = 0$.

Proof. We consider the quadratic form

$$\begin{aligned} Q : L^2(0,T) &\to \mathbb{R} \\ v \mapsto Q(v) &:= \int_0^T v(t) \int_0^t v(\tau) h(t,\tau)\, d\tau\, dt. \end{aligned}$$

It is easy to prove that

$$\nabla Q(v) = t \mapsto \int_0^T v(\tau)\big\{h(t,\tau)1_{\tau<t} + h(\tau,t)1_{\tau>t}\big\}\, d\tau.$$

Since $Q \equiv 0$, we have $\nabla Q \equiv 0$, i.e.

$$\int_0^T v(\tau)\big\{h(t,\tau)1_{\tau<t} + h(\tau,t)1_{\tau>t}\big\}\, d\tau = 0, \quad \text{a.e. } t \in [0,T],\ \forall v \in L^2(0,T).$$

Thus, $h(t,\tau) = 0$, a.e. $(t,\tau) \in D$. $\square$

Proof of Proposition 7. Let $\mu \in H^2(0,1)$ be such that (5) and (7) hold and $T \in (0,2]$. We assume that, either $T=2$, or $T=(2r-1)/p$ with $p, r \in \mathbb{N}^*$.

First step: *Let us present the global strategy of the proof.* Let $\xi : \mathbb{Z} \to \mathbb{Z}$ be such that

$$\xi(-k) = -\xi(k), \quad \forall k \in \mathbb{N}^*, \tag{35}$$

and $(e^{-i\xi(k)\pi t})_{k\in\mathbb{Z}}$ is a Riesz basis of $L^2(0,T)$ (see Proposition 5 for $T<2$ and take $\xi(k)=k$ for $T=2$). There exists a unique sequence $(\alpha_k)_{k\in\mathbb{Z}} \in l^2(\mathbb{Z},\mathbb{C})$ such that

$$T - t = \Re\left[\sum_{k=0}^{\infty} \alpha_k e^{-i\xi(k)\pi t}\right] \quad \text{in } L^2(0,T). \tag{36}$$

We have seen in the proof of Theorem 4 that the image R_T of the linear map $d\Theta_T(0)$ is contained in the vector space

$$\tilde{R}_T := \left\{ (W_f, \dot{W}_f) \in H^3_{(0)} \times H^2_{(0)}(0,1);\ d_{-1}(W_f, \dot{W}_f) = \Re\left[\sum_{k=0}^{\infty} \alpha_k d_{\xi(k)}(W_f, \dot{W}_f)\right]\right\},$$

where $(d_k(W_f, \dot{W}_f))_{k \geqslant -1}$ is defined by (31). In order to prove Proposition 7, it is sufficient to prove that the image of the quadratic form $d^2\Theta_T(0)$ is not contained in $\tilde{R}_T$.

Second step: *Let us state an equivalent property for* "$d^2\Theta_T(0).(v,v) \in \tilde{R}_T$". Let $v \in L^2(0,T)$ and W, ν be the weak solutions of (30) and (34). We have

$$W(t) = \left(\langle \mu, \varphi_0\rangle \int_0^t (t-\tau) v(\tau)\, d\tau\right)\varphi_0 + \sum_{k=1}^{\infty}\left(\frac{\langle \mu, \varphi_k\rangle}{\sqrt{\lambda_k}} \int_0^t v(\tau)\sin\left[\sqrt{\lambda_k}(t-\tau)\right] d\tau\right)\varphi_k, \tag{37}$$

$$\begin{aligned} \nu(T) = {} & \left(\int_0^T (T-t) v(t) \langle \mu W(t), \varphi_0\rangle\, dt\right)\varphi_0 \\ & + \sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{\lambda_k}} \int_0^T v(t)\langle \mu W(t), \varphi_k\rangle \sin\left[\sqrt{\lambda_k}(T-t)\right] dt\right)\varphi_k \end{aligned} \tag{38}$$

and

$$\frac{\partial \nu}{\partial t}(T) = \left(\int_0^T v(t)\langle \mu W(t), \varphi_0\rangle\, dt\right)\varphi_0 + \sum_{k=1}^{\infty}\left(\int_0^T v(t)\langle \mu W(t), \varphi_k\rangle \cos\left[\sqrt{\lambda_k}(T-t)\right] dt\right)\varphi_k. \tag{39}$$

Let us assume that $d^2\Theta_T(0).(v,v) \in \tilde{R}_T$. Then we have

$$\frac{\langle \nu(T), \varphi_0\rangle}{\langle \mu, \varphi_0\rangle} = \Re\left[\sum_{k=0}^{\infty} \alpha_k \frac{e^{-i\xi(k)\pi T}}{\langle \mu, \varphi_{\xi(k)}\rangle}\left(\langle \dot{\nu}(T), \varphi_{\xi(k)}\rangle + i\sqrt{\lambda_{\xi(k)}}\langle \nu(T), \varphi_{\xi(k)}\rangle\right)\right]. \tag{40}$$

Thanks to (38) and (36), we have

$$\frac{\langle \nu(T),\varphi_0\rangle}{\langle \mu,\varphi_0\rangle} = \frac{1}{\langle \mu,\varphi_0\rangle}\int_0^T (T-t)v(t)\big\langle \mu W(t),\varphi_0\big\rangle dt$$
$$= \frac{1}{\langle \mu,\varphi_0\rangle}\int_0^T \Re\left[\sum_{k=0}^{\infty}\alpha_k e^{-i\xi(k)\pi t}\right] v(t)\big\langle \mu W(t),\varphi_0\big\rangle dt$$
$$= \int_0^T v(t)\Re\left[\sum_{k=0}^{\infty}\alpha_k \frac{\langle \mu W(t),\varphi_0\rangle}{\langle \mu,\varphi_0\rangle} e^{-i\xi(k)\pi t}\right] dt.$$

Thanks to (38) and (39), we have

$$\frac{e^{-i\xi(k)\pi T}}{\langle \mu,\varphi_{\xi(k)}\rangle}\Big(\big\langle \dot{\nu}(T),\varphi_{\xi(k)}\big\rangle + i\sqrt{\lambda_{\xi(k)}}\big\langle \nu(T),\varphi_{\xi(k)}\big\rangle\Big)$$
$$= \frac{e^{-i\xi(k)\pi T}}{\langle \mu,\varphi_{\xi(k)}\rangle}\int_0^T v(t)\big\langle \mu W(t),\varphi_{\xi(k)}\big\rangle e^{i\sqrt{\lambda_{\xi(k)}}(T-t)}\, dt$$
$$= \int_0^T v(t)\frac{\langle \mu W(t),\varphi_{\xi(k)}\rangle}{\langle \mu,\varphi_{\xi(k)}\rangle} e^{-i\xi(k)\pi t}\, dt.$$

Therefore, the equality (40) gives

$$\int_0^T v(t)\Re\left[\sum_{k=0}^{\infty}\alpha_k\left(\frac{\langle \mu W(t),\varphi_0\rangle}{\langle \mu,\varphi_0\rangle} - \frac{\langle \mu W(t),\varphi_{\xi(k)}\rangle}{\langle \mu,\varphi_{\xi(k)}\rangle}\right) e^{-i\xi(k)\pi t}\right] dt = 0, \tag{41}$$

or, equivalently,

$$\int_0^T v(t)\Re\left[\sum_{k=0}^{\infty}\alpha_k\left\langle \mu W(t),\frac{\varphi_0}{\langle \mu,\varphi_0\rangle} - \frac{\varphi_{\xi(k)}}{\langle \mu,\varphi_{\xi(k)}\rangle}\right\rangle e^{-i\xi(k)\pi t}\right] dt = 0. \tag{42}$$

Noticing that

$$\left\langle \mu\varphi_0,\frac{\varphi_0}{\langle \mu,\varphi_0\rangle} - \frac{\varphi_{\xi(k)}}{\langle \mu,\varphi_{\xi(k)}\rangle}\right\rangle = 0$$

(because $\varphi_0 = 1$) and using (37), we get

$$\left\langle \mu W(t),\frac{\varphi_0}{\langle \mu,\varphi_0\rangle} - \frac{\varphi_{\xi(k)}}{\langle \mu,\varphi_{\xi(k)}\rangle}\right\rangle$$
$$= \sum_{j=1}^{\infty}\frac{\langle \mu,\varphi_j\rangle}{\sqrt{\lambda_j}}\int_0^t v(\tau)\sin\big[\sqrt{\lambda_j}(t-\tau)\big]\, d\tau \left\langle \mu\varphi_j,\frac{\varphi_0}{\langle \mu,\varphi_0\rangle} - \frac{\varphi_{\xi(k)}}{\langle \mu,\varphi_{\xi(k)}\rangle}\right\rangle.$$

Therefore, the equality (42) gives

$$\int_0^T v(t) \int_0^t v(\tau) h(t, t-\tau)\, d\tau\, dt = 0, \tag{43}$$

where, for every $s \in \mathbb{R}$, $t \in [0, T]$,

$$h(t,s) := \Re\left[\sum_{k=0}^{\infty} \alpha_k e^{-i\xi(k)\pi t} \sum_{j=1}^{\infty} \frac{\langle \mu, \varphi_j\rangle}{\sqrt{\lambda_j}} \sin[\sqrt{\lambda_j} s] \left\langle \mu\varphi_j, \frac{\varphi_0}{\langle \mu, \varphi_0\rangle} - \frac{\varphi_{\xi(k)}}{\langle \mu, \varphi_{\xi(k)}\rangle} \right\rangle\right].$$

Third step: *Let us prove that* $h \in C^0(\mathbb{R}_s, L^2(0,T)_t)$. We introduce the decomposition $h = h_1 - h_2$ where

$$h_1(t,s) := \Re\left[\sum_{k=0}^{\infty} \alpha_k e^{-i\xi(k)\pi t} \sum_{j=1}^{\infty} \frac{\langle \mu, \varphi_j\rangle}{\sqrt{\lambda_j}} \sin[\sqrt{\lambda_j} s] \frac{\langle \mu\varphi_j, \varphi_0\rangle}{\langle \mu, \varphi_0\rangle}\right],$$

$$h_2(t,s) := \Re\left[\sum_{k=0}^{\infty} \alpha_k g_k(s) e^{-i\xi(k)\pi t}\right] \quad \text{and} \quad g_k(s) := \sum_{j=1}^{\infty} \frac{\langle \mu, \varphi_j\rangle}{\sqrt{\lambda_j}} \sin[\sqrt{\lambda_j} s] \frac{\langle \mu\varphi_j, \varphi_{\xi(k)}\rangle}{\langle \mu, \varphi_{\xi(k)}\rangle}.$$

Using (36), we get

$$h_1(t,s) = \frac{(T-t)}{\langle \mu, \varphi_0\rangle} f(s) \quad \text{where } f(s) := \sum_{j=1}^{\infty} \frac{\langle \mu, \varphi_j\rangle^2}{\sqrt{\lambda_j}} \sin[\sqrt{\lambda_j} s].$$

Integrations by parts and the Riemann–Lebesgue Lemma show that $\langle \mu, \varphi_j\rangle = O(1/j^2)$ when $j \to +\infty$, thus

$$\exists C > 0 \quad \text{such that} \quad \frac{\langle \mu, \varphi_j\rangle^2}{\sqrt{\lambda_j}} \leqslant \frac{C}{j^5}, \quad \forall j \in \mathbb{N}^*.$$

Therefore $f \in C^0(\mathbb{R}_s, \mathbb{R})$ and $h_1 \in C^0(\mathbb{R}_s, L^2(0,T)_t)$. Explicit computations also show that

$$\exists C > 0 \quad \text{such that} \quad \left|\langle \mu\varphi_j, \varphi_K\rangle\right| \leqslant \frac{C}{(j-K)_*^2}, \quad \forall j, K \in \mathbb{N},$$

thus

$$\exists C > 0 \quad \text{such that} \quad \left|\frac{\langle \mu, \varphi_j\rangle\langle \mu\varphi_j, \varphi_K\rangle}{\sqrt{\lambda_j}}\right| \leqslant \frac{C}{j_*^3 (j-K)_*^2}, \quad \forall j, K \in \mathbb{N},$$

thus

$$\left|g_k(s)\right| \leqslant C\xi(k)^2 \sum_{j=1}^{\infty} \frac{1}{j^2 (j-\xi(k))_*^2}, \quad \forall k \in \mathbb{N},\ \forall s \in \mathbb{R}.$$

The decomposition

$$\frac{1}{j^2(j-K)^2} = \frac{2}{K^3}\left(\frac{1}{j} - \frac{1}{j-K}\right) + \frac{1}{K^2}\left(\frac{1}{j^2} + \frac{1}{(j-K)^2}\right)$$

allows to prove that

$$\exists C > 0 \quad \text{such that} \quad \sum_{j=1}^{\infty} \frac{1}{j^2(j-K)_*^2} \leqslant \frac{C}{K^2}, \quad \forall K \in \mathbb{N},$$

thus,

$$\exists C > 0 \quad \text{such that} \quad \left|g_k(s)\right| \leqslant C, \quad \forall s \in \mathbb{R},\ \forall k \in \mathbb{N}.$$

We have $\alpha_k g_k(\sigma) \to \alpha_k g_k(s)$ when $\sigma \to s$, for every $k \in \mathbb{N}$. Moreover, $|\alpha_k g_k(\sigma)| \leqslant C|\alpha_k|$, for every $k \in \mathbb{N}$, $\sigma \in \mathbb{R}$, and the sequence $(|\alpha_k|)_{k\in\mathbb{N}}$ belongs to $l^2(\mathbb{N})$. The dominated convergence theorem ensures that the sequence $(\alpha_k g_k(\sigma))_{k\in\mathbb{N}}$ converges to $(\alpha_k g_k(s))_{k\in\mathbb{N}}$ in $l^2(\mathbb{N},\mathbb{C})$ when $\sigma \to s$. Since $(e^{-i\xi(k)\pi t})_{k\in\mathbb{Z}}$ is a Riesz basis of $L^2(0,T)$, we deduce that $h_2(.,\sigma) \to h_2(.,s)$ in $L^2(0,T)$, when $\sigma \to s$. This ends the proof of the third step.

Fourth step: *Now, let us prove Proposition* 7. Let us assume that $d^2\Theta_T(0).(v,v) \in \tilde{R}_T$ for every $v \in L^2(0,T)$. Then, thanks to the second step, the equality (43) holds for every $v \in L^2(0,T)$. Moreover $h \in L^2(D,\mathbb{R})$ (see the third step), so we can apply Lemma 3, which gives $h = 0$ in $L^2(D)$. Therefore, we also have $\partial h/\partial s = 0$ in the sense of distributions over D. In the sense of distributions on $\mathbb{R}_s \times [0,T]_t$, we have

$$\frac{\partial h}{\partial s}(t,s) = \Re\left[\sum_{k=0}^{\infty} \alpha_k e^{-i\xi(k)\pi t} \sum_{j=1}^{\infty} \langle \mu, \varphi_j\rangle \cos[\sqrt{\lambda_j} s]\left\langle \mu\varphi_j, \frac{\varphi_0}{\langle \mu, \varphi_0\rangle} - \frac{\varphi_{\xi(k)}}{\langle \mu, \varphi_{\xi(k)}\rangle}\right\rangle\right]. \tag{44}$$

Moreover, working as in the third step, one can prove that $\frac{\partial h}{\partial s} \in C^0(\mathbb{R}_s, L^2(0,T)_t)$. Thus, the equality (44) holds for every $s \in \mathbb{R}$ in $L^2(0,T)_t$. In particular, with $s = 0$, we get

$$\Re\left[\sum_{k=0}^{\infty} \alpha_k e^{-i\xi(k)\pi t} \sum_{j=1}^{\infty} \langle \mu, \varphi_j\rangle \left\langle \mu\varphi_j, \frac{\varphi_0}{\langle \mu, \varphi_0\rangle} - \frac{\varphi_{\xi(k)}}{\langle \mu, \varphi_{\xi(k)}\rangle}\right\rangle\right] = 0 \quad \text{in } L^2(0,T)_t,$$

or, equivalently,

$$\Re\left[\sum_{k=0}^{\infty} \alpha_k e^{-i\xi(k)\pi t} \left\langle \mu^2, \frac{\varphi_0}{\langle \mu, \varphi_0\rangle} - \frac{\varphi_{\xi(k)}}{\langle \mu, \varphi_{\xi(k)}\rangle}\right\rangle\right] = 0 \quad \text{in } L^2(0,T)_t.$$

But $(e^{-i\xi(k)\pi t})_{k\in\mathbb{Z}}$ is a Riesz basis of $L^2(0,T)$ and (35) holds, thus the previous equality implies

$$\alpha_k \left\langle \mu^2, \frac{\varphi_0}{\langle \mu, \varphi_0\rangle} - \frac{\varphi_{\xi(k)}}{\langle \mu, \varphi_{\xi(k)}\rangle}\right\rangle = 0, \quad \forall k \in \mathbb{N}. \tag{45}$$

Let us assume temporarily that the number of integers $p \in \mathbb{N}$ such that

$$\left\langle \mu^2, \frac{\varphi_0}{\langle \mu, \varphi_0\rangle} - \frac{\varphi_p}{\langle \mu, \varphi_p\rangle}\right\rangle = 0 \tag{46}$$

is finite. Then the equality (45) implies that only a finite number of α_k may be different from zero. But this is in contradiction with (36) because, for every $N \in \mathbb{N}$, the family $\{t, e^{ik\pi t}; -N \leqslant k \leqslant N\}$ is linearly independent. Therefore, the image of $d^2\Theta_T(0)$ is not contained in $\tilde{R}_T$.

Now let us prove that the number of integers $p \in \mathbb{N}$ such that (46) holds is finite. Integrations by parts give

$$\langle \mu, \varphi_p \rangle = \frac{\sqrt{2}}{(p\pi)^2}\big((-1)^p \mu'(1) - \mu'(0)\big) - \frac{\sqrt{2}}{(p\pi)^2}\int_0^1 \mu''(x)\cos(p\pi x)\,dx,$$

$$\langle \mu^2, \varphi_p \rangle = \frac{\sqrt{2}}{(p\pi)^2}\big((-1)^p \big(\mu^2\big)'(1) - \big(\mu^2\big)'(0)\big) - \frac{\sqrt{2}}{(p\pi)^2}\int_0^1 \big(\mu^2\big)''(x)\cos(p\pi x)\,dx.$$

Since $\mu'(1) \pm \mu'(0) \neq 0$, we have

$$\frac{\langle \mu^2, \varphi_p \rangle}{\langle \mu, \varphi_p \rangle} \sim \frac{(-1)^p (\mu^2)' - (\mu^2)'(0)}{(-1)^p \mu'(1) - \mu'(0)} \quad \text{when } p \to +\infty.$$

Thus the assumption (7) implies that the number of integers $p \in \mathbb{N}$ such that (46) holds is finite. □

5. Proof of Theorem 1

Proof of Theorem 1. Let $\mu \in H^2(0,1)$ be such that (5) holds.

(1) Let $T > 2$. The map

$$\Theta_T : L^2(0,T) \to H^3_{(0)} \times H^2_{(0)}(0,1)$$

is C^1 (see Theorem 3), and $d\Theta_T(0)$ has a continuous right inverse (see Theorem 4(1))

$$d\Theta_T(0)^{-1} : H^3_{(0)} \times H^2_{(0)}(0,1) \to L^2(0,T).$$

Thus, thanks to the inverse mapping theorem, Θ_T has a local C^1 right inverse.

(3) Let $T < 2$. First, let us assume that $T = (2r-1)/p$ with $p, r \in \mathbb{N}^*$. The set R_T defined by (33) is a closed vector subspace of the Hilbert space $H^3_{(0)} \times H^2_{(0)}(0,1)$. Thus, we have the orthogonal decomposition

$$H^3_{(0)} \times H^2_{(0)}(0,1) = R_T \oplus R_T^{\perp}.$$

We consider the map

$$F_T : L^2(0,T) \times R_T^{\perp} \to H^3_{(0)} \times H^2_{(0)}(0,1)$$
$$(u, y) \mapsto \Theta_T(u) + y.$$

Thanks to Theorem 3, F_T is C^1. Thanks to Theorem 4(3), the continuous linear map

$$dF_T(0,0) : L^2(0,T) \times R_T^{\perp} \to H^3_{(0)} \times H^2_{(0)}(0,1)$$

has a continuous inverse

$$dF_T(0,0)^{-1} : H^3_{(0)} \times H^2_{(0)}(0,1) \to L^2(0,T) \times R_T^{\perp}$$

defined by

$$dF_T(0,0)^{-1}.(W_f, \dot{W}_f) := \big(d\Theta_T(0)^{-1}.\mathcal{P}_{R_T}(W_f, \dot{W}_f), \mathcal{P}_{R_T^{\perp}}(W_f, \dot{W}_f)\big)$$

where $\mathcal{P}_{R_T}$ (resp. $\mathcal{P}_{R_T^{\perp}}$) is the orthogonal projection from $H^3_{(0)} \times H^2_{(0)}(0,1)$ to R_T (resp. $R_T^{\perp}$). Thanks to the inverse mapping theorem, the map F_T has a local inverse: there exist $\delta, r > 0$ and a C^1-map

$$F_T^{-1} : \mathcal{V}_T \to L^2(0,T) \times R_T^{\perp}$$

where

$$\mathcal{V}_T := \big\{(w_f, \dot{w}_f) \in H^3_{(0)} \times H^2_{(0)}(0,1);\ \|w_f - 1\|_{H^3_{(0)}} + \|\dot{w}_f\|_{H^2_{(0)}} < \delta\big\}$$

such that $F_T^{-1}(1,0) = (0,0)$, $F_T^{-1}[F_T(u,y)] = (u,y)$, for every $(u,y) \in L^2(0,T) \times R_T^{\perp}$ with $\|u\|_{L^2} + \|y\|_{H^3_{(0)} \times H^2_{(0)}} < r$ and $F_T[F_T^{-1}(z)] = z$, for every $z \in \mathcal{V}_T$. Let us denote by G_T the second component of F_T^{-1}. Then, the map

$$G_T : \mathcal{V}_T \to R_T^{\perp}$$

is locally surjective and we have

$$G_T\big[\Theta_T(u)\big] = 0, \quad \forall u \in L^2(0,T) \text{ with } \|u\|_{L^2} < r.$$

This proves that the image of Θ_T is locally a C^1-submanifold of $H^3_{(0)} \times H^2_{(0)}(0,1)$ with infinite codimension. This submanifold does not coincide with its tangent space at $(1,0)$ thanks to Proposition 7.

Now, let us consider an arbitrary $T \in (0,2)$. Let $T' \in (T,2)$ be such that $T' = (2r-1)/p$ for some $p, r \in \mathbb{N}^*$. We extend the controls defined on $(0,T)$ by zero on (T,T'). Applying the previous result, we get

$$G_{T'}\big[e^{\mathcal{A}(T'-T)}\Theta_T(u)\big] = 0, \quad \forall u \in B_r\big[L^2(0,T)\big].$$

Thus, the map $G_T := G_{T'} \circ e^{A(T'-T)}$ gives the conclusion.

(2) Let $T = 2$. First, let us prove that the nonlinear system is locally controllable up to codimension one. We consider the map

$$\begin{aligned} \tilde{\Theta}_T : L^2(0,T) &\to \tilde{\mathcal{V}}_T \\ u &\mapsto \left(w(T) - \int_0^1 w(T,x)\,dx, \frac{\partial w}{\partial t}(T)\right), \end{aligned}$$

where

$$\tilde{\mathcal{V}}_T := \left\{(\tilde{w}_f, \dot{w}_f) \in H^3_{(0)} \times H^2_{(0)}(0,1);\ \int_0^1 \tilde{w}_f(x)\,dx = 0\right\}.$$

Thanks to Theorem 3, $\tilde{\Theta}_T$ is C^1. Thanks to Theorem 4(2), the continuous linear map

$$d\tilde{\Theta}_T(0) : L^2(0,T) \to \tilde{\mathcal{V}}_T$$

has a continuous inverse

$$d\tilde{\Theta}_T(0)^{-1} : \tilde{\mathcal{V}}_T \to L^2(0,T).$$

Thanks to the inverse mapping theorem, $\tilde{\Theta}_T$ has a local C^1 inverse. This proves the local controllability up to codimension one of (1) in time $T=2$, in $H^3_{(0)} \times H^2_{(0)}(0,1)$, with $L^2(0,T)$-controls.

Working as in the proof of (3), we get a locally surjective C^1-map

$$G_T : H^3_{(0)} \times H^2_{(0)} \to \mathbb{R}$$

such that, for every $u \in L^2(0,T)$ small enough, $G_T[\Theta_T(u)] = 0$. Thus, the image of Θ_T is a C^1-submanifold of $H^3_{(0)} \times H^1_{(0)}$ with codimension one. Thanks to Proposition 7, this submanifold does not coincide with its tangent space at $(1,0)$. $\square$

6. Conclusion, open problems, perspectives

6.1. Same system, other reference trajectory

In this article, we have studied the local controllability of the system (1) around the reference trajectory

$$\big(w_{ref}(t,x) = 1, u_{ref}(t) = 0\big). \tag{47}$$

One may study the local controllability of the same system around other reference trajectories, for example

$$\big(w_{ref}(t,x) := \sin(K\pi t)\varphi_K(x), u_{ref}(t) = 0\big) \quad \text{for } K \in \mathbb{N}^*. \tag{48}$$

Let us explain why this problem is more difficult than the one solved in this article. The difficulty relies in the controllability of the linearized system. The linearized system of (1) around the reference trajectory (48) is

$$\begin{cases} \dfrac{\partial^2 W}{\partial t^2} = \dfrac{\partial^2 W}{\partial x^2} + u(t)\mu(x)w_{ref}(t,x), \quad x \in (0,1),\ t \in (0,T), \\ \dfrac{\partial W}{\partial x}(t,0) = \dfrac{\partial W}{\partial x}(t,1) = 0, \\ W(0,x) = \dfrac{\partial W}{\partial t}(0,x) = 0. \end{cases} \tag{49}$$

Working as in the proof of Theorem 4, one may prove that, for every $(W_f, \dot{W}_f) \in H^3_{(0)} \times H^2_{(0)}(0,1)$, the equality

$$\left(W, \frac{\partial W}{\partial t}\right)(T) = (W_f, \dot{W}_f)$$

is equivalent to the moment problem

$$\int_0^T (T-t)v(t)\sin(K\pi t)\,dt = d_{-1}(W_f, \dot{W}_f),$$

$$\int_0^T v(t)\sin(K\pi t)e^{-ik\pi t}\,dt = d_k(W_f, \dot{W}_f), \quad \forall k \in \mathbb{N}, \tag{50}$$

where $(d_k(W_f, \dot{W}_f))_{k \geqslant -1}$ is defined by

$$d_{-1}(W_f, \dot{W}_f) := \frac{\langle W_f, \varphi_0\rangle}{\langle \mu\varphi_K, \varphi_0\rangle},$$

$$d_k(W_f, \dot{W}_f) := \frac{e^{-i\sqrt{\lambda_k}T}}{\langle \mu\varphi_K, \varphi_k\rangle}\big(\langle \dot{W}_f, \varphi_k\rangle + i\sqrt{\lambda_k}\langle W_f, \varphi_k\rangle\big), \quad \forall k \in \mathbb{N}.$$

In order to apply the same strategy as in this article, one would need to prove that the family $\{t\sin(K\pi t), \sin(K\pi t)e^{-ik\pi t};\ k \in \mathbb{Z}\}$ satisfies the Riesz-basis property in $L^2(0,T)$, for every $T > 2$. However, this is false. Thus, the study if the local controllability of (1) around the reference trajectory (48) needs additional tools.

The same problem appears with Dirichlet boundary conditions, instead of Neumann boundary conditions in (1).

6.2. Conjecture for 2D and 3D bilinear Schrödinger equations

As emphasized in Section 1.3, the system (1) is a toy model for 2D Schrödinger equations, with bilinear controls (16). We conjecture that the behavior of this system concerning the exact controllability in time T (locally around the ground state) depends on the position of T with respect to $2\pi/d$, where d is the density of the eigenvalues of the Dirichlet–Laplacian operator on Ω, in the Weyl formula (17). Precisely, we conjecture that, generically with respect to (Ω, μ),

- for every $T > 2\pi/d$, the system (16) is locally exactly controllable around the ground state (or any eigenstate) in some function space (to be defined),
- for every $T < 2\pi/d$, the system (16) is not locally exactly controllable around the ground state: the reachable set is contained in a non-flat submanifold of some functional space (to be defined), with infinite codimension.

Similarly, for 3D Schrödinger equations with bilinear control (i.e. Eq. (16) with Ω a bounded open subset of $\mathbb{R}^3$), we conjecture that, for every $T > 0$, the reachable set is a non-flat submanifold of some functional space, with infinite codimension.

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Appendix A. Genericity of the assumption on μ

The goal of this section is the proof of the following result.

Proposition 8. *The set* $\{\mu \in H^2(0,1);\ (5)$ *and* (7) *hold*$\}$ *is dense in* $H^2(0,1)$.

The following lemma will be useful in the proof of Proposition 8.

Lemma 4. *Let $\Phi_\pm, \Phi : H^2(0,1) \to \mathbb{R}$ be defined by $\Phi(\mu) := \Phi_+(\mu)\Phi_-(\mu)$ and*

$$\Phi_\pm(\mu) := \left[\left(\mu^2\right)'(1) \pm \left(\mu^2\right)'(0)\right] \int_0^1 \mu(x)\,dx - \left[\mu'(1) \pm \mu'(0)\right] \int_0^1 \mu(x)^2\,dx.$$

For every $\mu \in H^2(0,1)$ such that $\mu'(1) \pm \mu'(0) \neq 0$ and $\Phi(\mu) = 0$, we have either $d\Phi(\mu) \neq 0$ or $d\Phi(\mu) = 0$ and $d^2\Phi(\mu) \neq 0$.

Proof. For every $\mu, \nu \in H^2(0,1)$, we have

$$\begin{aligned} d\Phi_\pm(\mu).\nu = {} & 2\left[(\mu\nu)'(1) \pm (\mu\nu)'(0)\right] \int_0^1 \mu + \left[\left(\mu^2\right)'(1) \pm \left(\mu^2\right)'(0)\right] \int_0^1 \nu \\ & - \left[\nu'(1) \pm \nu'(0)\right] \int_0^1 \mu^2 - \left[\mu'(1) \pm \mu'(0)\right] \int_0^1 2\mu\nu. \end{aligned}$$

In particular, for every $\nu \in C_c^\infty(0,1)$ such that $\int_0^1 \nu = 0$, we have

$$d\Phi_\pm(\mu).\nu = -2\left[\mu'(1) \pm \mu'(0)\right] \int_0^1 \mu\nu.$$

Let $\mu \in H^2(0,1)$ be such that $\mu'(1) \pm \mu'(0) \neq 0$ and $\Phi(\mu) = 0$.

First case: *We assume $\Phi_+(\mu) = 0$ and $\Phi_-(\mu) \neq 0$.* Then, for every $\nu \in C_c^\infty(0,1)$ such that $\int_0^1 \nu = 0$ and $\int_0^1 \mu\nu \neq 0$, we have

$$d\Phi(\mu).\nu = \left[d\Phi_+(\mu).\nu\right]\Phi_-(\nu) = -2\Phi_-(\nu)\left[\mu'(1) + \mu'(0)\right] \int_0^1 \mu\nu \neq 0.$$

The case $\Phi_-(\mu) = 0$ and $\Phi_+(\mu) \neq 0$ may be treated similarly.

Second case: *We assume $\Phi_+(\mu) = \Phi_-(\mu) = 0$.* Then, $d\Phi(\mu) = 0$ and, for every $\nu \in C_c^\infty(0,1)$ such that $\int_0^1 \nu = 0$ and $\int_0^1 \mu\nu \neq 0$, we have

$$\begin{aligned} d^2\Phi(\mu).\nu &= \left[d\Phi_+(\mu).\nu\right]\left[d\Phi_-(\mu).\nu\right] \\ &= 4\left[\mu'(1) - \mu'(0)\right]\left[\mu'(1) + \mu'(0)\right]\left(\int_0^1 \mu\nu\right)^2 \neq 0. \qquad \square \end{aligned}$$

Proof of Proposition 8. First, let us notice that

$$\mathcal{W} := \left\{\mu \in H^2(0,1);\ \mu'(0) \pm \mu'(1) \neq 0\right\}$$

is a dense open subset of $H^2(0,1)$. Thanks to Lemma 4, the set

$$\mathcal{V} := \{\mu \in \mathcal{W};\ (7)\ \text{holds}\}$$

is a dense open subset of $\mathcal{W}$. Now, let us prove that the set

$$\mathcal{U} := \{\mu \in \mathcal{V};\ \langle \mu, \varphi_k \rangle \neq 0,\ \forall k \in \mathbb{N}\}$$

is dense in $\mathcal{V}$. For $n \in \mathbb{N}$, we introduce the set

$$\mathcal{U}_n := \{\mu \in \mathcal{V};\ \langle \mu, \varphi_k \rangle \neq 0,\ \forall k \in \{0, \dots, n\}\},$$

with the convention $\mathcal{U}_{-1} = \mathcal{V}$. Then, the sequence $\mathcal{U}_n$ is decreasing and

$$\mathcal{U} = \bigcap_{n=-1}^{\infty} \mathcal{U}_n.$$

We apply Baire Lemma: it is sufficient to prove that, for every $n \geqslant -1$, $\mathcal{U}_{n+1}$ is dense in $\mathcal{U}_n$ for the $H^2(0,1)$-topology. Let $n \geqslant -1$ and let $\mu \in \mathcal{U}_n - \mathcal{U}_{n+1}$. Then $\langle \mu\varphi_1, \varphi_k \rangle \neq 0$ for $k = 0, \dots, n$ and $\langle \mu\varphi_1, \varphi_{n+1} \rangle = 0$. There exists $\epsilon^* > 0$ such that, for every $\epsilon \in (0, \epsilon^*)$, $\mu + \epsilon x^2 \in \mathcal{V}$, because $\mathcal{V}$ is an open subset of $W^{2,\infty}(0,1)$. Thanks to (9), $\mu + \epsilon x^2 \in \mathcal{U}_{n+1}$ for every $\epsilon \in (0, \epsilon^*)$ such that

$$\epsilon \neq -\frac{\langle \mu\varphi_1, \varphi_j \rangle}{\langle x^2\varphi_1, \varphi_j \rangle}, \quad \forall j \in \{0, \dots, n\}.$$

Thus $\mathcal{U}_{n+1}$ is dense in $\mathcal{U}_n$. We have proved that $\mathcal{U}$ is dense in $H^2(0,1)$.

Now, let us emphasize that

$$\mathcal{U} \subset \{\mu \in H^2(0,1);\ (5)\ \text{and}\ (7)\ \text{hold}\}.$$

Indeed, for $\mu \in \mathcal{U}$ and $k \in \mathbb{N}^*$, integrations by parts give (8). Since $\mu'(0) \pm \mu'(1) \neq 0$, there exists $N \in \mathbb{N}$ such that, for every $k \geqslant N$,

$$\left|\langle \mu, \varphi_k \rangle\right| \geqslant \frac{1}{(k\pi)^2} \max\{\left|\mu'(1) + \mu'(0)\right|, \left|\mu'(1) - \mu'(0)\right|\}.$$

Since $\langle \mu, \varphi_k \rangle \neq 0, \forall k \in \mathbb{N}$, there exists $c > 0$ such that

$$\left|\langle \mu, \varphi_k \rangle\right| \geqslant \frac{c}{k_*^2}, \quad \forall k \in \mathbb{N}. \qquad \square$$

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