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Local controllability of a 1-D Schrödinger equation

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Abstract

We consider a nonrelativistic charged particle in a 1-D box of potential. This quantum system is subject to a control, which is a uniform electric field. It is represented by a complex probability amplitude solution of a Schrödinger equation. We prove the local controllability of this nonlinear system around the ground state. Our proof uses the return method, a Nash–Moser implicit function theorem and moment theory.

Résumé

On considère une particule chargée non relativiste dans un puits de potentiel en dimension un d'espace. Ce système quantique est soumis à un champ électrique uniforme, qui constitue un contrôle. Il est représenté par une densité de probabilité complexe, solution d'une équation de Schrödinger. On démontre la contrôlabilité locale de ce système non-linéaire au voisinage de l'état fondamental. La démonstration utilise la méthode du retour, un théorème de Nash–Moser et la théorie des moments.

Keywords: Controllability; Schrödinger equation; Nash–Moser theorem; Moment theory

1. Introduction

We consider a nonrelativistic single charged particle in a one dimension space, with a potential V, in a uniform electric field $t \mapsto u(t)$. Assuming the mass of the particle is

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1 and the constant $\hbar$ is equal to 1, it is represented by a probability complex amplitude $q \in \mathbb{R} \mapsto \psi(t,q)$ solution of the Schrödinger equation:

$$\mathrm{i}\frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi}{\partial q^2} + \big(V(q) - u(t)q\big)\psi.$$

We study this quantum system in the case of a box potential: $V(q) = 0$ for $q \in I := (-1/2, 1/2)$ and $V(q) = +\infty$ for q outside I. Therefore our system is:

$$\mathrm{i}\frac{\partial \psi}{\partial t}(t,q) = -\frac{1}{2}\frac{\partial^2 \psi}{\partial q^2}(t,q) - u(t)q\psi(t,q), \quad q \in I, \tag{1.1}$$

$$\psi(0,q) = \psi_0(q), \tag{1.2}$$

$$\psi(t,-1/2) = \psi(t,1/2) = 0. \tag{1.3}$$

This is a control system, denoted (Σ), where

- the state is ψ, with $\int_I |\psi(t,q)|^2 \,\mathrm{d}q = 1$ for every t,
- the control is the electric field $t \mapsto u(t) \in \mathbb{R}$.

Definition 1. Let T_1 and T_2 be two real numbers satisfying $T_1 \leqslant T_2$, $u : [T_1, T_2] \to \mathbb{R}$ be a continuous function and $\psi_0 \in H^2 \cap H^1_0(I, \mathbb{C})$ be such that $\|\psi_0\|_{L^2} = 1$. A function $\psi : [T_1, T_2] \times I \to \mathbb{C}$ is a solution of the system (Σ) if:

- ψ belongs to $C^0([T_1, T_2], (H^2 \cap H^1_0)(I, \mathbb{C})) \cap C^1([T_1, T_2], L^2(I, \mathbb{C}))$,
- the equality (1.1) is true in $L^2(I, \mathbb{C})$ for every $t \in [T_1, T_2]$,
- the equality (1.2) is true in $L^2(I, \mathbb{C})$.

Then, we say that (ψ, u) is a trajectory of the control system (Σ).

Note that Eq. (1.1) guarantees the conservation of the L^2-norm of ψ, since u is real valued. Indeed, using the notation,

$$\langle f, g \rangle = \int_I f(q)\overline{g(q)}\,\mathrm{d}q,$$

and Eq. (1.1), we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi(t)\|^2_{L^2} = \left\langle \frac{\partial \psi}{\partial t}, \psi \right\rangle + \left\langle \psi, \frac{\partial \psi}{\partial t} \right\rangle = 0.$$

Our main result states that this control system is locally controllable around the ground state for $u \equiv 0$, which is the function:

$$\psi_1(t,q) := \varphi_1(q)\mathrm{e}^{-\mathrm{i}\lambda_1 t}.$$

Here, $\lambda_1 := \pi^2/2$ is the smallest eigenvalue of the operator A defined on $D(A) := (H^2 \cap H_0^1)(I, \mathbb{C})$, by $A\varphi := -(1/2)\varphi''$. The function $\varphi_1(q) := \sqrt{2}\cos(\pi q)$ is the associated eigenvector. This property was stated for the first time by P. Rouchon in [18].

Let us introduce the unitary sphere of $L^2(I, \mathbb{C})$,

$$S := \{\varphi \in L^2(I, \mathbb{C});\ \|\varphi\|_{L^2} = 1\},$$

and the closed subspace of the Sobolev space $H^7(I, \mathbb{C})$ defined by:

$$H^7_{(0)}(I, \mathbb{C}) := \{\varphi \in H^7(I, \mathbb{C});\ \varphi^{(2n)}(1/2) = \varphi^{(2n)}(-1/2) = 0,\ \text{for } n = 0, 1, 2, 3\}.$$

Theorem 1. *Let $\phi_0, \phi_1 \in \mathbb{R}$. There exist $T > 0$ and $\eta > 0$ such that, for every ψ_0, ψ_f in $S \cap H^7_{(0)}(I, \mathbb{C})$ satisfying,*

$$\|\psi_0 - \varphi_1 e^{i\phi_0}\|_{H^7} < \eta, \qquad \|\psi_f - \varphi_1 e^{i\phi_1}\|_{H^7} < \eta,$$

there exists a trajectory (ψ, u) of the control system (Σ) on $[0, T]$ such that $\psi(0) = \psi_0$, $\psi(T) = \psi_f$ and $u \in H_0^1((0, T), \mathbb{R})$.

The first remark concerns the regularity assumption on the initial and final states. Following arguments from J.M. Ball, J.E. Marsden and M. Slemrod in [1], it has been pointed out by G. Turinici in [9, Chapter 4] that the control system (Σ) is not controllable in $H^2 \cap H_0^1$. More precisely, whatever the initial data is, the set of reachable sets has a dense complement in the L^2-sphere S. Thus, in order to have controllability, it is necessary to put stronger regularity assumptions on the initial and final states.

The proof given in this article gives the controllability of (Σ) in H^7. The exponent 7 is purely technical and related to the application of the Nash–Moser theorem. With the same strategy and strengthened estimates in the Nash–Moser theorem, it should be possible to get the controllability in spaces H^s with $s < 7$ (for example, for any $s > 6$). We conjecture that the local controllability of the nonlinear system (Σ) holds in $H^3 \cap H_0^1$ with control in L^2 because it is the case for the linearized system considered in Section 3.1.

The second remark concerns the time of control. In this article, we prove the local controllability in time larger than $4/\pi$ and rather long, because we use quasi-static transformations in Section 4. However, we do not think a so long time is necessary. The existence of a minimal time for the control is an open problem.

Usually, the controllability of systems involving the Schrödinger equation does not require a positive minimal time of control because this equation has an infinite propagation speed. Nevertheless, the existence of a positive minimal time for the control of (Σ) is not excluded.

In order to understand why, let us consider, as in [6], the following toy model:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + u, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = -x_3 + 2x_1x_2. \end{cases} \tag{$\mathcal{T}$}$$

The linearized system around $(x_1 \equiv 0, x_2 \equiv 0, x_3 \equiv 0, x_4 \equiv 0, u \equiv 0)$ is not controllable. For $\gamma \neq 0$, the linearized system around $(x_1 \equiv \gamma, x_2 \equiv 0, x_3 \equiv 0, x_4 \equiv 0, u \equiv \gamma)$ is controllable in time arbitrarily small. Nevertheless, the nonlinear system $(\mathcal{T})$ is not small time controllable. Indeed, if $(x, u) : [0, T] \to \mathbb{R}^4 \times \mathbb{R}$ is a trajectory of the control system $(\mathcal{T})$ such that $x(0) = 0$, then

$$x_3(T) = \int_0^T x_1^2(t) \cos(T - t)\,\mathrm{d}t, \qquad x_4(T) = x_1^2(T) - \int_0^T x_1^2(t) \sin(T - t)\,\mathrm{d}t.$$

In particular, if $x_1(T) = 0$ and $T \leqslant \pi$ then $x_4(T) \leqslant 0$ thus $(\mathcal{T})$ is not controllable in time $T \leqslant \pi$. Moreover, it is proved in [6] that $(\mathcal{T})$ is locally controllable in time T around zero if and only if $T > \pi$.

The system (Σ) is similar to $(\mathcal{T})$. Indeed, the linearized system around the ground state ψ_1, for $u \equiv 0$ is not controllable. The linearized system around the ground state $\psi_{1,\gamma}$, for $u \equiv \gamma$, studied in Section 3.1, is controllable in time arbitrarily small.

Thus, we conjecture there exists a positive minimal time for the control of (Σ). The method introduced by J.-M. Coron and E. Crépeau in [7] could be used in order to know what is the minimal time for controllability.

It is quite important to get a control u with $u(0) = u(T) = 0$. Indeed, if ψ_f is on the ground state and if we stop the control at $t = T$, then ψ stays on the ground state.

For other results about the controllability of Schrödinger equations, we refer to the survey [20].

2. Sketch of the proof

A classical approach to get local controllability consists in proving the controllability of the linearized system around the point studied and concluding using an inverse mapping theorem. This method does not work here: Pierre Rouchon proved in [18] that around any state of definite energy, the linear tangent approximate system is not controllable, but is "steady-state" controllable, with the state (ψ, s, D) where $\dot{s} = u$, $\dot{D} = s$.

The proof of Theorem 1 relies on the return method, a method introduced in [2] to solve a stabilization problem, together with quasi-static transformations as in [5]. The return method has already been used for controllability problems by J.-M. Coron in [5,3,4], by A.V. Fursikov and O.Yu. Imanuvilov in [10], by O. Glass in [11,12], by Th. Horsin in [16] and by E. Sontag in [19]. We find a trajectory $(\tilde{\psi}, \tilde{u})$ of the control system (Σ) such that the linearized control system around $(\tilde{\psi}, \tilde{u})$ is controllable in time T. Using an implicit function theorem, we get the local controllability in time T of the nonlinear dynamics around $(\tilde{\psi}(0), \tilde{\psi}(T))$: there exist a neighbourhood V_0 of $\tilde{\psi}(0)$ and a neighbourhood V_T of $\tilde{\psi}(T)$ such that the system (Σ) can be moved in time T from any state in V_0 to any state in V_T.

Then for two states ψ_0, ψ_f closed enough to $\varphi_1 \mathrm{e}^{\mathrm{i}\phi_0}$, $\varphi_1 \mathrm{e}^{\mathrm{i}\phi_1}$, we prove the system (Σ) can be moved:

– from ψ_0 to a point $\psi_2 \in V_0$, using quasi-static transformations,
– from one point $\psi_3 \in V_T$ to ψ_f, using again quasi-static transformations,
– from ψ_2 to ψ_3 using the local controllability around $(\tilde{\psi}(0), \tilde{\psi}(T))$.

Let us give an example of such a family of trajectories $(\tilde{\psi}, \tilde{u})$. For this, we need few notations.

For a given real constant γ, we write $A_\gamma : D(A_\gamma) \to L^2(I, \mathbb{C})$ the operator defined by:

$$D(A_\gamma) := H^2 \cap H_0^1(I, \mathbb{C}), \qquad A_\gamma \varphi := -\frac{1}{2}\varphi'' - \gamma q \varphi.$$

The space $L^2(I, \mathbb{C})$ admits a complete orthonormal system $(\varphi_{k,\gamma})_{k \in \mathbb{N}^*}$ of eigenfunctions for A_γ:

$$-\frac{1}{2}\frac{\mathrm{d}^2\varphi_{k,\gamma}}{\mathrm{d}q^2} - \gamma q \varphi_{k,\gamma} = \lambda_{k,\gamma}\varphi_{k,\gamma},$$

where $(\lambda_{k,\gamma})_{k \in \mathbb{N}^*}$ is an increasing sequence of positive real numbers. Then the function $\psi_{1,\gamma}(t,q) := \varphi_{1,\gamma}(q)\mathrm{e}^{-\mathrm{i}\lambda_{1,\gamma}t}$ is a solution of the system (Σ) with control $u \equiv \gamma$. It is the ground state for $u \equiv \gamma$.

Using the notation,

$$\begin{cases} \psi(t,q) = \psi_{1,\gamma}(t,q) + \Psi(t,q), \\ u(t) = \gamma + w(t), \end{cases}$$

the linearized system around $(\psi_{1,\gamma}, \gamma)$ is:

$$\begin{cases} \mathrm{i}\frac{\partial \Psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \Psi}{\partial q^2} - \gamma q \Psi - w(t) q \psi_{1,\gamma}, \\ \Psi(0) = \Psi_0, \\ \Psi(t, -1/2) = \Psi(t, 1/2) = 0, \end{cases}$$

where the state is Ψ and the control is $w : [0, T] \to \mathbb{R}$. Note that the first equation on Ψ guarantees,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\Re\left(\int_I \psi_{1,\gamma}(t,q)\overline{\Psi(t,q)}\,\mathrm{d}q\right)\right) = 0,$$

where $\Re(z)$ denotes the real part of the complex number z. Therefore, when Ψ_0 belongs to the tangent space to S at $\varphi_{1,\gamma}$,

$$\Re\left(\int_I \varphi_{1,\gamma}(q)\overline{\Psi_0(q)}\,\mathrm{d}q\right) = 0,$$

then $\Psi(t)$ belongs to the tangent space to S at $\psi_{1,\gamma}(t)$ for every time t,

$$\Re\left(\int_I \psi_{1,\gamma}(t,q)\overline{\Psi(t,q)}\,\mathrm{d}q\right)=0.$$

We will see, in Section 3.1, using moment theory, that when γ is small enough but different from zero, this linear control system is controllable in any positive time T. However, the classical implicit function theorem is not sufficient to conclude the local controllability in time T of the nonlinear system around $(\psi_{1,\gamma}(0),\psi_{1,\gamma}(T))$. Indeed, the map Φ_γ which associates to any couple of initial condition and control (ψ_0,v) the couple of initial and final conditions (ψ_0,ψ_T) for the system (Σ) with $u=\gamma+v$,

$$\Phi_\gamma:\big[S\cap H_0^1(I,\mathbb{C})\big]\times L^2\big((0,T),\mathbb{R}\big)\to\big[S\cap H_0^1(I,\mathbb{C})\big]\times\big[S\cap H_0^1(I,\mathbb{C})\big],$$
$$(\psi_0,v)\mapsto(\psi_0,\psi_T),$$

is well defined and of class C^1. Its differential application $\mathrm{d}\Phi_\gamma(\varphi_{1,\gamma},0)$ at the point $(\varphi_{1,\gamma},0)$ maps the space,

$$E:=\big[T_S(\varphi_{1,\gamma})\cap H_0^1(I,\mathbb{C})\big]\times L^2\big((0,T),\mathbb{R}\big),$$

into the space,

$$F:=\big[T_S\big(\psi_{1,\gamma}(0)\big)\cap H_0^1(I,\mathbb{C})\big]\times\big[T_S\big(\psi_{1,\gamma}(T)\big)\cap H_0^1(I,\mathbb{C})\big],$$

where $T_S(\xi)$ is the tangent space to the L^2-sphere S at the point ξ. It admits a right inverse, written $\mathrm{d}\Phi_\gamma(\varphi_{1,\gamma},0)^{-1}$, but this right inverse does not map F into E. We only know that $\mathrm{d}\Phi_\gamma(\varphi_{1,\gamma},0)^{-1}$ maps:

$$\big[T_S\big(\psi_{1,\gamma}(0)\big)\cap H^3_{(0)}(I,\mathbb{C})\big]\times\big[T_S\big(\psi_{1,\gamma}(T)\big)\cap H^3_{(0)}(I,\mathbb{C})\big],$$

into,

$$\big[T_S(\varphi_{1,\gamma})\cap H^3_{(0)}(I,\mathbb{C})\big]\times L^2\big((0,T),\mathbb{R}\big),$$

where $H^3_{(0)}(I,\mathbb{C})$ is a closed subspace of $H^3(I,\mathbb{C})$. We deal with this loss of regularity using a Nash–Moser implicit function theorem given by Hörmander in [15]. We get the following theorem, proved in Section 3.

Theorem 2. *Let $T=4/\pi$. There exists a constant $\gamma_1>0$ such that, for every $\gamma\in(0,\gamma_1]$, there exists a constant $\eta>0$ such that, for every $(\psi_0,\psi_T)\in S\cap H^7_{(\gamma)}(I,\mathbb{C})$ satisfying,*

$$\big\|\psi_0-\psi_{1,\gamma}(0)\big\|_{H^7}\leqslant\eta,\qquad\big\|\psi_T-\psi_{1,\gamma}(T)\big\|_{H^7}\leqslant\eta,$$

there exists a trajectory (ψ,u) of the control system (Σ) satisfying $\psi(0)=\psi_0$, $\psi(T)=\psi_T$ and $(u-\gamma)\in H_0^1((0,T),\mathbb{R})$.

In this theorem, $H^7_{(\gamma)}(I,\mathbb{C})$ denotes the closed subspace of $H^7(I,\mathbb{C})$ containing $\varphi_{1,\gamma}$ defined by:

$$H^7_{(\gamma)}(I,\mathbb{C}) := \{\varphi \in H^7(I,\mathbb{C});\ A^l_\gamma\varphi(-1/2) = A^l_\gamma\varphi(1/2) = 0 \text{ for } l = 0,1,2,3\}.$$

The use of the Nash–Moser theorem is motivated because we work on Sobolev spaces. However, we do not think the use of the Nash–Moser theorem is necessary: there exists probably spaces on which the classical inverse mapping theorem can be applied but we do not know them for the moment.

In the last part of the proof, we construct explicitly, for $\gamma > 0$ small enough, trajectories $(\psi, u) : [0, T^1] \to H^7(I,\mathbb{C}) \times \mathbb{R}$ such that

$$u(0) = 0, \quad u(T^1) = \gamma,$$
$$\psi(0) = \varphi_1 e^{i\phi_0}, \quad \psi(T^1) \in H^7_{(\gamma)}(I,\mathbb{C}), \quad \|\psi(T^1) - \varphi_{1,\gamma}\|_{H^7} < \eta/2.$$

Then, for $\psi_0 \in H^7_{(0)}(I,\mathbb{C})$ closed enough to $\varphi_1 e^{i\phi_0}$, the same control moves the system from ψ_0 to ψ_2 which satisfies:

$$\psi_2 \in H^7_{(\gamma)}(I,\mathbb{C}) \quad \text{and} \quad \|\psi_2 - \varphi_{1,\gamma}\|_{H^7} < \eta,$$

thanks to the continuity with respect to initial condition. We also construct trajectories $(\psi, u) : [T^1 + T, T^1 + T + T^2] \to H^7(I,\mathbb{C}) \times \mathbb{R}$ such that

$$u(T^1 + T) = \gamma, \quad u(T^1 + T + T^2) = 0,$$
$$\psi(T^1 + T) \in H^7_{(\gamma)}(I,\mathbb{C}), \quad \|\psi(T^1 + T) - \varphi_{1,\gamma} e^{-i\lambda_{1,\gamma} T}\|_{H^7} < \eta/2,$$
$$\psi(T^1 + T + T^2) = \varphi_1 e^{i\phi_1}.$$

Then, for $\psi_f \in H^7_{(0)}(I,\mathbb{C})$ closed enough to $\varphi_1 e^{i\phi_1}$, the same control moves the system from ψ_3 to ψ_f, where ψ_3 satisfies,

$$\psi_3 \in H^7_{(\gamma)}(I,\mathbb{C}) \quad \text{and} \quad \|\psi_3 - \varphi_{1,\gamma} e^{-i\lambda_{1,\gamma} T}\|_{H^7} < \eta.$$

Our idea is that, starting from an initial point $\varphi_1 e^{i\phi_0}$ on the ground state for $u \equiv 0$, $t \mapsto \varphi_1 e^{i\lambda_1 t}$, if we change sufficiently slowly the value of the control u from 0 to γ, the state of the system will stay very closed, at each time t_1, to a point on the ground state for an electric field constant in $u(t_1)$, $t \mapsto \varphi_{1,u(t_1)} e^{i\lambda_{1,u(t_1)} t}$. Therefore, the final value of the state will be very closed to $\varphi_{1,\gamma}$, up to a phase factor. More precisely, we have the following theorems, proved in Section 4.

Theorem 3. *Let $\gamma_0 \in \mathbb{R}$. We consider the solution ψ_ε of the following system*:

$$\begin{cases} i\dot{\psi}_\varepsilon = -\frac{1}{2}\psi''_\varepsilon - \gamma_0 f(\varepsilon t) q \psi_\varepsilon, \\ \psi_\varepsilon(0) = \varphi_1 e^{i\phi_0}, \\ \psi_\varepsilon(t, -1/2) = \psi_\varepsilon(t, 1/2) = 0, \end{cases}$$

where $f\in C^\infty([0,1],\mathbb{R})$ *satisfies* $f^{(k)}(0)=0$ *for every* $k\in\mathbb{N}$, $f(1)=1$, $0\leqslant f\leqslant 1$ *and* $\phi_0\in[0,2\pi)$. *Let* $(\varepsilon_n)_{n\in\mathbb{N}^*}$ *be defined by*:

$$\frac{1}{\varepsilon_n}\int_0^1 \lambda_{1,\gamma_0 f(t)}\,\mathrm{d}t=\phi_0+2n\pi,$$

for every $n\in\mathbb{N}^*$. *There exists* $\gamma^*>0$ *such that, for every* $\gamma_0\in(-\gamma^*,\gamma^*)$, *for every* $s\in\mathbb{N}$, $(\psi_{\varepsilon_n}(1/\varepsilon_n))_{n\in\mathbb{N}}$ *converges to* φ_{1,γ_0} *in* $H^s(I,\mathbb{C})$.

Theorem 4. *Let* $\gamma_0\in\mathbb{R}$. *We consider the solution* ξ_ε *of the following system*:

$$\begin{cases} \mathrm{i}\dot{\xi_\varepsilon}=-\frac{1}{2}\xi_\varepsilon''-\gamma_0 f(1-\varepsilon t)q\xi_\varepsilon,\\ \xi_\varepsilon(1/\varepsilon)=\varphi_1\mathrm{e}^{\mathrm{i}\phi_1},\\ \xi_\varepsilon(t,-1/2)=\xi_\varepsilon(t,1/2)=0,\end{cases}$$

where $\phi_1\in(-2\pi,0]$. *Let* $(\varepsilon_n)_{n\in\mathbb{N}^*}$ *be defined by:*

$$\frac{1}{\varepsilon_n}\int_0^1 \lambda_{1,\gamma_0 f(t)}\,\mathrm{d}t=-\lambda_{1,\gamma_0}T-\phi_1+2(n+1)\pi,$$

for every $n\in\mathbb{N}^*$, *where* $T:=4/\pi$. *There exists* $\gamma^*>0$ *such that, for every* $\gamma\in(-\gamma^*,\gamma)$, *for every* $s\in\mathbb{N}$, $(\xi_{\varepsilon_n}(0))_{n\in\mathbb{N}^*}$ *converges to* $\varphi_{1,\gamma_0}\mathrm{e}^{-\mathrm{i}\lambda_{1,\gamma_0}T}$ *in* $H^s(I,\mathbb{C})$.

The constant γ^* is such that every proposition in Appendix A is true with $\gamma\in(-\gamma^*,\gamma^*)$.

3. Local controllability of the nonlinear system around the ground state for $u\equiv\gamma$

3.1. Controllability of the linearized system around $(\psi_{1,\gamma},\gamma)$

The linearized system around $(\psi_{1,\gamma},\gamma)$ is the following one,

$$\mathrm{i}\frac{\partial\Psi}{\partial t}=-\frac{1}{2}\frac{\partial^2\Psi}{\partial q^2}-\gamma q\Psi-w(t)q\psi_{1,\gamma},\tag{3.1}$$

$$\Psi(0)=\Psi_0,\tag{3.2}$$

$$\Psi(t,-1/2)=\Psi(t,1/2)=0,\tag{3.3}$$

where the state is $\Psi(t)$ and the control is $w(t)$. We know (see Appendix B, Proposition 45) that for every couple $(\Psi_0,w)\in H_0^1(I,\mathbb{C})\times L^2((0,T),\mathbb{R})$, there exists a unique generalized solution $\Psi \in C^0([0,T], H_0^1(I,\mathbb{C}))$ of (3.1)–(3.3), in the sense that it satisfies the following equality in $L^2(I,\mathbb{C})$, for every $t \in [0,T]$,

$$\Psi(t) = T_\gamma(t)\Psi_0 + \int_0^t T_\gamma(t-s)\big[\mathrm{i}w(s)q\psi_{1,\gamma}(s)\big]\,\mathrm{d}s. \tag{3.4}$$

In this formula, $(T_\gamma(t))_{t\geqslant 0}$ is the group of isometries of $L^2(I,\mathbb{C})$ with infinitesimal generator $-\mathrm{i}A_\gamma$. More explicitly, for $\varphi \in L^2(I,\mathbb{C})$ and $t \in \mathbb{R}$,

$$T_\gamma(t)\varphi := \sum_{k=1}^{+\infty} \langle \varphi, \varphi_{k,\gamma}\rangle \mathrm{e}^{-\mathrm{i}\lambda_{k,\gamma}t}\varphi_{k,\gamma}.$$

We assume Ψ_0 satisfies $\Re(\langle \Psi_0, \varphi_{1,\gamma}\rangle) = 0$. Then, for every $t \in [0,T]$,

$$\big\langle \Psi(t), \psi_{1,\gamma}(t)\big\rangle = \langle \Psi_0, \varphi_{1,\gamma}\rangle + \mathrm{i}\int_0^t w(s)\langle q\varphi_{1,\gamma}, \varphi_{1,\gamma}\rangle\,\mathrm{d}s \in \mathrm{i}\mathbb{R},$$

so this generalized solution satisfies $\Psi(t) \in T_S(\psi_{1,\gamma}(t))$ for every $t \in [0,T]$.

If $T > 0$ and $\Psi_T \in T_S(\psi_{1,\gamma}(T))$, the equality $\Psi(T) = \Psi_T$ is equivalent to:

$$\mathrm{i}b_{k,\gamma}\int_0^T w(t)\mathrm{e}^{\mathrm{i}(\lambda_{k,\gamma}-\lambda_{1,\gamma})t}\,\mathrm{d}t = \langle \Psi_T, \varphi_{k,\gamma}\rangle \mathrm{e}^{\mathrm{i}\lambda_{k,\gamma}T} - \langle \Psi_0, \varphi_{k,\gamma}\rangle, \quad \text{for every } k \in \mathbb{N}^*, \tag{3.5}$$

where $b_{k,\gamma} := \langle q\varphi_{1,\gamma}, \varphi_{k,\gamma}\rangle$. If $b_{k,\gamma} \neq 0$ for every $k \in \mathbb{N}^*$, this is a moment problem in $L^2((0,T),\mathbb{R})$,

$$\int_0^T w(t)\mathrm{e}^{\mathrm{i}(\lambda_{k,\gamma}-\lambda_{1,\gamma})t}\,\mathrm{d}t = d_{k,\gamma}, \quad \text{for every } k \in \mathbb{N}^*.$$

Thanks to standard results about trigonometric moment problems, we will prove this moment problem has a solution $w \in L^2((0,T),\mathbb{R})$ as soon as the right-hand side $(d_{k,\gamma})_{k\in\mathbb{N}^*}$ belongs to $l^2(\mathbb{N}^*,\mathbb{C})$, when γ is small enough, different from zero and T is positive.

The noncontrollability result when $\gamma = 0$, proved by Pierre Rouchon in [18] is related to the behaviour of the coefficients $b_{k,0}$: $b_{k,0} = 0$ for every odd integer k. When $\gamma = 0$, we only control half of the projections. The controllability when γ is small enough and different from zero is possible because as soon as $\gamma \neq 0$, we have $b_{k,\gamma} \neq 0$ for every k.

In this article, we use the same letter C to design various constants. The value of C can change from one expression to another one.

Proposition 1. *There exists $\gamma_0 > 0$ such that, for every $\gamma \in (0, \gamma_0]$ and for every $k \in \mathbb{N}^*$, $b_{k,\gamma} \neq 0$. There exist $\gamma_1 > 0$ and $C > 0$ such that, for every $\gamma \in (0, \gamma_1]$ and for every even integer $k \geqslant 2$,*

$$\left| b_{k,\gamma} - \frac{(-1)^{k/2+1} 8k}{\pi^2 (k^2-1)^2} \right| < \frac{C\gamma}{k^3},$$

and for every odd integer $k \geqslant 3$,

$$\left| b_{k,\gamma} - \gamma \frac{2(-1)^{(k-1)/2}(k^2+1)}{\pi^4 k (k^2-1)^2} \right| < \frac{C\gamma^2}{k^3}.$$

Proof. We use results on $\varphi_{k,\gamma}$ presented in Appendix A. In particular, $\varphi_{k,\gamma}$ is an analytic function of γ:

$$\begin{aligned} \varphi_{k,\gamma} &= \varphi_k + \gamma \varphi_k^{(1)} + \gamma^2 \varphi_k^{(2)} + \cdots, \\ b_{k,\gamma} &= \langle q\varphi_k, \varphi_1 \rangle + \gamma \big(\langle q\varphi_k^{(1)}, \varphi_1 \rangle + \langle q\varphi_k, \varphi_1^{(1)} \rangle \big) + \cdots. \end{aligned} \tag{3.6}$$

When k is odd (respectively, even) the first (respectively, the second) term of the right-hand side of (3.6) vanishes, because of the parity of the functions involved.

Study of $b_{1,\gamma}$. We have $b_{1,\gamma} = 2\gamma \langle q\varphi_1^{(1)}, \varphi_1 \rangle + o(\gamma)$, when $\gamma \to 0$. Using (A.5) and (A.2) we get:

$$\langle q\varphi_1^{(1)}, \varphi_1 \rangle = \frac{128}{\pi^6} \sum_{j=1}^{+\infty} \frac{(2j)^2}{(1+2j)^5 (2j-1)^5},$$

which is a positive real number.

Study of $b_{k,\gamma}$ when k is even. When k is a fixed even integer, we have:

$$b_{k,\gamma} = \langle q\varphi_k, \varphi_1 \rangle + o(\gamma) = \frac{8(-1)^{k/2+1} k}{\pi^2 (k^2-1)^2} + o(\gamma).$$

Let us prove that there exists a positive constant C such that, for every even integer k,

$$\big| \langle q\varphi_{k,\gamma}, \varphi_{1,\gamma} \rangle - \langle q\varphi_k, \varphi_1 \rangle \big| \leqslant \frac{C\gamma}{k^3}.$$

Using integrations by parts, we get:

$$\begin{aligned} \langle q\varphi_{k,\gamma}, \varphi_{1,\gamma} \rangle - \langle q\varphi_k, \varphi_1 \rangle = {} & \left(\frac{1}{\lambda_{k,\gamma}} - \frac{1}{\lambda_k} \right) \langle \varphi_{k,\gamma}, A_\gamma(q\varphi_{1,\gamma}) \rangle + \frac{1}{\lambda_k} \langle \varphi_{k,\gamma} - \varphi_k, A_\gamma(q\varphi_{1,\gamma}) \rangle \\ & + \frac{1}{\lambda_k} \langle \varphi_k, A_\gamma(q\varphi_{1,\gamma}) - A(q\varphi_1) \rangle. \end{aligned}$$

We deal with the two first terms of the right-hand side of the above equality using (A.13) and (A.7). In the third term, the scalar product is a Fourier coefficient of a C^1 function f_γ such that, for every $\gamma \in [-\gamma^*, \gamma^*]$, $\|f_\gamma\|_{C^1} \leqslant C\gamma$.

Study of $b_{k,\gamma}$ when k is odd. When k is a fixed odd integer, we have:

$$b_{k,\gamma} = \gamma\big(\langle q\varphi_k^{(1)}, \varphi_1\rangle + \langle q\varphi_k, \varphi_1^{(1)}\rangle\big) + \mathrm{o}(\gamma).$$

Using (A.5) and (A.2), we get:

$$\langle q\varphi_k^{(1)}, \varphi_1\rangle = \frac{128(-1)^{(k+1)/2}k}{\pi^6}\sum_{j=1}^{+\infty}\frac{(2j)^2}{(1+2j)^2(1-2j)^2(k+2j)^3(k-2j)^3}.$$

In order to compute this sum, we decompose the fraction,

$$F(X) = \frac{X^2}{(k+X)^3(k-X)^3(1+X)^2(1-X)^2},$$

in the following way:

$$\begin{aligned}F(X) = {} & \frac{15k^4+10k^2-1}{16k^3(k^2-1)^4}\left(\frac{1}{X+k} - \frac{1}{X-k}\right) + \frac{7k^2+1}{16k^2(k^2-1)^3}\left(\frac{1}{(X+k)^2} + \frac{1}{(X-k)^2}\right) \\ & + \frac{1}{8k(k^2-1)^2}\left(\frac{1}{(X+k)^3} - \frac{1}{(X-k)^3}\right) \\ & - \frac{k^2+5}{4(k^2-1)^4}\left(\frac{1}{X+1} - \frac{1}{X-1}\right) + \frac{1}{4(k^2-1)^3}\left(\frac{1}{(X+1)^2} + \frac{1}{(X-1)^2}\right).\end{aligned}$$

We sum each term and we get:

$$\langle q\varphi_k^{(1)}, \varphi_1\rangle = \frac{2(-1)^{(k+1)/2}(11k^2+1)}{\pi^4 k(k^2-1)^3}.$$

In the same way, we have:

$$\langle q\varphi_k, \varphi_1^{(1)}\rangle = \frac{2(-1)^{(k-1)/2}k(k^2+11)}{\pi^4(k^2-1)^3}.$$

Therefore,

$$\langle q\varphi_k^{(1)}, \varphi_1\rangle + \langle q\varphi_k, \varphi_1^{(1)}\rangle = \frac{2(-1)^{(k-1)/2}(k^2+1)}{\pi^4 k(k^2-1)^2}.$$

Let us prove that there exists a constant $C > 0$ such that, for every odd integer k,

$$\big|b_{k,\gamma} - \gamma\big(\langle q\varphi_k^{(1)}, \varphi_1\rangle + \langle q\varphi_k, \varphi_1^{(1)}\rangle\big)\big| \leqslant \frac{C\gamma^2}{k^3}.$$

Thanks to parity arguments, this inequality can be written:

$$|\Delta_{k,\gamma}| \leqslant \frac{C\gamma^2}{k^3}, \quad \text{where } \Delta_{k,\gamma} := \langle q\varphi_{k,\gamma}, \varphi_{1,\gamma}\rangle - \langle q\widetilde{\varphi}_{k,\gamma}, \widetilde{\varphi}_{1,\gamma}\rangle,$$

with $\widetilde{\varphi}_{k,\gamma} := \varphi_k + \gamma\varphi_k^{(1)}$. Using (A.1) and (A.6) and integrations by parts, we get:

$$\Delta_{k,\gamma} = \left(\frac{1}{\lambda_{k,\gamma}} - \frac{1}{\lambda_k}\right)\big\langle\varphi_{k,\gamma}, A_\gamma(q\varphi_{1,\gamma})\big\rangle + \frac{1}{\lambda_k}\big\langle\varphi_{k,\gamma} - \widetilde{\varphi}_{k,\gamma}, A_\gamma(q\varphi_{1,\gamma})\big\rangle$$
$$+ \frac{1}{\lambda_k}\big\langle\widetilde{\varphi}_{k,\gamma}, A_\gamma\big(q[\varphi_{1,\gamma} - \widetilde{\varphi}_{1,\gamma}]\big)\big\rangle - \frac{\gamma^2}{\lambda_k}\big\langle\varphi_k^{(1)}, q^2\widetilde{\varphi}_{1,\gamma}\big\rangle.$$

We deal with the first term of the right-hand side of this equality using (A.13), with the second one using (A.8) and with the fourth one using (A.18). Using the notation $f_\gamma := A_\gamma(q[\varphi_{1,\gamma} - \widetilde{\varphi}_{1,\gamma}])$, we decompose the third term in the following way:

$$\frac{1}{\lambda_k}\langle\widetilde{\varphi}_{k,\gamma}, f_\gamma\rangle = \frac{1}{\lambda_k}\langle\varphi_k, f_\gamma\rangle + \frac{\gamma}{\lambda_k}\big\langle\varphi_k^{(1)}, f_\gamma\big\rangle.$$

The first term of the right-hand side of this equality is a Fourier coefficient of a C^1-function f_γ satisfying $\|f_\gamma\|_{C^1} \leqslant \gamma^2$. We get a suitable bound on the second term of the right-hand side of this equality using (A.18) and $\|f_\gamma\|_{L^2} \leqslant \gamma^2$. □

We introduce the space:

$$H^3_{(0)}(I,\mathbb{C}) := \big\{\Psi \in H^3(I,\mathbb{C});\ \Psi(q) = \Psi''(q) = 0 \text{ for } q = -1/2, 1/2\big\}.$$

Theorem 5. *There exists $\gamma_1 > 0$ such that, for every $\gamma \in (0,\gamma_1]$, for every $T > 0$ and for every $\Psi_0, \Psi_T \in H^3_{(0)}(I,\mathbb{C})$ satisfying,*

$$\Re\big(\langle\Psi_0, \psi_{1,\gamma}(0)\rangle\big) = \Re\big(\langle\Psi_T, \psi_{1,\gamma}(T)\rangle\big) = 0, \tag{3.7}$$

there exists $w \in L^2((0,T),\mathbb{R})$ solution of the moment problem (3.5).

Proof. Thanks to (A.11), we have $\lim_{\gamma\to 0}\lambda_{j,\gamma} = \lambda_j = (j\pi)^2/2$, uniformly with respect to $j \in \mathbb{N}^*$. Thus, there exists $\gamma_1 > 0$ such that, for every $\gamma \in [0,\gamma_1]$, for every $j \in \mathbb{N}^*$, $\lambda_{j+1,\gamma} - \lambda_{j,\gamma} > 0$ and $\lim_{j\to+\infty}(\lambda_{j+1,\gamma} - \lambda_{j,\gamma}) = +\infty$.

Let $\gamma \in (0,\gamma_1]$ and $T > 0$. We know from [14] that for every $d = (d_k)_{k\in\mathbb{N}^*} \in l^2(\mathbb{N}^*,\mathbb{C})$, such that $d_1 \in \mathbb{R}$, there exists exactly one $w \in L^2((0,T),\mathbb{C})$ minimum L^2-norm solution of the moment problem:

$$\int_0^T w(t)e^{i(\lambda_{k,\gamma}-\lambda_{1,\gamma})t}\,dt = d_{k,\gamma}, \quad \int_0^T w(t)e^{-i(\lambda_{k,\gamma}-\lambda_{1,\gamma})t}\,dt = \overline{d_{k,\gamma}}, \quad \forall k \in \mathbb{N}^*.$$

Thanks to the uniqueness, w is real valued.

Let $\Psi_0, \Psi_T \in H^3_{(0)}(I,\mathbb{C})$, satisfying (3.7). Then the sequence,

$$(d_k)_{k\in\mathbb{N}^*} := \left(\frac{1}{\mathrm{i}b_{k,\gamma}}\big(\langle\Psi_T,\varphi_{k,\gamma}\rangle \mathrm{e}^{\mathrm{i}\lambda_{k,\gamma}T} - \langle\Psi_0,\varphi_{k,\gamma}\rangle\big)\right)_{k\in\mathbb{N}^*}, \tag{3.8}$$

satisfies $d_1 \in \mathbb{R}$. Let us prove that $(d_k) \in l^2(\mathbb{N}^*,\mathbb{C})$, which ends the proof. It is sufficient to prove that, if $\Psi \in H^3_{(0)}(I,\mathbb{C})$, then

$$\left(\frac{1}{b_{k,\gamma}}\langle\Psi,\varphi_{k,\gamma}\rangle\right)_{k\in\mathbb{N}^*} \in l^2(\mathbb{N}^*,\mathbb{C}). \tag{3.9}$$

Let $\Psi \in H^3_{(0)}(I,\mathbb{C})$. Then, $c_k := \langle\Psi,\varphi_{k,\gamma}\rangle$ satisfies:

$$c_k = \frac{1}{\lambda_{k,\gamma}^2}\left(\frac{1}{2}\big\langle(A_\gamma\Psi)',\varphi'_{k,\gamma}\big\rangle - \gamma\langle qA_\gamma\Psi,\varphi_{k,\gamma}\rangle\right).$$

Thanks to (A.12), we get

$$k^3|c_k| \leqslant \frac{C}{k}\big(\big|\big\langle(A_\gamma\Psi)',\varphi'_k\big\rangle\big| + \big\|(A_\gamma\Psi)'\big\|_{L^2}\|\varphi'_{k,\gamma}-\varphi'_k\|_{L^2} + \|qA_\gamma\Psi\|_{L^2(I)}\big).$$

Since $(\frac{1}{l\pi}\varphi'_l)_{l\in\mathbb{N}^*}$ is an orthonormal family of $L^2(I,\mathbb{C})$, the first term of the right-hand side of this inequality belongs to $l^2(\mathbb{N}^*,\mathbb{C})$. The second term of the right-hand side of this inequality also belongs to $l^2(\mathbb{N}^*,\mathbb{C})$ because of (A.9). We have proved (3.9). □

Remark. The assumption $\Psi_0, \Psi_T \in H^2 \cap H^1_0(I,\mathbb{C})$, is not sufficient to get (3.8) in $l^2(\mathbb{N}^*,\mathbb{C})$.

Let us introduce the map:

$$\Phi_\gamma : (\psi_0, v) \mapsto (\psi_0, \psi_T),$$

where ψ is the generalized solution of (Σ) with $u = \gamma + v$ and $\psi_T = \psi(T)$. The map Φ_γ is well defined and of class C^1

- from $[S \cap (H^2 \cap H^1_0)(I,\mathbb{C})] \times L^2((0,T),\mathbb{R})$ to $[S \cap (H^2 \cap H^1_0)(I,\mathbb{C})] \times [S \cap (H^2 \cap H^1_0)(I,\mathbb{C})]$,
- from $[S \cap H^3_{(0)}(I,\mathbb{C})] \times H^1_0((0,T),\mathbb{R})$ to $[S \cap H^3_{(0)}(I,\mathbb{C})] \times [S \cap H^3_{(0)}(I,\mathbb{C})]$.

To get the local controllability of the nonlinear system (Σ) around $\psi_{1,\gamma}$ from the standard implicit function theorem, we consider $\Psi_0, \Psi_T \in L^2(I,\mathbb{C})$ satisfying (3.7), one needs to construct a control bringing the system from Ψ_0 to Ψ_T which belongs:

– either to $L^2((0,T),\mathbb{R})$ when $\Psi_0, \Psi_T \in (H^2 \cap H^1_0)(I,\mathbb{C})$,
– or to $H^1_0((0,T),\mathbb{R})$ when $\Psi_0, \Psi_T \in H^3_{(0)}(I,\mathbb{C})$.

The previous remark explains why it does not seem to be possible.

3.2. The Nash–Moser implicit function theorem used

To get the local controllability of the nonlinear system around $\psi_{1,\gamma}$, we use the Nash–Moser implicit function theorem given by Hörmander in [15]. We need small changes in Hörmander's assumptions. Those changes do not influence much his proof. In this subsection, we first recall the context of the Nash–Moser theorem stated by Hörmander in [15]. Then, we state a Nash–Moser theorem which is a little bit different from Hörmander's one and can be directly applied to our problem. We repeat Hörmander's proof in order to justify our changes in the assumptions. Finally, we give explicitly a local diffeomorphism from the L^2-sphere S to $L^2(I,\mathbb{C})$, which allows us to use the Nash–Moser theorem on the manifold S, instead of a whole space.

We consider a decreasing family of Hilbert spaces $(E_a)_{a\in\{1,\dots,9\}}$ with continuous injections $E_b \to E_a$ of norm $\leqslant 1$ when $b \geqslant a$. Suppose we have given linear operators $S_\theta : E_1 \to E_9$ for $\theta \geqslant 1$. We assume there exists a constant $K > 0$ such that for every $a \in \{1,\dots,9\}$, for every $\theta \geqslant 1$ and for every $u \in E_a$ we have:

$$\|S_\theta u\|_b \leqslant K\|u\|_a, \quad \forall b \in \{1,\dots,a\}, \tag{3.10}$$

$$\|S_\theta u\|_b \leqslant K\theta^{b-a}\|u\|_a, \quad \forall b \in \{a+1,\dots,9\}, \tag{3.11}$$

$$\|u - S_\theta u\|_b \leqslant K\theta^{b-a}\|u\|_a, \quad \forall b \in \{1,\dots,a-1\}, \tag{3.12}$$

$$\left\|\frac{\mathrm{d}}{\mathrm{d}\theta} S_\theta u\right\|_b \leqslant K\theta^{b-a-1}\|u\|_a, \quad \forall b \in \{1,\dots,9\}. \tag{3.13}$$

Then, we have the convexity of the norms (see [15] for the proof): there exists a constant $c \geqslant 1$ such that, for every $\lambda \in [0,1]$, for every $a, b \in \{1,\dots,9\}$ such that $a \leqslant b$, $\lambda a + (1-\lambda)b \in \mathbb{N}$ and for every $u \in E_b$,

$$\|u\|_{\lambda a+(1-\lambda)b} \leqslant c\|u\|_a^\lambda \|u\|_b^{1-\lambda}.$$

We fix a sequence $1 = \theta_0 < \theta_1 < \cdots \to \infty$ of the form $\theta_j = (j+1)^\delta$ where $\delta > 0$. We set $\Delta_j := \theta_{j+1} - \theta_j$ and we introduce:

$$R_j u := \frac{1}{\Delta_j}(S_{\theta_{j+1}} - S_{\theta_j})u \text{ if } j > 0 \quad \text{and} \quad R_0 u := \frac{1}{\Delta_0} S_{\theta_1} u.$$

Thanks to (3.12), we have:

$$u = \sum_{j=0}^{\infty} \Delta_j R_j u,$$

with convergence in E_b when $u \in E_a$ and $a > b$. Moreover, (3.13) gives, for $u \in E_a$ and for every $b \in \{1, \dots, 9\}$:

$$\|R_j u\|_b \leqslant K_{a,b} \theta_j^{b-a-1} \|u\|_a,$$

where

$$K_{a,b} := K \max\{2^{\delta(b-a-1)}, 1\}, \quad \text{when } b \neq a,$$
$$K_{a,a} := K \max\left\{ \frac{\ln(\theta_{j+1}/\theta_j)}{(\theta_{j+1}/\theta_j) - 1};\ j \in \mathbb{N} \right\}.$$

This maximum is finite because, for every $\delta > 0$,

$$\lim_{j \to \infty} \left(\frac{\ln((1 + 1/j)^\delta)}{(1 + 1/j)^\delta - 1} \right) = 1.$$

Let $K' := \max\{K_{a,b};\ a, b \in \{1, \dots, 9\}\}$.

Let $a_1, a_2 \in \mathbb{N}$ and $a \in \mathbb{R}$ be such that $1 \leqslant a_1 < a < a_2 \leqslant 9$. We define the space:

$$E'_a := \left\{ \sum_{j=0}^{\infty} \Delta_j u_j;\ u_j \in E_{a_2},\ \exists M > 0 \;\middle|\; \forall j,\ \|u_j\|_b \leqslant M \theta_j^{b-a-1} \text{ for } b = a_1, a_2 \right\},$$

with the norm $\|u\|'_a$ given by the infimum of M over all such decomposition of u. This space does not depend on the choice of a_1 and a_2 (see [15] for the proof). The norm $\|.\|'_a$ is stronger than the norm $\|.\|_b$ when $b < a$ because,

$$\|u\|_b \leqslant c \left(\sum_{j=0}^{\infty} \Delta_j \theta_j^{b-a-1} \right) \|u\|'_a, \tag{3.14}$$

and $\|.\|'_a$ is weaker than $\|.\|_a$ because,

$$\|u\|'_a \leqslant K' \|u\|_a.$$

There exists a constant K'' such that, for every $a \in \{1, \dots, 9\}$, for every $\theta \geqslant 1$, for every $b < a$ and for every $u \in E'_a$ we have:

$$\|u - S_\theta u\|_b \leqslant K'' \theta^{b-a} \|u\|'_a. \tag{3.15}$$

Indeed, let $a \in [1, 9]$, $b, a_1, a_2 \in \{1, \dots, 9\}$ be such that $b < a_1 < a < a_2$. Let $u \in E'_a$ and a decomposition:

$$u = \sum \Delta_j u_j \quad \text{with } \|u_j\|_{a_i} \leqslant M \theta_j^{a_i - a - 1} \text{ for } i = 1, 2.$$

We have:

$$u - S_\theta u = \sum \Delta_j (u_j - S_\theta u_j),$$
$$\|u_j - S_\theta u_j\|_b \leqslant K M \theta^{b-a_i} \theta_j^{a_i - a - 1} \quad \text{for } i = 1, 2.$$

We sum for $\theta_j < \theta$ with $i = 2$ and for $\theta_j \geqslant \theta$ with $i = 1$ and we get (3.15) for

$$K'' := K\left(\frac{2^{\delta(a+1-a_1)}}{a - a_1} + \max\left\{1, \frac{2^{\delta(a+1-a_2)}}{a_2 - a}\right\}\right).$$

Note that, when b and a are fixed, if we need (3.15), it is sufficient to know (3.12) for two values a_1 and a_2 satisfying $b < a_1 < a < a_2$. We will use this remark in the construction of smoothing operators for our problem.

We have another family $(F_a)_{a \in \{1,\dots,9\}}$ with the same properties as above, we use the same notations for the smoothing operators. Moreover, we assume the injection $F_b \to F_a$ is compact when $b > a$.

Theorem 6. *Let α and β be fixed positive real numbers such that*

$$4 < \alpha < \beta < 7 \quad \textit{and} \quad \beta - \alpha \geqslant 2. \tag{3.16}$$

Let V be a convex $E_\alpha^{'}$-neighbourhood of 0 and Φ a map from $V \cap E_7$ to F_β which is twice differentiable and satisfies,

$$\left\|\Phi''(u; v, w)\right\|_7 \leqslant C \sum \left(1 + \|u\|_{m_j'}\right) \|v\|_{m_j''} \|w\|_{m_j'''}, \tag{3.17}$$

where the sum is finite, all the subscripts belong to $\{1, 3, 5, 7\}$ and satisfy:

$$\max(m_j' - \alpha, 0) + \max(m_j'', 2) + m_j''' < 2\alpha, \quad \forall j. \tag{3.18}$$

We assume that $\Phi : E_a \to F_a$ is continuous for $a = 1, 3$. We also assume that $\Phi'(v)$, for $v \in V \cap E_9$, has a right inverse $\psi(v)$ mapping F_9 to E_7, that $(v, g) \mapsto \psi(v)g$ is continuous from $(V \cap E_9) \times F_9$ to E_7 and that there exists a constant C such that for every $(v, g) \in (V \cap E_9) \times F_9$,

$$\left\|\psi(v)g\right\|_1 \leqslant C\|g\|_3, \tag{3.19}$$

$$\left\|\psi(v)g\right\|_3 \leqslant C\left[\|g\|_5 + \|v\|_5 \|g\|_3\right], \tag{3.20}$$

$$\left\|\psi(v)g\right\|_5 \leqslant C\left[\|g\|_7 + \|v\|_5\|g\|_5 + \left(\|v\|_7 + \|v\|_5^2\right)\|g\|_3\right], \tag{3.21}$$

$$\begin{aligned}\left\|\psi(v)g\right\|_7 \leqslant C\Big[&\|g\|_9 + \|v\|_5\|g\|_7 + \left(\|v\|_7 + \|v\|_5^2\right)\|g\|_5 \\ &+ \left(\|v\|_9 + \|v\|_7\|v\|_5 + \|v\|_5^3\right)\|g\|_3\Big].\end{aligned} \tag{3.22}$$

For every $f \in F_\beta'$ with sufficiently small norm one can find a sequence $u_j \in V \cap E_7$ which converges in E_b for every $b < \alpha$ to u satisfying $\Phi(u) = \Phi(0) + f$.

Remark. The main difference with Hörmander's statement concerns the bounds (3.19)–(3.22).

Proof. Let $g \in F'_\beta$. There exists a decomposition:

$$g = \sum \Delta_j g_j \quad \text{with } \|g_j\|_b \leqslant K'\theta_j^{b-\beta-1}\|g\|'_\beta \text{ for every } b \in \{1, \dots, 9\}. \tag{3.23}$$

We claim that if $\|g\|'_\beta$ is small enough we can define a sequence $u_j \in E_7 \cap V$ with $u_0 = 0$ by the recursive formula:

$$u_{j+1} := u_j + \Delta_j \dot{u}_j, \quad \dot{u}_j := \psi(v_j) g_j, \quad v_j := S_{\theta_j} u_j. \tag{3.24}$$

We also claim that there exist constants C_1, C_2, C_3 such that for every $j \in \mathbb{N}$,

$$\|\dot{u}_j\|_a \leqslant C_1 \|g\|'_\beta \theta_j^{a-\alpha-1}, \quad a \in \{1, 3, 5, 7\}, \tag{3.25}$$

$$\|v_j\|_a \leqslant C_2 \|g\|'_\beta \theta_j^{a-\alpha}, \quad a \in \{5, 7, 9\}, \tag{3.26}$$

$$\|u_j - v_j\|_a \leqslant C_3 \|g\|'_\beta \theta_j^{a-\alpha}, \quad a \in \{1, 3, 5, 7\}. \tag{3.27}$$

More precisely, we prove by induction on k the following property $\mathcal{P}_k$:

- u_j is well defined for $j = 0, \dots, k+1$,
- (3.25) is satisfied for $j = 0, \dots, k$,
- (3.26), (3.27) are satisfied for $j = 0, \dots, k+1$.

Let $k \in \mathbb{N}^*$. We suppose the property $\mathcal{P}_{k-1}$ is true, and we prove $\mathcal{P}_k$. We introduce a real number $\rho > 0$ such that, for every $u \in E'_\alpha$, $\|u\|'_\alpha \leqslant \rho$ implies $u \in V$. We have:

$$u_k = \sum_{j=0}^{k-1} \Delta_j \dot{u}_j,$$

so (3.25) gives

$$\|u\|'_\alpha \leqslant C_1 \|g\|'_\beta. \tag{3.28}$$

We also have,

$$v_k = \sum_{j=0}^{k-1} \Delta_j S_{\theta_k} \dot{u}_j,$$

so using (3.10) and (3.25) for $j = 0, \dots, k-1$, we get:

$$\|S_{\theta_k} \dot{u}_j\|_a \leqslant K C_1 \|g\|'_\beta \theta_j^{a-\alpha-1} \quad \text{for } j = 0, \dots, k-1 \text{ and } a = 1, 3, 5, 7,$$

thus

$$\|v_k\|'_\alpha \leqslant KC_1\|g\|'_\beta.$$

Therefore, when $\|g\|'_\beta \leqslant \rho/KC_1$, $v_k \in V$ and u_{k+1} is defined.

We prove (3.25) for $j = k$ by application of (3.19)–(3.22). For the case $a = 1$, using (3.19) and (3.23), we get:

$$\|\dot{u}_k\|_1 \leqslant CK'\theta_k^{2-\beta}\|g\|'_\beta,$$

which gives (3.25) with any constant $C_1 \geqslant CK'$ because $\beta - \alpha \geqslant 2$. For the case $a = 3$, using (3.20), (3.23) and (3.26) for $j = k$, we get:

$$\|\dot{u}_k\|_3 \leqslant CK'\|g\|'_\beta\left(\theta_k^{4-\beta} + C_2\|g\|'_\beta\theta_k^{5-\alpha}\theta_k^{2-\beta}\right).$$

This gives (3.25) with any constant $C_1 \geqslant 2CK'$ when $\|g\|'_\beta \leqslant 1/C_2$, because $\beta - \alpha \geqslant 2$ and $\beta \geqslant 5$. For $a = 5$, using (3.21), (3.23) and (3.26) for $j = k$, we get:

$$\|\dot{u}_k\|_5 \leqslant CK'\|g\|'_\beta\left[\theta_k^{6-\beta} + C_2\|g\|'_\beta\left(\theta_k^{5-\alpha}\theta_k^{4-\beta} + \theta_k^{7-\alpha}\theta_k^{2-\beta}\right) + C_2^2\|g\|'^2_\beta\theta_k^{10-2\alpha}\theta_k^{2-\beta}\right].$$

This gives (3.25) with any constant $C_1 \geqslant 4CK'$, when $\|g\|'_\beta \leqslant 1/C_2$, because $\beta - \alpha \geqslant 2$, $\beta \geqslant 5$ and $\beta + \alpha \geqslant 8$. For the case $a = 7$, using (3.22), (3.23) and (3.26), we get:

$$\begin{aligned}\|\dot{u}_k\|_7 \leqslant CK'\|g\|'_\beta\Big[&\theta_k^{8-\beta} + C_2\|g\|'_\beta\left(\theta_k^{5-\alpha}\theta_k^{6-\beta} + \theta_k^{7-\alpha}\theta_k^{4-\beta} + \theta_k^{9-\alpha}\theta_k^{2-\beta}\right)\\ &+ \left(C_2\|g\|'_\beta\right)^2\left(\theta_k^{10-2\alpha}\theta_k^{4-\beta} + \theta_k^{7-\alpha}\theta_k^{5-\alpha}\theta_k^{2-\beta}\right) + \left(C_2\|g\|'_\beta\right)^3\theta_k^{15-3\alpha}\theta_k^{2-\beta}\Big].\end{aligned}$$

This gives (3.25) with any constant $C_1 \geqslant 7CK'$ when $\|g\|'_\beta \leqslant 1/C_2$, because $\beta - \alpha \geqslant 2$, $\beta \geqslant 5$, $\alpha + \beta \geqslant 8$ and $2\alpha + \beta \geqslant 11$. Finally, we have proved (3.25) for $j = k$ with any constant $C_1 \geqslant 7CK'$, when $\|g\|'_\beta \leqslant 1/C_2$ and $\|g\|'_\beta \leqslant \rho/C_1K$.

Now, we prove (3.26) for $j = k + 1$. Let $a \in \{5, 7\}$. Using (3.10) and (3.25), we have:

$$\|v_{k+1}\|_a \leqslant KC_1\|g\|'_\beta\sum_{j=0}^{k}\Delta_j\theta_j^{a-\alpha-1}.$$

We find an upper bound for the sum in the cases $a = 5$ and $a = 7$, for $a = 9$, we use (3.11) and we get (3.26) with

$$C_2 := KC_1\max\left\{\frac{1}{7-\alpha}, \frac{2^{\delta(\alpha-4)}}{5-\alpha}\right\}. \tag{3.29}$$

Now, we prove (3.27) for $j = k + 1$. Thanks to the convexity of the norms, it is sufficient to prove the inequality for $a = 7$ and for $a = 1$. Using (3.10) and (3.25), we get:

$$\|u_{k+1} - v_{k+1}\|_7 \leqslant (1+K)C_1\|g\|'_\beta \sum_{j=0}^{k} \Delta_j \theta_j^{6-\alpha} \leqslant \frac{1+K}{7-\alpha} C_1 \|g\|'_\beta \theta_{k+1}^{7-\alpha}.$$

Using (3.15) with $b = 1$, $a = \alpha$, we get:

$$\|u_{k+1} - v_{k+1}\|_1 \leqslant K'' C_1 \|g\|'_\beta \theta_{k+1}^{1-\alpha}.$$

Finally, we get (3.27) for $j = k+1$ with

$$C_3 := c \max\left\{\frac{1+K}{7-\alpha} C_1, K'' C_1\right\}. \tag{3.30}$$

In conclusion, $\mathcal{P}_k$ is true for every $k \in \mathbb{N}$ with

$$C_1 := 7CK',\ C_2 \text{ defined by (3.29)},\ C_3 \text{ defined by (3.30) and } \|g\|'_\beta \leqslant \min\left\{\frac{1}{C_2}, \frac{\rho}{KC_1}\right\}.$$

The inequality (3.25) proves (u_k) is a Cauchy sequence in E_a for $a = 1, 3$ so $u_k \to u$ in E_a for $a = 1, 3$. The continuity of Φ gives $\Phi(u_k) \to \Phi(u)$ in F_a, for $a = 1, 3$.

Now, let us consider the limit of $(\Phi(u_k))_{k\in\mathbb{N}}$. We have:

$$\Phi(u_{j+1}) - \Phi(u_j) = \Phi(u_j + \Delta_j \dot{u}_j) - \Phi(u_j) = \Delta_j(e'_j + e''_j + g_j),$$

where

$$e'_j := \frac{1}{\Delta_j}\big(\Phi(u_j + \Delta_j \dot{u}_j) - \Phi(u_j) - \Phi'(u_j)\Delta_j \dot{u}_j\big),$$
$$e''_j := \big(\Phi'(u_j) - \Phi'(v_j)\big)\dot{u}_j.$$

Let us study e'_j. We have:

$$e'_j = \Delta_j \int_0^1 (1-t)\Phi''(u_j + t\Delta_j \dot{u}_j; \dot{u}_j, \dot{u}_j)\,\mathrm{d}t.$$

Using (3.17), we get:

$$\|e'_j\|_7 \leqslant C\Delta_j \sum_l \big(1 + \|u_j\|_{m'_l} + \Delta_j \|\dot{u}_j\|_{m'_l}\big)\|\dot{u}_j\|_{m''_l}\|\dot{u}_j\|_{m'''_l}.$$

For $a \in \{1, 3\}$, using (3.14) and (3.28) we get, for every $j \in \mathbb{N}$, $\|u_j\|_a \leqslant \widetilde{C}\|g\|'_\beta$, with some constant $\widetilde{C}$. For $a \in \{5, 7\}$, with the same proof as for (3.26), we get, for every $j \in \mathbb{N}$, $\|u_j\|_a \leqslant \widetilde{C}\|g\|'_\beta \theta_j^{a-\alpha}$, with some constant $\widetilde{C}$. Those bounds, together with (3.17) leads to,

$$\|e'_j\|_7 \leqslant \widetilde{C}\Delta_j \sum_l \big(1 + \|g\|'_\beta \theta_j^{\max(m'_l-\alpha,0)} + \Delta_j \|g\|'_\beta \theta_j^{m'_l-\alpha-1}\big)\|g\|'^2_\beta \theta_j^{m''_l+m'''_l-2\alpha-2},$$

with a new constant $\widetilde{C}$. Let $\varepsilon > 0$ be such that (3.46) is true with $2\alpha - \varepsilon$ on the right-hand side. Then, there exists a constant $C_4 > 0$ such that, for every $j \in \mathbb{N}$,

$$\|e'_j\|_7 \leqslant C_4 \|g\|'^2_\beta \theta_j^{-1-\varepsilon}. \tag{3.31}$$

Let us prove a similar bound on e''_j. We have:

$$e''_j = \int_0^1 \Phi''\big(v_j + t(u_j - v_j); u_j - v_j, \dot{u}_j\big)\, \mathrm{d}t.$$

Let us recall that, for $a \in \{1, 3\}$, thanks to (3.14),

$$\|v_j\|_a \leqslant \widetilde{C} \|v_j\|'_\alpha \leqslant \widetilde{C} K C_1 \|g\|'_\beta.$$

Using this bound, together with (3.25)–(3.27), we get, with a new constant $\widetilde{C}$,

$$\begin{aligned}
\|e''_j\|_7 &\leqslant C \sum_k \big(1 + \|v_j\|_{m'_k} + \|u_j - v_j\|_{m'_k}\big) \|u_j - v_j\|_{m''_k} \|\dot{u}_j\|_{m'''_k} \\
&\leqslant \widetilde{C} \sum_k \big(1 + \|g\|'_\beta \theta_j^{\max(m'_k - \alpha, 0)} + \|g\|'_\beta \theta_j^{m'_k - \alpha}\big) \|g\|'^2_\beta \theta_j^{m''_k + m''_k - 2\alpha - 1}.
\end{aligned}$$

Thanks to (3.46), we get the existence of a constant $C_5 > 0$ such that

$$\|e''_j\|_7 \leqslant C_5 \|g\|'^2_\beta \theta_j^{-1-\varepsilon}. \tag{3.32}$$

Using $\theta_j = (1 + j)^\delta$ with $\delta > 0$, it is easy to get the convergence in F_7 of $\sum \Delta_j (e'_j + e''_j)$. Let us denote $T(g)$ this limit,

$$T(g) := \sum_{j=0}^{+\infty} \Delta_j (e'_j + e''_j).$$

Thanks to (3.32) and (3.31), there exists a constant $C_6 > 0$ such that

$$\big\|T(g)\big\|_7 \leqslant C_6 \|g\|'^2_\beta.$$

The uniqueness of the limit of $\Phi(u_k)$ gives the following equality in F_a for $a = 1, 3$,

$$\Phi(u) = \Phi(0) + T(g) + g.$$

Let us fix $f \in F'_\beta$. We search u such that $\Phi(u) = \Phi(0) + f$. It is sufficient to find $g \in F'_\beta$ such that $g + T(g) = f$. It is equivalent to prove the existence of a fixed point for the map:

$$F : F'_\beta \to F'_\beta,$$
$$g \mapsto f - T(g).$$

We conclude by applying the Leray–Schauder fixed-point theorem. □

Remark. It can be useful to have the continuity of the right inverse of the map Φ. This can be obtained by using the Banach fixed point theorem, instead of the Leray–Schauder fixed point theorem, in the previous proof. In order to do this, we need more assumptions than in Theorem 6. We propose a proof of this other version of the Nash–Moser theorem and its application to the controllability of (Σ) in Appendix C.

We will apply this theorem to $\Phi_\gamma : (\psi_0, v) \mapsto (\psi_0, \psi_T)$ defined in Section 3.1. In a neighbourhood of $(\varphi_{1,\gamma}, 0)$. Our spaces are:

$$\begin{aligned}
E_1^\gamma &:= \big[H_0^1(I,\mathbb{C}) \cap S\big] \times L^2\big((0,T),\mathbb{R}\big), & F_1^\gamma &:= \big[H_0^1(I,\mathbb{C}) \cap S\big] \times \big[H_0^1(I,\mathbb{C}) \cap S\big],\\
E_3^\gamma &:= \big[H_{(\gamma)}^3(I,\mathbb{C}) \cap S\big] \times H_0^1\big((0,T),\mathbb{R}\big), & F_3^\gamma &:= \big[H_{(\gamma)}^3(I,\mathbb{C}) \cap S\big] \times \big[H_{(\gamma)}^3(I,\mathbb{C}) \cap S\big],\\
E_5^\gamma &:= \big[H_{(\gamma)}^5(I,\mathbb{C}) \cap S\big] \times H_0^2\big((0,T),\mathbb{R}\big), & F_5^\gamma &:= \big[H_{(\gamma)}^5(I,\mathbb{C}) \cap S\big] \times \big[H_{(\gamma)}^5(I,\mathbb{C}) \cap S\big],\\
E_7^\gamma &:= \big[H_{(\gamma)}^7(I,\mathbb{C}) \cap S\big] \times H_0^3\big((0,T),\mathbb{R}\big), & F_7^\gamma &:= \big[H_{(\gamma)}^7(I,\mathbb{C}) \cap S\big] \times \big[H_{(\gamma)}^7(I,\mathbb{C}) \cap S\big],\\
E_9^\gamma &:= \big[H_{(\gamma)}^9(I,\mathbb{C}) \cap S\big] \times H_0^4\big((0,T),\mathbb{R}\big), & F_9^\gamma &:= \big[H_{(\gamma)}^9(I,\mathbb{C}) \cap S\big] \times \big[H_{(\gamma)}^9(I,\mathbb{C}) \cap S\big],
\end{aligned}$$

where

$$H_{(\gamma)}^s(I,\mathbb{C}) := \left\{\Psi \in H^s(I,\mathbb{C});\ A_\gamma^l \Psi\left(-\frac{1}{2}\right) = A_\gamma^l \Psi\left(\frac{1}{2}\right) = 0 \text{ for } l = 0, \dots, (s-1)/2\right\}.$$

Our E'_α-neighbourhood V of $(\varphi_{1,\gamma}, 0)$ is a E_3^γ-bowl.

We work on the manifold S instead of a whole space. It does not matter because we can move the problem to an hyperplane of $L^2(I,\mathbb{C})$ by studying:

$$\widetilde{\Phi}_\gamma := q_\gamma o \Phi_\gamma o r_\gamma, \quad \text{where } r_\gamma(x,u) := \big(p_\gamma^{-1}(x), u\big),$$
$$q_\gamma(\psi_0, \psi_T) := \big(p_\gamma(\psi_0), p_\gamma(\psi_T)\big),$$

and p_γ is a suitable local diffeomorphism from a neighbourhood of the trajectory $\psi_{1,\gamma}$ in the sphere S into an hyperplane of $L^2(I,\mathbb{C})$, which does not change too much the H^s-norm. For example, we can use the following one:

Proposition 2. *Let* $U_\gamma := \{\psi \in L^2(I,\mathbb{C});\ \exists t \in [0,2\pi],\ \|\psi - \varphi_{1,\gamma} e^{it}\|_{L^2} < 1/12\}$, $\mathcal{H}_\gamma := \{\psi \in L^2(I,\mathbb{C});\ \Re(\langle \psi, \varphi_{2,\gamma}\rangle) = 0\}$ *and* $p_\gamma : L^2(I,\mathbb{C}) \to \mathcal{H}_\gamma$ *be defined by*:

$$p_\gamma(\psi) := \psi - \Re\big(\langle \psi, \varphi_{2,\gamma}\rangle\big)\varphi_{2,\gamma} + \Re\big(\langle \psi, \varphi_{2,\gamma}\rangle\big)\langle \psi, \varphi_{1,\gamma}\rangle \varphi_{1,\gamma}.$$

Then p_γ is a C^1 diffeomorphism from U_γ to an open subset of $\mathcal{H}_\gamma$. Moreover, the norm of $\mathrm{d}p_\gamma(\psi)$ as a linear operator from $(T_{\psi_\gamma} S, \|.\|_{H^s})$ to $(\mathcal{H}_\gamma, \|.\|_{H^s})$ is uniformly bounded on U_γ for every integer $s \in [1,7]$.

Proof. Let us introduce the orthogonal projection:

$$P_\gamma : L^2(I,\mathbb{C}) \to (\mathbb{R}\varphi_{2,\gamma} \oplus \mathbb{C}\varphi_{1,\gamma})^\perp.$$

We first prove that p_γ is injective. Let $\psi, \tilde{\psi} \in S$ be such that $p_\gamma(\psi) = p_\gamma(\tilde{\psi})$. Then, $P_\gamma(\psi) = P_\gamma(\tilde{\psi})$ and

$$\big(1 + \Re\big(\langle \psi, \varphi_{2,\gamma}\rangle\big)\big)\langle \psi, \varphi_{1,\gamma}\rangle = \big(1 + \Re\big(\langle \tilde{\psi}, \varphi_{2,\gamma}\rangle\big)\big)\langle \tilde{\psi}, \varphi_{1,\gamma}\rangle. \tag{3.33}$$

We have:

$$1 = \|\psi\|_{L^2}^2 = \big\|P_\gamma(\psi)\big\|_{L^2}^2 + \Re\big(\langle \psi, \varphi_{2,\gamma}\rangle\big)^2 + \big|\langle \psi, \varphi_{1,\gamma}\rangle\big|^2,$$
$$1 = \|\tilde{\psi}\|_{L^2}^2 = \big\|P_\gamma(\tilde{\psi})\big\|_{L^2}^2 + \Re\big(\langle \tilde{\psi}, \varphi_{2,\gamma}\rangle\big)^2 + \big|\langle \tilde{\psi}, \varphi_{1,\gamma}\rangle\big|^2,$$

so

$$\Re\big(\langle \psi, \varphi_{2,\gamma}\rangle\big)^2 + \big|\langle \psi, \varphi_{1,\gamma}\rangle\big|^2 = \Re\big(\langle \tilde{\psi}, \varphi_{2,\gamma}\rangle\big)^2 + \big|\langle \tilde{\psi}, \varphi_{1,\gamma}\rangle\big|^2.$$

Using (3.33), we get:

$$\begin{aligned}&\big|\langle \psi, \varphi_{1,\gamma}\rangle\big|^2\Big(\big(1 + \Re\big(\langle \tilde{\psi}, \varphi_{2,\gamma}\rangle\big)\big)^2 - \big(1 + \Re\big(\langle \psi, \varphi_{2,\gamma}\rangle\big)\big)^2\Big)\\ &\quad = \Big(\Re\big(\langle \tilde{\psi}, \varphi_{2,\gamma}\rangle\big)^2 - \Re\big(\langle \psi, \varphi_{2,\gamma}\rangle\big)^2\Big)\big(1 + \Re\big(\langle \tilde{\psi}, \varphi_{2,\gamma}\rangle\big)\big)^2.\end{aligned}$$

We assume $\psi \neq \tilde{\psi}$. Then $\Re(\langle \psi, \varphi_{2,\gamma}\rangle) \neq \Re(\langle \tilde{\psi}, \varphi_{2,\gamma}\rangle)$ so $\Re(\langle \tilde{\psi}, \varphi_{2,\gamma}\rangle)$ is a solution in $[-1/12, 1/12]$ of the equation $f(y) = 0$, where

$$f(y) := (1+y)^2(b+y) - a^2(2+y+b),$$
$$a := \big|\langle \psi, \varphi_{1,\gamma}\rangle\big|,\ b := \Re\big(\langle \psi, \varphi_{2,\gamma}\rangle\big).$$

Using $a \in [11/12, 1]$ and $b \in [-1/12, 1/12]$ it is easy to prove that $f(y) \leqslant [13^2 - 11^3]/(6 * 12^2) < 0$ for every $y \in [-1/12, 1/12]$. This is a contradiction. Therefore $\psi = \tilde{\psi}$ and p_γ is injective.

Now, we prove that for every $\psi \in U_\gamma$, $\mathrm{d}p_\gamma(\psi)$ is an isomorphism from $T_\psi S$ to $\mathcal{H}_\gamma$. We recall the orthogonality is related to the scalar product $\Re(\langle \psi, \tilde{\psi}\rangle)$. Let $\psi \in U_\gamma$ and $\xi \in \mathcal{H}_\gamma$. For $h \in L^2(I,\mathbb{C})$ the statement $\mathrm{d}p_\gamma(\psi)h = \xi$ and $h \in T_\psi S$ is equivalent to $P_\gamma(h) = P_\gamma(\xi)$ and $AX = b$, where

$$A := \begin{pmatrix} 1+\Re(\langle\psi,\varphi_{2,\gamma}\rangle) & 0 & \Re(\langle\psi,\varphi_{1,\gamma}\rangle) \\ 0 & 1+\Re(\langle\psi,\varphi_{2,\gamma}\rangle) & \Im(\langle\psi,\varphi_{1,\gamma}\rangle) \\ \Re(\langle\psi,\varphi_{1,\gamma}\rangle) & \Im(\langle\psi,\varphi_{1,\gamma}\rangle) & \Re(\langle\psi,\varphi_{2,\gamma}\rangle) \end{pmatrix},$$

$$X := \begin{pmatrix} \Re(\langle h,\varphi_{1,\gamma}\rangle) \\ \Im(\langle h,\varphi_{1,\gamma}\rangle) \\ \Re(\langle h,\varphi_{2,\gamma}\rangle) \end{pmatrix},$$

$$b := \begin{pmatrix} \Re(\langle\xi,\varphi_{1,\gamma}\rangle) \\ \Im(\langle\xi,\varphi_{1,\gamma}\rangle) \\ -\Re(\langle P(\xi), P(\psi)\rangle) \end{pmatrix}.$$

Using,

$$\left|\langle\psi,\varphi_{1,\gamma}\rangle\right| = \left|\left\langle\psi-\varphi_{1,\gamma}\mathrm{e}^{\mathrm{i}t},\varphi_{1,\gamma}\right\rangle + \mathrm{e}^{\mathrm{i}t}\right| > 1-1/12,$$
$$\left|\langle\psi,\varphi_{2,\gamma}\rangle\right| = \left|\left\langle\psi-\varphi_{1,\gamma}\mathrm{e}^{\mathrm{i}t},\varphi_{2,\gamma}\right\rangle\right| < 1/12,$$

we get $|\det(A)| > 1/2$. We conclude thanks to the inverse mapping theorem.

It is clear that $\|\mathrm{d}p_\gamma(\psi)\|_{H^s\to H^s} \leqslant 4$. Since $\|P_\gamma(\xi)\|_{H^s} \leqslant \|\xi\|_{H^s}$ and $\|A\| = \|\det(A)^{-1}\mathrm{Com}(A)^t\|$, then $\|A^{-1}\|$ is uniformly bounded with respect to $\psi \in U_\gamma$ and $\|\mathrm{d}p_\gamma(\psi)^{-1}\|_{H^s\to H^s}$ also. □

3.3. *Smoothing operators*

In this subsection, we construct smoothing operators on the spaces E_a^γ and F_b^γ defined in the previous subsection. In the proof of the Nash–Moser theorem, we use on the spaces F_b^γ smoothing operators,

$$S_\theta : F_1^\gamma \to F_9^\gamma,$$

with the properties (3.10)–(3.13) and on the spaces E_a^γ smoothing operators,

$$S_\theta : E_1^\gamma \to E_7^\gamma$$

with the properties (3.10) and (3.15) for $b=1$ and $a=\alpha$. Therefore, it is sufficient to check the properties (3.10) and (3.12) with $b=1$ and $a=3,5$ on the smoothing operators on the spaces E_a^γ. The construction proposed for the smoothing operators on the controls v could also be used for the wave function ψ. We propose in the next paragraph a simpler one.

3.3.1. *Smoothing operators on the spaces F_b^γ*

We do not need smoothing operators preserving the L^2-sphere, because we can move our problem on the hyperplane of $L^2(I,\mathbb{C})$ defined by:

$$\mathcal{H}_\gamma = \left\{\psi \in L^2(I,\mathbb{C});\ \Re\left(\langle\psi,\varphi_{2,\gamma}\rangle\right) = 0\right\}.$$

In this paragraph, we construct smoothing operators preserving $\mathcal{H}_\gamma$.

Let $s \in C^{\infty}(\mathbb{R}, \mathbb{R})$ be such that

$$s = 1 \text{ on } [0,1], \quad 0 \leqslant s \leqslant 1, \quad s = 0 \text{ on } [2, \infty).$$

Let $\gamma \in (0, \gamma_*]$. For $\theta \in [1, +\infty)$ and $\varphi \in L^2(I, \mathbb{C})$, we define:

$$S_\theta \varphi := \sum_{k=1}^{\infty} s\left(\frac{k}{\theta}\right) \langle \varphi, \varphi_{k,\gamma} \rangle \varphi_{k,\gamma}.$$

Proposition 3. *There exists a constant K such that, for every $a \in \{1, \dots, 9\}$, for every $\varphi \in H^a_{(\gamma)}(I, \mathbb{C})$ and for every $\theta \geqslant 1$, we have*:

$$\|S_\theta \varphi\|_{H^b} \leqslant K \|\varphi\|_{H^a}, \quad b \in \{1, \dots, a\}, \tag{3.34}$$

$$\|S_\theta \varphi\|_{H^b} \leqslant K \theta^{b-a} \|\varphi\|_{H^a}, \quad b \in \{a+1, \dots, 9\}, \tag{3.35}$$

$$\|\varphi - S_\theta \varphi\|_{H^b} \leqslant K \theta^{b-a} \|\varphi\|_{H^a}, \quad b \in \{1, \dots, a-1\}, \tag{3.36}$$

$$\left\| \frac{\mathrm{d}}{\mathrm{d}\theta} S_\theta \varphi \right\|_{H^b} \leqslant K \theta^{b-a-1} \|\varphi\|_{H^a}, \quad b \in \{1, \dots, 9\}. \tag{3.37}$$

In order to prove this proposition, we need the following lemma which will be proved later.

Lemma 1. *There exist $\gamma_* > 0$, $Q_1 > 0$ and $Q_2 > 0$ such that, for every $\gamma \in (0, \gamma_*]$, for every $s \in \{1, 3, 5, 7, 9\}$ and for every $\varphi \in H^s_{(\gamma)}(I, \mathbb{C})$, we have*:

$$Q_1 \|\varphi\|_{H^s} \leqslant \left(\sum_{k=1}^{\infty} \left| k^s \langle \varphi, \varphi_{k,\gamma} \rangle \right|^2 \right)^{1/2} \leqslant Q_2 \|\varphi\|_{H^s}. \tag{3.38}$$

Proof of Proposition 3. Let $a \in \{1, \dots, 9\}$, $\varphi \in H^a_{(\gamma)}(I, \mathbb{C})$ and $\theta \geqslant 1$. Using $0 \leqslant s \leqslant 1$ and (3.38), we get:

$$\|S_\theta \varphi\|_{H^a} \leqslant \frac{1}{Q_1} \left(\sum_{k=1}^{\infty} \left| k^a s\left(\frac{k}{\theta}\right) \langle \varphi, \varphi_{k,\gamma} \rangle \right|^2 \right)^{1/2} \leqslant \frac{Q_2}{Q_1} \|\varphi\|_a.$$

Let $b \in \{a+1, \dots, 9\}$. Using $s = 0$ on $[2, +\infty)$, $0 \leqslant s \leqslant 1$ and (3.38), we get:

$$\|S_\theta \varphi\|_{H^b} \leqslant \frac{1}{Q_1} (2\theta)^{b-a} \left(\sum_{1 \leqslant k < 2\theta} \left| k^a \langle \varphi, \varphi_{k,\gamma} \rangle \right|^2 \right)^{1/2} \leqslant \frac{Q_2}{Q_1} (2\theta)^{b-a} \|\varphi\|_{H^a}.$$

Let $b \in \{1, \dots, a-1\}$. Using $s = 1$ on $[0, 1]$ and (3.38), we get:

$$\|\varphi - S_\theta \varphi\|_{H^b} \leqslant \frac{1}{Q_1}\theta^{b-a}\left(\sum_{k\geqslant\theta}\left|k^a\langle\varphi,\varphi_{k,\gamma}\rangle\right|^2\right)^{1/2} \leqslant \frac{Q_2}{Q_1}\theta^{b-a}\|\varphi\|_{H^a}.$$

Let $b \in \{1,\dots,9\}$. We have:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}S_\theta\varphi = -\sum_{k=1}^{\infty}\frac{k}{\theta^2}\dot{s}\left(\frac{k}{\theta}\right)\langle\varphi,\varphi_{k,\gamma}\rangle\varphi_{k,\gamma}.$$

Using $\dot{s}=0$ on $[0,1]$ and $[2,\infty)$, we get:

$$\left\|\frac{\mathrm{d}}{\mathrm{d}\theta}S_\theta\varphi\right\|_{H^b} \leqslant \frac{Q_2}{Q_1}\|\dot{s}\|_\infty \max\left\{1, 2^{b-a+1}\right\}\theta^{b-a-1}\|\varphi\|_{H^a}. \quad \square$$

Proof of Lemma 1. There exist positive constants γ_0, $\mathcal{P}$, $\mathcal{C}_-$, $\mathcal{C}_+$ such that for every $s \in \{1,3,5,7,9\}$, for every $\gamma \in [0,\gamma_0]$ and for every $\varphi \in H^s_{(\gamma)}(I,\mathbb{C})$,

$$\left\|\varphi^{(s)}\right\|_{L^2} \leqslant \|\varphi\|_{H^s} \leqslant \mathcal{P}\left\|\varphi^{(s)}\right\|_{L^2},$$
$$\mathcal{C}_-\|\varphi\|_{H^s} \leqslant \left\|\left(A_\gamma^{(s-1)/2}\varphi\right)'\right\|_{L^2} \leqslant \mathcal{C}_+\|\varphi\|_{H^s},$$
$$\left\|A_\gamma^{(s-1)/2}\varphi\right\|_{L^2} \leqslant \mathcal{C}_+\|\varphi\|_{H^s}.$$

Let $s \in \{1,3,5,7,9\}$ and $\sigma \in \{0,1,2,3,4\}$ be such that $s = 2\sigma+1$. We first study the case $\gamma=0$. Let $\varphi \in H^s_{(0)}(I,\mathbb{C})$. Using integrations by parts, we get, for every $k \in \mathbb{N}^*$,

$$\langle\varphi,\varphi_k\rangle = \frac{(-1)^\sigma}{(k\pi)^s}\left\langle\varphi^{(s)}, \frac{1}{k\pi}\varphi_k'\right\rangle.$$

The family $((1/k\pi)\varphi_k')_{k\in\mathbb{N}^*}$ is an orthonormal basis of $L^2(I,\mathbb{C})$, so

$$\left\|\varphi^{(s)}\right\|_{L^2} = \pi^s\left\|k^s\langle\varphi,\varphi_k\rangle\right\|_{l^2}.$$

Finally, we get (3.38) for $\gamma=0$ with any constants Q_1 and Q_2 satisfying:

$$0 < Q_1 \leqslant \frac{1}{\mathcal{P}\pi^9}, \qquad Q_2 \geqslant \frac{1}{\pi}.$$

Now we study the case $\gamma \neq 0$. Let $\varphi \in H^s_{(\gamma)}(I,\mathbb{C})$. Using integrations by parts, we get:

$$\langle\varphi,\varphi_{k,\gamma}\rangle = \frac{k\pi}{2\lambda_k\lambda_{k,\gamma}^\sigma}\left\langle(A_\gamma^\sigma\varphi)', \frac{1}{k\pi}\varphi_k'\right\rangle + \frac{1}{\lambda_{k,\gamma}^\sigma}\langle A_\gamma^\sigma\varphi, \varphi_{k,\gamma}-\tilde{\varphi}_{k,\gamma}\rangle + \frac{\gamma}{\lambda_{k,\gamma}^\sigma}\left\langle A_\gamma^\sigma\varphi, \left.\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\right|_0\right\rangle.$$

Using (A.12), (A.8), and Lemma 2 (proved in Section 3.6.2), we get (3.38) for $\gamma \in [0,\gamma_1]$, where $\gamma_1 := \min\{\gamma_0,\gamma^*\}$ with any constants Q_1, Q_2 satisfying:

$$0 < Q_1 \leqslant \frac{2^\sigma}{\pi^s}\left(\frac{\mathcal{C}_-}{C^{*\sigma}} - \gamma^2\frac{C^*\mathcal{C}_+\pi^2}{C_*^\sigma\sqrt{6}} - \gamma\frac{\pi\mathcal{C}_+C}{C_*^\sigma}\right),$$
$$Q_2 \geqslant \frac{2^\sigma}{C_*^\sigma\pi^s}\mathcal{C}_+\left(1+\gamma^2\frac{C^*\pi^2}{\sqrt{6}} + \gamma\pi C\right).$$

We can assume $C_* \leqslant 1 \leqslant C^*$. There exists $\gamma_2 \in (0, \gamma_1]$ such that

$$\frac{\mathcal{C}_-}{2C^{*4}} \geqslant \gamma_2^2\frac{C^*\mathcal{C}_+\pi^2}{C_*^4\sqrt{6}} + \gamma_2\pi\mathcal{C}_+.$$

In conclusion, for every $\gamma \in [0, \gamma_2]$, for every $s \in \{1, 3, 5, 7, 9\}$ and for every $\varphi \in H^s_{(\gamma)}(I, \mathbb{C})$, we have (3.38) with

$$Q_1 = \min\left\{\frac{1}{\mathcal{P}\pi}, \frac{\mathcal{C}_-}{2\pi^9C^{*4}}\right\},$$
$$Q_2 = \max\left\{\frac{1}{\pi}, \frac{2^4}{C_*^4\pi}\mathcal{C}_+\left(1+\gamma_2^2\frac{C^*\pi^2}{\sqrt{6}} + \gamma_2\pi C\right)\right\}. \quad \square$$

In conclusion, for $(\psi_0, \psi_1) \in F_1^\gamma$ and $\theta \geqslant 1$, we define:

$$S_\theta(\psi_0, \psi_1) := (S_\theta\psi_0, S_\theta\psi_1)$$

and this operator satisfies (3.10)–(3.13).

3.3.2. Smoothing operators on the spaces E_a^γ

In this section, we construct smoothing operators for the controls:

$$S_\theta : L^2\big((0,T),\mathbb{R}\big) \to H_0^3\big((0,T),\mathbb{R}\big),$$
$$v \mapsto S_\theta v,$$

for which there exists a constant $\mathcal{K}$ such that, for every $\theta \geqslant 1$, for every $c \in \{0, 1, 2, 3\}$ and for every $v \in H^c((0,T),\mathbb{R})$,

$$\|S_\theta v\|_{H^c} \leqslant \mathcal{K}\|v\|_{H^c}, \tag{3.39}$$

for every $v \in H_0^1((0,T),\mathbb{R})$,

$$\|v - S_\theta v\|_{L^2} \leqslant \mathcal{K}\theta^{-2}\|v\|_{H^1}, \tag{3.40}$$

and for every $v \in H_0^2((0,T),\mathbb{R})$

$$\|v - S_\theta v\|_{L^2} \leqslant \mathcal{K}\theta^{-4}\|v\|_{H^2}. \tag{3.41}$$

Then, the operator defined for $(\psi_0, v) \in E_1^\gamma = H_0^1(I,\mathbb{C}) \times L^2((0,T),\mathbb{R})$ by:

$$S_\theta(\psi_0, v) := (S_\theta \psi_0, S_\theta v),$$

satisfies (3.10) and (3.12) with $b = 1$ and $a \in \{3, 5\}$.

We can assume $T = 1$. We will use convolution products on $\mathbb{R}$ in order to construct the smoothing operators, as in [13]. The next proposition justifies that instead of dealing with functions $v : [0,1] \to \mathbb{R}$, we can deal with functions $f : \mathbb{R}_+ \to \mathbb{R}$, with $\operatorname{Supp}(f) \subset [0,1]$ and which belong to $H^c(\mathbb{R}_+,\mathbb{R})$ when $v \in H^c((0,1),\mathbb{R})$ for some $c \in \{0,1,2,3\}$. Then, considering $f : \mathbb{R}_+ \to \mathbb{R}$, we construct an extension $\tilde{f} : \mathbb{R} \to \mathbb{R}$ of f, with the same regularity. Those two first steps are the same as in [13]. Finally, we use a convolution product of $\tilde{f}$ with a smooth function ρ_θ to get a regular function and we truncate with a smooth function vanishing on 0 and 1 in order to get the boundary conditions. For this last step, our arguments are a little bit different from [13].

Proposition 4. *Let $h_1, h_2 \in C^\infty(\mathbb{R},\mathbb{R})$ be such that $0 \leqslant h_1, h_2 \leqslant 1$, $h_1 + h_2 = 1$ on $[0,1]$, $\operatorname{Supp}(h_1) \subset [-1/4, 3/4]$, $\operatorname{Supp}(h_2) \subset [1/4, 5/4]$. Let $v \in L^2((0,1),\mathbb{R})$ and $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = v(x)$ for $x \in [0,1]$ and $f(x) = 0$ for $x \in (-\infty, 0) \cup (1, +\infty)$. We define the functions*:

$$\begin{aligned} f_1 &: \mathbb{R}_+ \to \mathbb{R}, \\ &\quad t \mapsto (f h_1)(t), \\ f_2 &: \mathbb{R}_+ \to \mathbb{R}, \\ &\quad t \mapsto (f h_2)(1-t). \end{aligned}$$

If $v \in H^s((0,1),\mathbb{R})$ for some $s \in \{0,1,2,3\}$ then $f_i \in H^s(\mathbb{R}_+,\mathbb{R})$ for $i = 1,2$. Moreover, there exists a constant c_1 such that for every $s \in \{0,1,2,3\}$, for every $v \in H^s((0,1),\mathbb{R})$, $\|f_i\|_{H^s(\mathbb{R}_+,\mathbb{R})} \leqslant c_1 \|v\|_{H^s((0,1),\mathbb{R})}$.

Proposition 5. *Let $f \in L^2(\mathbb{R}+,\mathbb{R})$ with $\operatorname{Supp}(f) \subset [0,1]$. Let $\tilde{f} : \mathbb{R} \to \mathbb{R}$ be defined by*:

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \geqslant 0, \\ 5f(-x) - 5f(-2x) + f(-4x) & \text{if } x < 0. \end{cases}$$

Then, $\operatorname{Supp}(\tilde{f}) \subset [-1,1]$. If $f \in H^s(\mathbb{R}_+,\mathbb{R})$ for some $s \in \{0,1,2,3\}$ then $\tilde{f} \in H^s(\mathbb{R},\mathbb{R})$. There exists a constant c_2 such that for every $s \in \{0,1,2,3\}$ and for every $f \in H^s(\mathbb{R}_+,\mathbb{R})$, $\|\tilde{f}\|_{H^s(\mathbb{R},\mathbb{R})} \leqslant c_2 \|f\|_{H^s(\mathbb{R}_+,\mathbb{R})}$.

Proof. The choice of the coefficients in $\tilde{f}$ gives $\tilde{f}^{(s)}(0) = f^{(s)}(0^+)$ for any $s \in \{0,1,2\}$ for which it has a sense. □

Let $\rho \in C^\infty(\mathbb{R},\mathbb{R})$ be such that

$$\text{Supp}(\rho) \subset [-1,1] \quad \text{and} \quad \int_{\mathbb{R}} \rho(x)\,\mathrm{d}x = 1.$$

For $\theta \geqslant 1$, we define $\rho_\theta(x) := \theta\rho(\theta x)$. For $\tilde{f} \in L^2(\mathbb{R},\mathbb{R})$, the function $\rho_\theta * \tilde{f} : \mathbb{R} \to \mathbb{R}$ is defined by:

$$\rho_\theta * \tilde{f}(t) := \int_{\mathbb{R}} \tilde{f}(t+\tau)\rho_\theta(\tau)\,\mathrm{d}\tau.$$

Proposition 6. *There exists a constant $\mathcal{K}'$ such that, for every $\tilde{f} \in H^1(\mathbb{R},\mathbb{R})$ with* $\text{Supp}(\tilde{f}) \subset [-1,1]$,

$$\|\tilde{f} - \rho_\theta * \tilde{f}\|_{L^2((0,1),\mathbb{R})} \leqslant \mathcal{K}'\frac{1}{\theta}\|\tilde{f}\|_{H^1(\mathbb{R},\mathbb{R})}, \tag{3.42}$$

for every $s \in \{1,2,3\}$, for every $\tilde{f} \in H^s(\mathbb{R},\mathbb{R})$ with $f^{(k)}(0) = f^{(k)}(1) = 0$ for $k = 0,\dots,s-1$ and $\text{Supp}(\tilde{f}) \subset [-1,1]$,

$$\|\rho_\theta * \tilde{f}\|_{L^2((0,1/\theta),\mathbb{R})} \leqslant \mathcal{K}'\frac{1}{\theta^s}\|\tilde{f}\|_{H^s(\mathbb{R},\mathbb{R})}, \tag{3.43}$$

$$\|\rho_\theta * \tilde{f}\|_{L^2((1-1/\theta,1),\mathbb{R})} \leqslant \mathcal{K}'\frac{1}{\theta^s}\|\tilde{f}\|_{H^s(\mathbb{R},\mathbb{R})}. \tag{3.44}$$

Proof. Let $\tilde{f} \in H^1(\mathbb{R},\mathbb{R})$ be such that $\text{Supp}(\tilde{f}) \subset [-1,1]$. For $t \in \mathbb{R}$, we have:

$$\big(\rho_\theta * \tilde{f} - \tilde{f}\big)(t) = \int_{\mathbb{R}} \int_0^1 \tau \tilde{f}'(t+\lambda\tau)\,\mathrm{d}\lambda\,\rho_\theta(\tau)\,\mathrm{d}\tau.$$

Using a function $h \in L^2(\mathbb{R},\mathbb{R})$ and Fubini's theorem in

$$\int_{\mathbb{R}} \big(\rho_\theta * \tilde{f} - \tilde{f}\big)(t)h(t)\,\mathrm{d}t$$

we get:

$$\|\rho_\theta * \tilde{f} - \tilde{f}\|_{L^2(\mathbb{R},\mathbb{R})} \leqslant \frac{1}{\theta}\|\tilde{f}\|_{H^1(\mathbb{R},\mathbb{R})}\left(\int_{\mathbb{R}} \big|y\rho(y)\big|\,\mathrm{d}y\right).$$

We first prove (3.43) for $s = 1$. Let $\tilde{f} \in H^1(\mathbb{R},\mathbb{R})$ with $\text{Supp}(\tilde{f}) \subset [-1,1]$, $\tilde{f}(0) = \tilde{f}(1) = 0$. Let $r : \mathbb{R} \to \mathbb{R}$ be such that $\dot{r} = \rho$ and $r(-1) = 0$. Using $\text{Supp}(\rho) \subset [-1,1]$ and an integration by parts, we get, for every $t \in \mathbb{R}$,

$$\left(\rho_\theta * \tilde{f}\right)(t) = \tilde{f}\left(t + \frac{1}{\theta}\right) - \int_{-1/\theta}^{1/\theta} \tilde{f}'(t+\tau) r(\theta\tau)\,\mathrm{d}\tau.$$

Thanks to the condition $\tilde{f}(0) = 0$, we get:

$$\int_0^{1/\theta} \left|\tilde{f}\left(t + \frac{1}{\theta}\right)\right|^2 \mathrm{d}t = \int_0^{1/\theta} \left|\int_0^{t+1/\theta} \tilde{f}'(s)\,\mathrm{d}s\right|^2 \mathrm{d}t \leqslant \frac{3}{2\theta^2} \|\tilde{f}'\|_{L^2}^2.$$

Using $h \in L^2((0,1/\theta),\mathbb{R})$ and Fubini's theorem in

$$\int_0^{1/\theta} h(t) \int_{-1/\theta}^{1/\theta} \tilde{f}'(t+\tau) r(\theta\tau)\,\mathrm{d}\tau\,\mathrm{d}t$$

we get:

$$\left\| \int_{-1/\theta}^{1/\theta} \tilde{f}'(t+\tau) r(\theta\tau)\,\mathrm{d}\tau \right\|_{L^2((0,1/\theta),\mathbb{R})} \leqslant \frac{1}{\theta}\left(\int_{-1}^{1} \left|r(x)\right| \mathrm{d}x\right) \|\tilde{f}'\|_{L^2}.$$

In conclusion, we have (3.43) for $s = 1$ with any constant $\mathcal{K}'$ such that

$$\mathcal{K}' \geqslant \frac{\sqrt{3}}{\sqrt{2}} + \int_{-1}^{1} \left|r(x)\right| \mathrm{d}x.$$

For the proof of (3.44) in the case $s = 1$, we use:

$$\int_{1-1/\theta}^{1} \left|\tilde{f}\left(t + \frac{1}{\theta}\right)\right|^2 \mathrm{d}t = \int_{1-1/\theta}^{1} \left|\int_1^{t+1/\theta} \tilde{f}'(s)\,\mathrm{d}s\right|^2 \mathrm{d}t,$$

and the same arguments.

Now, we prove (3.43) in the case $s = 2$. Let $\tilde{f} \in H^2(\mathbb{R},\mathbb{R})$ with $\mathrm{Supp}(\tilde{f}) \subset [-1,1]$, $f^{(k)}(0) = f^{(k)}(1) = 0$ for $k = 0,1$. Let $R : \mathbb{R} \to \mathbb{R}$ be such that $\dot{R} = r$ and $R(-1) = 0$. Using integrations by parts, we get:

$$\left(\rho_\theta * \tilde{f}\right)(t) = \tilde{f}\left(t + \frac{1}{\theta}\right) - \frac{R(1)}{\theta} \tilde{f}'\left(t + \frac{1}{\theta}\right) + \frac{1}{\theta} \int_{-1/\theta}^{1/\theta} \tilde{f}''(t+\tau) R(\theta\tau)\,\mathrm{d}\tau.$$

Thanks to $\tilde{f}(0) = \tilde{f}'(0) = 0$, we get:

$$\int_0^{1/\theta} \left| \tilde{f}\left(t+\frac{1}{\theta}\right)\right|^2 \mathrm{d}t \leqslant \frac{5}{4\theta^4}\|\tilde{f}''\|_{L^2}^2,$$

$$\int_0^{1/\theta} \left| \tilde{f}'\left(t+\frac{1}{\theta}\right)\right|^2 \mathrm{d}t \leqslant \frac{3}{2\theta^2}\|\tilde{f}''\|_{L^2}^2.$$

We conclude as in the proof of (3.43) for $s=1$ that (3.43) for $s=2$ is true with any constant $\mathcal{K}'$ such that

$$\mathcal{K}' \geqslant \frac{\sqrt{5}}{2} + \frac{\sqrt{3}}{\sqrt{2}}\big|R(1)\big| + \int_{-1}^{1}\big|R(x)\big|\,\mathrm{d}x.$$

The proof of (3.44) in the case $s=2$ is similar to the proof of (3.43) in the case $s=2$ using $\tilde{f}(1)=\tilde{f}'(1)=0$ instead of $\tilde{f}(0)=\tilde{f}'(0)=0$.

For the proof of (3.43) in the case $s=3$, we use another integration by parts in $(\rho_\theta * \tilde{f})(t)$. □

For every $\theta \geqslant 1$, we consider a function $g_\theta \in C^\infty(\mathbb{R},\mathbb{R})$ such that

$$\mathrm{Supp}(g_\theta) \subset [0,1], \qquad g_\theta^{(k)}(0) = g_\theta^{(k)}(1) = 0 \quad \text{for } k=0,1,2,$$
$$g_\theta = 1 \quad \text{on } [1/\theta, 1-1/\theta], \qquad \big\|g_\theta^{(k)}\big\|_\infty \leqslant C\theta^k \quad \text{for } k=0,1,2,3,$$

where C is a constant which does not depend on θ. We define, for $\tilde{f} \in L^2(\mathbb{R},\mathbb{R})$,

$$R_\theta \tilde{f} := g_\theta\big(\rho_\theta * \tilde{f}\big).$$

Proposition 7. *There exists a constant $\mathcal{K}$ such that, for every $\tilde{f} \in H^1(\mathbb{R},\mathbb{R})$ with* $\mathrm{Supp}(\tilde{f}) \subset [-1,1]$*, we have*:

$$\|\tilde{f} - R_\theta \tilde{f}\|_{L^2((0,1),\mathbb{R})} \leqslant \mathcal{K}\frac{1}{\theta}\|\tilde{f}\|_{H^1(\mathbb{R},\mathbb{R})},$$

for every $s \in \{0,1,2,3\}$, for every $\tilde{f} \in H^s(\mathbb{R},\mathbb{R})$ with $\mathrm{Supp}(\tilde{f}) \subset [-1,1]$ *and $f^{(k)}(0) = f^{(k)}(1) = 0$ for $k = 0,\ldots,s-1$, we have*:

$$\big\|R_\theta \tilde{f}\big\|_{H^s((0,1),\mathbb{R})} \leqslant \mathcal{K}\big\|\tilde{f}\big\|_{H^s(\mathbb{R},\mathbb{R})}.$$

Proof. Let $\tilde{f} \in H^1(\mathbb{R},\mathbb{R})$ with $\mathrm{Supp}(\tilde{f}) \subset [-1,1]$. We have:

$$\begin{aligned}
&\|\tilde{f} - R_\theta \tilde{f}\|_{L^2((0,1),\mathbb{R})} \\
&\quad \leqslant \|(1-g_\theta)(\rho_\theta * \tilde{f})\|_{L^2((0,1),\mathbb{R})} + \|\rho_\theta * \tilde{f} - \tilde{f}\|_{L^2((0,1),\mathbb{R})} \\
&\quad \leqslant \|\rho_\theta * \tilde{f}\|_{L^2((0,1/\theta),\mathbb{R})} + \|\rho_\theta * \tilde{f}\|_{L^2((1-1/\theta,1),\mathbb{R})} + \|\rho_\theta * \tilde{f} - \tilde{f}\|_{L^2((0,1),\mathbb{R})}.
\end{aligned}$$

Using the previous proposition, we get the first inequality.

Let $s \in \{0,1,2,3\}$ and $\tilde{f} \in H^s(\mathbb{R},\mathbb{R})$ be such that $\mathrm{Supp}(\tilde{f}) \subset [-1,1]$ and $f^{(k)}(0) = f^{(k)}(1) = 0$ for $k = 0,\dots,s-1$. Let $\sigma \in \{0,\dots,s\}$. Using the Leibniz's formula, and the previous proposition on the derivatives of $\tilde{f}$, we get:

$$\|(R_\theta \tilde{f})^{(\sigma)}\|_{L^2((0,1),\mathbb{R})} \leqslant \left(2^{\sigma+1} C\mathcal{K}' + \int_0^1 |\rho(x)|\,\mathrm{d}x\right) \|\tilde{f}\|_{H^s(\mathbb{R},\mathbb{R})}.$$

Therefore,

$$\|R_\theta \tilde{f}\|_{H^s((0,1),\mathbb{R})} \leqslant \left(2C\mathcal{K}'\left(\frac{4^{s+1}-1}{3}\right)^{1/2} + \sqrt{s}\int_0^1 |\rho(x)|\,\mathrm{d}x\right) \|\tilde{f}\|_{H^s(\mathbb{R},\mathbb{R})}. \quad \square$$

Finally, for $v \in L^2((0,1),\mathbb{R})$, we define:

$$(S_\theta v)(t) := (R_{\theta^4} \tilde{f}_1)(t) + (R_{\theta^4} \tilde{f}_2)(1-t).$$

Now, it is easy to get the inequalities (3.39), (3.40) and (3.41).

3.4. The map Φ_γ is twice differentiable and satisfies (3.17)

Using the results in Appendix B, it is easy to prove the following proposition:

Proposition 8. *Let $\gamma > 0$ and $T > 0$ be such that $T\gamma < \sqrt{2}/\sqrt{17}$. We define*:

$$B_{\gamma,T} := \left\{ v \in L^2((0,T),\mathbb{R});\ T\gamma + \|v\|_{L^1((0,T),\mathbb{R})} < \frac{\sqrt{2}}{\sqrt{17}} \right\}.$$

For every $s \in \{1,3,5,7\}$, Φ_γ is a continuous map from

$$\left[H^s_{(\gamma)}(I,\mathbb{C}) \cap S\right] \times \left[H_0^{(s-1)/2}((0,T),\mathbb{R}) \cap B_{\gamma,T}\right]$$

into

$$\left[H^s_{(\gamma)}(I,\mathbb{C}) \cap S\right] \times \left[H^s_{(\gamma)}(I,\mathbb{C}) \cap S\right].$$

Proposition 9. *Let $\gamma > 0$ and $T > 0$ be such that $T\gamma < \sqrt{2}/\sqrt{17}$. The map,*

$$\Phi_\gamma : \left[H^7_{(\gamma)}(I,\mathbb{C}) \cap S\right] \times \left[H^3_0\big((0,T),\mathbb{R}\big) \cap B_{\gamma,T}\right] \to F^\gamma_7,$$

is differentiable and for every

$$(\psi_0, v) \in \left[H^7_{(\gamma)}(I,\mathbb{C}) \cap S\right] \times \left[H^3_0\big((0,T),\mathbb{R}\big) \cap B_{\gamma,T}\right],$$

for every

$$(\phi_0, \nu) \in \left[T_{\psi_0} S \cap H^7_{(\gamma)}(I,\mathbb{C})\right] \times H^3_0\big((0,T),\mathbb{R}\big),$$

we have:

$$\Phi'_\gamma(\psi_0, v).(\phi_0, \nu) = (\phi_0, \phi_T),$$

where ϕ is the solution of

$$\begin{cases} \mathrm{i}\dot{\phi} = -\frac{1}{2}\phi'' - u(t)q\phi - \nu(t)q\psi, \\ \phi(0) = \phi_0, \\ \phi(t, -\frac{1}{2}) = \phi(t, \frac{1}{2}) = 0, \end{cases}$$

with $u(t) = \gamma + v(t)$ and ψ is the solution of

$$\begin{cases} \mathrm{i}\dot{\psi} = -\frac{1}{2}\psi'' - u(t)q\psi, \\ \psi(0) = \psi_0, \\ \psi(t, -\frac{1}{2}) = \psi(t, \frac{1}{2}) = 0. \end{cases}$$

Proof. Let us introduce the solution ξ of

$$\begin{cases} \mathrm{i}\dot{\xi} = -\frac{1}{2}\xi'' - (u+\nu)(t)q\xi, \\ \xi(0) = \psi_0 + \phi_0, \\ \xi(t, -\frac{1}{2}) = \xi(t, \frac{1}{2}) = 0. \end{cases}$$

Then

$$\Phi_\gamma(\psi_0 + \phi_0, v + \nu) - \Phi_\gamma(\psi_0, v) - (\phi_0, \phi_T) = \big(0, \Delta(T)\big),$$

where $\Delta := \xi - \psi - \phi$ solves:

$$\begin{cases} \mathrm{i}\dot{\Delta} = -\frac{1}{2}\Delta'' - u(t)q\Delta - \nu(t)q(\xi - \psi), \\ \Delta(0) = 0, \\ \Delta(t, -\frac{1}{2}) = \Delta(t, \frac{1}{2}) = 0. \end{cases}$$

Let $f := \nu q \eta$ where $\eta := \xi - \psi$. Using Proposition 51 in Appendix B, we get:

$$\big\|\Delta(T)\big\|_{H^7} \leqslant C A_u(f),$$

with

$$A_u(f) := \|f\|_{C^0([0,T],H^5)} + \|\dot f\|_{C^0([0,T],H^3)} + \|\ddot f\|_{L^1((0,T),H^2)} + \left\|\frac{\partial^3 f}{\partial t^3}\right\|_{L^1((0,T),H^1)}$$
$$+ \|u\|_{W^{2,1}}\|f\|_{W^{1,1}((0,T),H^1)} + \|u\|_{W^{3,1}}\|f\|_{L^1((0,T),H^1)}. \tag{3.45}$$

There exists a constant C such that

$$\|f\|_{C^0([0,T],H^5)} \leqslant C\|\nu\|_{H^1}\|\eta\|_{H^5},$$
$$\|\dot f\|_{C^0([0,T],H^3)} \leqslant C\big[\|\nu\|_{H^2}\|\eta\|_{H^3} + \|\nu\|_{H^1}\|\dot\eta\|_{H^3}\big],$$
$$\|\ddot f\|_{L^1((0,T),H^2)} \leqslant C\big[\|\nu\|_{H^2}\|\eta\|_{H^2} + \|\nu\|_{H^1}\|\dot\eta\|_{H^2} + \|\nu\|_{L^2}\|\ddot\eta\|_{H^2}\big],$$
$$\left\|\frac{\partial^3 f}{\partial t^3}\right\|_{L^1((0,T),H^1)} \leqslant C\left[\|\nu\|_{H^3}\|\eta\|_{H^1} + \|\nu\|_{H^2}\|\dot\eta\|_{H^1} + \|\nu\|_{H^1}\|\ddot\eta\|_{H^1} + \|\nu\|_{L^2}\left\|\frac{\partial^3\eta}{\partial t^3}\right\|_{H^1}\right],$$

where $\|\frac{\partial^k\eta}{\partial t^k}\|_{H^s} := \|\frac{\partial^k\eta}{\partial t^k}\|_{C^0([0,T],H^s)}$. The function η satisfies the equations:

$$\begin{cases} \mathrm{i}\dot\eta = -\frac{1}{2}\eta'' - (u+\nu)(t)q\eta - \nu(t)q\psi, \\ \eta(0) = \phi_0, \\ \eta(t,-\frac{1}{2}) = \eta(t,\frac{1}{2}) = 0. \end{cases}$$

We work with ν small in $H_0^3((0,T),\mathbb{R})$, so we can assume $\|u+\nu\|_{L^1} < \sqrt{2}/\sqrt{17}$ and apply the bounds given in Appendix B on η. Thanks to Proposition 45, we get:

$$\|\eta\|_{H^1} \leqslant q_1(\eta) := C\big(\|\phi_0\|_{H^1} + \|\nu\|_{L^2}\|\psi_0\|_{H^1}\big).$$

Thanks to Proposition 47 in Appendix B, the norms $\|\dot\eta\|_{H^1}$ and $\|\eta\|_{H^3}$ can be dominated by the same quantity:

$$q_3(\eta) := C\big[\|\phi_0\|_{H^3} + \|\nu\|_{H^1}\|\psi_0\|_{H^3}\big].$$

Thanks to Proposition 49 in Appendix B, the norms $\|\ddot\eta\|_{H^1}$, $\|\dot\eta\|_{H^3}$, $\|\eta\|_{H^5}$ can be dominated by the same quantity:

$$q_5(\eta) := C\big[\|\phi_0\|_{H^5} + \|\nu\|_{H^2}\|\psi_0\|_{H^5} + \|u+\nu\|_{H^2}\big(\|\phi_0\|_{H^1} + \|\nu\|_{L^2}\|\psi_0\|_{H^1}\big)\big].$$

Thanks to Propositions 49 and 51 in Appendix B, the norms $\|\partial^3\eta/\partial t^3\|_{H^1}$ and $\|\ddot\eta\|_{H^2}$ can be dominated by the same quantity:

$$q_7(\eta) := C\big[\|\phi_0\|_{H^7} + \|\nu\|_{H^3}\|\psi_0\|_{H^7} + \|u+\nu\|_{H^3}\big(\|\phi_0\|_{H^3} + \|\nu\|_{H^1}\|\psi_0\|_{H^3}\big)\big].$$

We have:

$$
\begin{aligned}
A_u(f) &\leqslant C\big[\|\nu\|_{L^2} q_7(\eta) + \|\nu\|_{H^1} q_5(\eta) + \|\nu\|_{H^2} q_3(\eta) + \|\nu\|_{H^3} q_1(\eta)\big] \\
&\leqslant 4C\|\nu\|_{H^3} q_7(\eta) \\
&\leqslant C\|\nu\|_{H^3}\big(\|\phi_0\|_{H^7} + \|\nu\|_{H^3}\big)\big(1 + \|\psi_0\|_{H^7} + \|u\|_{H^3}\big).
\end{aligned}
$$

We have proved that, for every $(\phi_0, \nu) \in E_7^\gamma$ small enough,

$$
\big\|\Phi_\gamma(\psi_0 + \phi_0, v + \nu) - \Phi_\gamma(\psi_0, v) - (\phi_0, \phi_T)\big\|_{F_7^\gamma} \leqslant C\big\|(\phi_0, \nu)\big\|_{E_7^\gamma}^2. \quad \square
$$

Theorem 7. *Let $T > 0$ and $\gamma > 0$ be such that $T\gamma < \sqrt{2}/\sqrt{17}$. The map*

$$
\Phi_\gamma : \big[H^7_{(\gamma)}(I, \mathbb{C}) \cap S\big] \times \big[H^3_0\big((0,T), \mathbb{R}\big) \cap B_{\gamma,T}\big] \to F_7^\gamma
$$

is twice differentiable and for every

$$
(\psi_0, v) \in \big[H^7_{(\gamma)}(I, \mathbb{C}) \cap S\big] \times \big[H^3_0\big((0,T), \mathbb{R}\big) \cap B_{\gamma,T}\big],
$$

for every

$$
(\phi_0, \nu), (\xi_0, \mu) \in \big[T_{\psi_0} S \cap H^7_{(\gamma)}(I, \mathbb{C})\big] \times H^3_0\big((0,T), \mathbb{R}\big),
$$

we have:

$$
\Phi_\gamma''(\psi_0, v).\big((\phi_0, \nu), (\xi_0, \mu)\big) = \big(0, h(T)\big),
$$

where h is the solution of

$$
\begin{cases}
\mathrm{i}\dot{h} = -\frac{1}{2}h'' - u(t)qh - \nu(t)q\xi - \mu(t)q\phi, \\
h(0) = 0, \\
h(t, -\frac{1}{2}) = h(t, \frac{1}{2}) = 0,
\end{cases}
$$

where $u(t) = \gamma + v(t)$, ξ, ϕ and ψ are the solutions of

$$
\begin{cases}
\mathrm{i}\dot{\xi} = -\frac{1}{2}\xi'' - u(t)q\xi - \mu(t)q\psi, \\
\xi(0) = \xi_0, \\
\xi(t, -\frac{1}{2}) = \xi(t, \frac{1}{2}) = 0,
\end{cases}
$$

$$
\begin{cases}
\mathrm{i}\dot{\phi} = -\frac{1}{2}\phi'' - u(t)q\phi - \nu(t)q\psi, \\
\phi(0) = \phi_0, \\
\phi(t, -\frac{1}{2}) = \phi(t, \frac{1}{2}) = 0,
\end{cases}
$$

$$
\begin{cases}
\mathrm{i}\dot{\psi} = -\frac{1}{2}\psi'' - u(t)q\psi, \\
\psi(0) = \psi_0, \\
\psi(t, -\frac{1}{2}) = \psi(t, \frac{1}{2}) = 0.
\end{cases}
$$

Proof. We prove the existence of a constant $C > 0$ such that, when $\|(\phi_0, \nu)\|_{E_7^\gamma} < 1$,

$$\big\|\big[\mathrm{d}\Phi_\gamma(\psi_0+\phi_0, v+\nu) - \mathrm{d}\Phi_\gamma(\psi_0, v)\big].(\xi_0,\mu) - \big(0, h(T)\big)\big\|_{F_7^\gamma} \\ \leqslant C\big\|(\phi_0,\nu)\big\|_{E_7}^2\big\|(\xi_0,\mu)\big\|_{E_7^\gamma}.$$

Let us introduce the solutions k and φ of the following systems:

$$\begin{cases} \mathrm{i}\dot{k} = -\frac{1}{2}k'' - (u+\nu)(t)qk - \mu q\varphi, \\ k(0) = \xi_0, \\ k(t,-\frac{1}{2}) = k(t,\frac{1}{2}) = 0, \end{cases}$$

$$\begin{cases} \mathrm{i}\dot{\varphi} = -\frac{1}{2}\varphi'' - (u+\nu)(t)q\varphi, \\ \varphi(0) = \psi_0 + \phi_0, \\ \varphi(t,-\frac{1}{2}) = \varphi(t,\frac{1}{2}) = 0, \end{cases}$$

so that

$$\mathrm{d}\Phi_\gamma(\psi_0+\phi_0, v+\nu).(\xi_0,\mu) = \big(\xi_0, k(T)\big).$$

We have:

$$\big[\mathrm{d}\Phi_\gamma(\psi_0+\phi_0, v+\nu) - \mathrm{d}\Phi_\gamma(\psi_0, v)\big].(\xi_0,\mu) - \big(0,h(T)\big) = \big(0,\Lambda(T)\big),$$

where $\Lambda := k - \xi - h$ solves:

$$\begin{cases} \mathrm{i}\dot{\Lambda} = -\frac{1}{2}\Lambda'' - (u+\nu)(t)q\Lambda - \nu(t)qh - \mu(t)q(\varphi - \psi - \phi), \\ \Lambda(0) = 0, \\ \Lambda(t,-\frac{1}{2}) = \Lambda(t,\frac{1}{2}) = 0. \end{cases}$$

Let $f := \nu(t)qh + \mu(t)q(\varphi - \psi - \phi)$. Thanks to Proposition 51, in Appendix B, we know that $\|\Lambda\|_{H^7} \leqslant CA_{(u+\nu)}(f)$ where A is defined in the previous proof by the expression (3.45). In the same way as in the previous proof, there exists a constant C_1 such that

$$A_{u+\nu}(f) \leqslant C_1\big[\|\nu\|_{H^3}q_7(h) + \|\mu\|_{H^3}q_7(\psi+\phi-\varphi)\big],$$

where $q_7(.)$ denotes the upper bound on the H^7-norm given in Proposition 51 in the general case. In particular, we have:

$$q_7(h) \leqslant C\big[\|\nu\|_{H^3}q_7(\xi) + \|\mu\|_{H^3}q_7(\phi)\big],$$

with

$$q_7(\xi) \leqslant C\big[\|\xi_0\|_{H^7} + \|\mu\|_{H^3}\|\psi_0\|_{H^7}\big],$$
$$q_7(\phi) \leqslant C\big[\|\phi_0\|_{H^7} + \|\nu\|_{H^3}\|\psi_0\|_{H^7}\big].$$

So there exists a constant C_2 depending only on (ψ_0, u) such that

$$q_7(h) \leqslant C_2\big(\|\phi_0\|_{H^7} + \|\nu\|_{H^3}\big)\big(\|\xi_0\|_{H^7} + \|\mu\|_{H^3}\big).$$

We have:

$$\begin{cases} \mathrm{i}\frac{\partial}{\partial t}(\psi + \phi - \varphi) = -\frac{1}{2}(\psi + \phi - \varphi)'' - u(t)q(\psi + \phi - \varphi) - \nu(t)q(\psi - \varphi), \\ (\psi + \phi - \varphi)(0) = 0, \\ (\psi + \phi - \varphi)(t, -\frac{1}{2}) = (\psi + \phi - \varphi)(t, \frac{1}{2}) = 0, \end{cases}$$

so

$$q_7(\psi + \phi - \varphi) \leqslant C\|\nu\|_{H^3} q_7(\psi - \varphi).$$

We have:

$$\begin{cases} \mathrm{i}\frac{\partial}{\partial t}(\psi - \varphi) = -\frac{1}{2}(\psi - \varphi)'' - (u + \nu)q(\psi - \varphi) + \nu q\psi, \\ (\psi - \varphi)(0) = \phi_0, \\ (\psi - \varphi)(t, -\frac{1}{2} = (\psi - \varphi)(t, \frac{1}{2}) = 0, \end{cases}$$

so

$$q_7(\psi - \varphi) \leqslant C\big[\|\phi_0\|_{H^7} + \|\nu\|_{H^3}\|\psi_0\|_{H^7}\big].$$

Finally, there exists a constant C_3 depending only on (ψ_0, u) such that

$$q_7(\psi + \phi - \varphi) \leqslant C_3\big(\|\phi_0\|_{H^7} + \|\nu\|_{H^3}\big)^2.$$

We conclude:

$$\big\|\Lambda(T)\big\|_{H^7} \leqslant CC_1(C_2 + C_3)\big(\|\phi_0\|_{H^7} + \|\nu\|_{H^3}\big)^2\big(\|\xi_0\|_{H^7} + \|\mu\|_{H^3}\big). \quad \square$$

Proposition 10. *Let $\gamma > 0$ and $T > 0$ be such that $T\gamma < \sqrt{2}/\sqrt{17}$. For every bounded subset $\mathcal{B}$ of E_3^γ, there exists a constant C such that for every*

$$(\psi_0, v) \in \big[H^7_{(\gamma)}(I, \mathbb{C}) \cap S\big] \times \big[H^3_0\big((0, T), \mathbb{R}\big) \cap B_{\gamma,T}\big] \quad \textit{with } (\psi_0, v) \in \mathcal{B},$$

for every

$$(\phi_0, \nu), (\xi_0, \mu) \in \big[T_{\psi_0}S \cap H^7_{(\gamma)}(I, \mathbb{C})\big] \times H^3_0\big((0, T), \mathbb{R}\big),$$

we have:

$$\big\|\Phi''_\gamma\big((\psi_0, v); (\phi_0, \nu), (\xi_0, \mu)\big)\big\|_{H^7 \times H^7} \leqslant C \sum\big(1 + \big\|(\psi_0, v)\big\|_{m'_j}\big)\big\|(\phi_0, \nu)\big\|_{m''_j}\big\|(\xi_0, \mu)\big\|_{m'''_j}$$

where the sum is finite, all the subscripts belong to $\{1,3,5,7\}$ *and satisfy for every* j

$$\max(m'_j-\alpha,0)+\max(m''_j,1)+m'''_j<2\alpha. \tag{3.46}$$

Proof. Let $f_1:=\nu q\xi$ and $f_2:=\mu q\phi$. Using Proposition 51, we get:

$$\|h\|_{H^7}\leqslant A_u(f_1)+A_u(f_2),$$

where $A_u(f)$ is defined by (3.45). In the same way as in the proof of the differentiability of Φ_γ, we have:

$$A(f_1)\leqslant \|\nu\|_{L^2}\big[q_7(\xi)+\|u\|_{H^2}\|\dot{\xi}\|_{H^1}+\|u\|_{H^3}\|\xi\|_{H^1}\big]+\|\nu\|_{H^1}\big[q_5(\xi)+\|u\|_{H^2}\|\xi\|_{H^1}\big] \\ +\|\nu\|_{H^2}q_3(\xi)+\|\nu\|_{H^3}q_1(\xi),$$

where $q_i(\xi)$ is the upper bound of the H^i-norm of ξ given in Propositions 45, 47, 49, 51. We have:

$$\begin{aligned} q_1(\xi)&\leqslant C\big[\|\xi_0\|_{H^1}+\|\mu\|_{L^2}\|\psi_0\|_{H^1}\big],\\ q_3(\xi)&\leqslant C\big[\|\xi_0\|_{H^3}+\|\mu\|_{H^1}\|\psi_0\|_{H^1}+\|\mu\|_{L^2}\|\psi_0\|_{H^3}\big],\\ q_5(\xi)&\leqslant C\big[\|\xi_0\|_{H^5}+\|\mu\|_{H^2}\|\psi_0\|_{H^1}+\|\mu\|_{H^1}\|\psi_0\|_{H^3}+\|\mu\|_{L^2}\|\psi_0\|_{H^5}\\ &\quad+\|u\|_{H^2}\big(\|\xi_0\|_{H^1}+\|\mu\|_{L^2}\|\psi_0\|_{H^1}\big)\big],\\ q_7(\xi)&\leqslant C\big[\|\xi_0\|_{H^7}+\|\mu\|_{H^3}\|\psi_0\|_{H^1}+\|\mu\|_{H^2}\|\psi_0\|_{H^3}+\|\mu\|_{H^1}\|\psi_0\|_{H^5}\\ &\quad+\|\mu\|_{L^2}\|\psi_0\|_{H^7}+\|u\|_{H^3}\big(\|\xi_0\|_{H^1}+\|\mu\|_{L^2}\|\psi_0\|_{H^1}\big)\\ &\quad+\|u\|_{H^2}\big(\|\xi_0\|_{H^3}+\|\mu\|_{H^1}\|\psi_0\|_{H^1}+\|\mu\|_{L^2}\|\psi_0\|_{H^3}\big)\big]. \end{aligned}$$

We get a bound on $A(f_2)$ just by exchanging (ϕ_0,ν) and (ξ_0,μ). Finally, we get the following values:

m'_j	1	1	1	1	3	3	3	5	5	7
m''_j	1	7	3	5	1	5	3	1	3	1
m'''_j	7	1	5	3	5	1	3	3	1	1

We check (3.46) by studying each column of this table. □

3.5. Controllability of the linearized system around $(\psi_{1,\gamma}(t),\gamma)$ and bounds (3.19)–(3.22) in this case

Let $\gamma>0$ and $T>0$ be such that $T\gamma<\sqrt{2}/\sqrt{17}$. Let $(\Psi_0,\Psi_T)\in F_7^\gamma$ be such that

$$\Re\big(\langle\Psi_0,\psi_{1,\gamma}(0)\rangle\big)=\Re\big(\langle\Psi_T,\psi_{1,\gamma}(T)\rangle\big)=0. \tag{3.47}$$

We are looking for $w \in H_0^3((0,T),\mathbb{R})$ such that

$$\int_0^T w(t)\mathrm{e}^{\mathrm{i}(\lambda_{k,\gamma}-\lambda_{1,\gamma})t}\,\mathrm{d}t = -\frac{\mathrm{i}}{b_{k,\gamma}}\big(\langle \Psi_0,\varphi_{k,\gamma}\rangle - \langle \Psi_T,\varphi_{k,\gamma}\rangle \mathrm{e}^{\mathrm{i}\lambda_{k,\gamma}T}\big), \quad \forall k \in \mathbb{N}^*, \tag{3.48}$$

$$\|w\|_{L^2} \leqslant C\big\|(\Psi_0,\Psi_T)\big\|_{H^3\times H^3}, \qquad \|w\|_{H_0^1} \leqslant C\big\|(\Psi_0,\Psi_T)\big\|_{H^5\times H^5},$$
$$\|w\|_{H_0^2} \leqslant C\big\|(\Psi_0,\Psi_T)\big\|_{H^7\times H^7}, \qquad \|w\|_{H_0^3} \leqslant C\big\|(\Psi_0,\Psi_T)\big\|_{H^9\times H^9},$$

with a constant C which does not depend on Ψ_0, Ψ_T, w. Our strategy is the following one. We give an explicit solution for the moment problem $Z(w)=d$ taken with $\gamma=0$ and $T=4/\pi$ which satisfies these estimates. Then we prove the linear maps Z and Z_γ are closed enough to get a right inverse for Z_γ which satisfies the same estimates.

There is no contradiction between the existence of a solution of the moment problem $Z(w)=d$ and the noncontrollability of the linearized system around $\psi_{1,0}$: there is no controllability because some coefficients $b_{k,0}$ vanish.

We introduce, for $s \in \mathbb{R}_+$, the space $h^s(\mathbb{N}^*,\mathbb{C})$ and its subspace $h_r^s(\mathbb{N}^*,\mathbb{C})$ defined by:

$$h^s(\mathbb{N}^*,\mathbb{C}) := \left\{ d=(d_k)_{k\in\mathbb{N}^*};\ \|d\|_{h^s} := \left(\sum_{k=1}^{+\infty}|k^s d_k|^2\right)^{1/2} < +\infty \right\},$$
$$h_r^s(\mathbb{N}^*,\mathbb{C}) := \big\{ d \in h^s(\mathbb{N}^*,\mathbb{C});\ d_1 \in \mathbb{R}\big\}.$$

We use the notation $l^2(\mathbb{N}^*,\mathbb{C})$ and $l_r^2(\mathbb{N}^*,\mathbb{C})$ instead of $h^0(\mathbb{N}^*,\mathbb{C})$ and $h_r^0(\mathbb{N}^*,\mathbb{C})$,

Proposition 11. *Let $T=4/\pi$ and $\gamma \in [0,\gamma^*]$. The linear map,*

$$Z_\gamma : w \mapsto \left(\int_0^T w(t)\mathrm{e}^{-\mathrm{i}(\lambda_{k,\gamma}-\lambda_{1,\gamma})t}\,\mathrm{d}t\right)_{k\in\mathbb{N}^*},$$

is continuous from $L^2((0,T),\mathbb{R})$ to $l_r^2(\mathbb{N}^*,\mathbb{C})$, from $H_0^1((0,T),\mathbb{R})$ to $h_r^2(\mathbb{N}^*,\mathbb{C})$, from $H_0^2((0,T),\mathbb{R})$ to $h_r^4(\mathbb{N}^*,\mathbb{C})$, from $H_0^3((0,T),\mathbb{R})$ to $h_r^6(\mathbb{N}^*,\mathbb{C})$.

Proof. Let $w \in L^2((0,T),\mathbb{R})$. We have:

$$\big\langle w,\mathrm{e}^{\mathrm{i}(\lambda_{k,\gamma}-\lambda_{1,\gamma})t}\big\rangle = \big\langle w,\mathrm{e}^{\mathrm{i}(\lambda_k-\lambda_1)t}\big\rangle + \big\langle w\big(\mathrm{e}^{\mathrm{i}\lambda_{1,\gamma}t}-\mathrm{e}^{\mathrm{i}\lambda_1 t}\big),\mathrm{e}^{\mathrm{i}\lambda_k t}\big\rangle$$
$$+\big\langle w\mathrm{e}^{\mathrm{i}\lambda_{1,\gamma}t},\mathrm{e}^{\mathrm{i}\lambda_{k,\gamma}t}-\mathrm{e}^{\mathrm{i}\lambda_k t}\big\rangle.$$

Since $(\frac{1}{T}\mathrm{e}^{\mathrm{i}\frac{1}{2}n\pi^2 t})_{n\in\mathbb{Z}}$ is an orthonormal family of $L^2((0,T),\mathbb{C})$, the two first terms of the right-hand side of this equality belong to $l^2(\mathbb{N}^*,\mathbb{C})$ and

$$\left\| \left\langle w, e^{i(\lambda_k - \lambda_1)t} \right\rangle \right\|_{l^2} \leqslant T\|w\|_{L^2},$$
$$\left\| \left\langle w\left(e^{i\lambda_1 t} - e^{i\lambda_{1,\gamma} t}\right), e^{i\lambda_k t} \right\rangle \right\|_{l^2} \leqslant T\left\| w\left(e^{i\lambda_1 t} - e^{i\lambda_{1,\gamma} t}\right)\right\|_{L^2}.$$

Using (A.11), we get:

$$\left\| \left\langle w\left(e^{i\lambda_1 t} - e^{i\lambda_{1,\gamma} t}\right), e^{i\lambda_k t} \right\rangle \right\|_{l^2} \leqslant C^* T^2 \gamma^2 \|w\|_{L^2},$$
$$\left\| \left\langle w e^{i\lambda_{1,\gamma} t}, e^{i\lambda_{k,\gamma} t} - e^{i\lambda_k t} \right\rangle \right\|_{l^2} \leqslant \|w\|_{L^2} C^* \gamma^2 T \sqrt{\frac{\pi^2}{6}}.$$

For $w \in H_0^1((0,T),\mathbb{R})$, we have:

$$Z_\gamma(w)_k = \frac{-i}{\lambda_{k,\gamma} - \lambda_{1,\gamma}} Z_\gamma(\dot{w})_{k-1}.$$

Thanks to (A.12) and the previous result, we conclude the existence of a constant C such that, for every $w \in H_0^1((0,T),\mathbb{C})$, $Z_\gamma(w) \in h^2(\mathbb{N}^*,\mathbb{C})$ and $\|Z_\gamma(w)\|_{h^2} \leqslant C\|\dot{w}\|_{L^2}$.

For $w \in H_0^2((0,T),\mathbb{R})$, we have:

$$Z_\gamma(w)_{k-1} = \frac{-1}{(\lambda_{k,\gamma} - \lambda_{1,\gamma})^2} Z_\gamma(\ddot{w})_{k-1}$$

and we conclude thanks to (A.12) and the previous result.

For $w \in H_0^3((0,T),\mathbb{R})$, we have:

$$Z_\gamma(w)_{k-1} = \frac{i}{(\lambda_{k,\gamma} - \lambda_{1,\gamma})^3} Z_\gamma\left(\frac{d^3 w}{dt^3}\right)_{k-1}$$

and we conclude thanks to (A.12) and the previous result. □

Proposition 12. *Let $T = 4/\pi$. There exists a continuous linear map,*

$$Z^{-1} : h_r^6(\mathbb{N}^*,\mathbb{C}) \to H_0^3\big((0,T),\mathbb{R}\big)$$

such that for every $d \in h_r^6(\mathbb{N}^*,\mathbb{C})$, $Z \circ Z^{-1}(d) = d$. Moreover, there exists a constant C_0 such that for every $d \in h_r^6(\mathbb{N}^*,\mathbb{C})$ the function $w := Z^{-1}(d)$ satisfies

$$\|w\|_{L^2} \leqslant C_0\|d\|_{l^2}, \quad \|w\|_{H_0^1} \leqslant C_0\|d\|_{h^2}, \quad \|w\|_{H_0^2} \leqslant C_0\|d\|_{h^4}, \|w\|_{H_0^3} \leqslant C_0\|d\|_{h^6}. \tag{3.49}$$

Proof. We introduce the notations, for $k \in \mathbb{N}$:

$$\omega_k := \lambda_{k+1} - \lambda_1, \qquad \omega_{-k} := -\omega_k.$$

Since $\lambda_j=(j\pi)^2/2$, $\forall j\in\mathbb{N}^*$ then, $\omega_{k+1}-\omega_k\geqslant 3\pi^2/2$, $\forall k\in\mathbb{Z}$.

Let $d\in h^6_r(\mathbb{N}^*,\mathbb{C})$. We define $\tilde{d}\in h^6(\mathbb{Z},\mathbb{C})$ by:

$$\tilde{d}_k:=d_{k+1},\quad \tilde{d}_{-k}:=\overline{d_{k+1}}\quad \text{for every } k\in\mathbb{N}.$$

Since $\tilde{d}\in h^6(\mathbb{Z},\mathbb{C})$ and $\tilde{d}_0\in\mathbb{R}$, the following expression:

$$w(t)=\left(\frac{1}{T}\sum_{k\in\mathbb{Z}}\tilde{d}_k\mathrm{e}^{\mathrm{i}\omega_k t}\right)\left(1-\mathrm{e}^{\mathrm{i}\frac{1}{2}\pi^2 t}\right)^2\left(1-\mathrm{e}^{-\mathrm{i}\frac{1}{2}\pi^2 t}\right)^2$$

defines $w\in H^3_0((0,T),\mathbb{R})$. The family $(\mathrm{e}^{\mathrm{i}\omega_j t}/T)_{j\in\mathbb{Z}}$ is orthonormal in $L^2((0,T),\mathbb{C})$. For every $k\in\mathbb{Z}$, $\mathrm{e}^{\mathrm{i}(\omega_k+\frac{1}{2}\pi^2)t}$ (respectively, $\mathrm{e}^{\mathrm{i}(\omega_k-\frac{1}{2}\pi^2)t}$, respectively, $\mathrm{e}^{\mathrm{i}(\omega_k+\pi^2)t}$, respectively, $\mathrm{e}^{\mathrm{i}(\omega_k-\pi^2)t}$) is orthogonal to $\overline{\mathrm{Span}\{\mathrm{e}^{\mathrm{i}\omega_j t};\ j\in\mathbb{Z}\}}$. Therefore w solves $Z(w)=d$ and satisfies (3.49). □

Proposition 13. *Let $T=4/\pi$. There exists a constant $C_1>0$ such that for every $\gamma\in[-\gamma^*,\gamma^*]$ and for every $w\in H^3_0((0,T),\mathbb{R})$,*

$$\left\|(Z_\gamma-Z)(w)\right\|_F\leqslant C_1\gamma^2\|w\|_E,\tag{3.50}$$

for every $(E,F)\in\{(L^2,l^2),(H^1_0,h^2),(H^2_0,h^4),(H^3_0,h^6)\}$.

Proof. For $(E,F)=(L^2((0,T),\mathbb{R}),l^2(\mathbb{N}^*,\mathbb{C}))$, we have:

$$(Z_\gamma-Z)(w)_k=\left\langle w\mathrm{e}^{\mathrm{i}\lambda_{1,\gamma}t},\mathrm{e}^{\mathrm{i}\lambda_{k,\gamma}t}-\mathrm{e}^{\mathrm{i}\lambda_k t}\right\rangle+\left\langle w\left(\mathrm{e}^{\mathrm{i}\lambda_{1,\gamma}t}-\mathrm{e}^{\mathrm{i}\lambda_1 t}\right),\mathrm{e}^{\mathrm{i}\lambda_k t}\right\rangle.$$

The second term of the right-hand side of this equality is a Fourier coefficient of the L^2-function $t\to w(t)(\mathrm{e}^{\mathrm{i}\lambda_{1,\gamma}t}-\mathrm{e}^{\mathrm{i}\lambda_1 t})$, it belongs to $l^2(\mathbb{N}^*,\mathbb{C})$ and thanks to (A.11), we get:

$$\left\|\left\langle w\left(\mathrm{e}^{\mathrm{i}\lambda_{1,\gamma}t}-\mathrm{e}^{\mathrm{i}\lambda_1 t}\right),\mathrm{e}^{\mathrm{i}\lambda_k t}\right\rangle\right\|_{l^2(\mathbb{N}^*,\mathbb{C})}\leqslant T^2C^*\gamma^2\|w\|_{L^2((0,T),\mathbb{C})}.$$

Using (A.11), we get:

$$\left\|\left\langle w\mathrm{e}^{\mathrm{i}\lambda_{1,\gamma}t},\mathrm{e}^{\mathrm{i}\lambda_{k,\gamma}t}-\mathrm{e}^{\mathrm{i}\lambda_k t}\right\rangle\right\|_{l^2}\leqslant C^*\gamma^2T\sqrt{\frac{\pi^2}{6}}\|w\|_{L^2}.$$

For $(E,F)=(H^1_0((0,T),\mathbb{R}),h^2(\mathbb{N}^*,\mathbb{C}))$, we have:

$$(Z_\gamma-Z)(w)_k=-\mathrm{i}\left(\frac{1}{\lambda_{k,\gamma}-\lambda_{1,\gamma}}-\frac{1}{\lambda_k-\lambda_1}\right)\left\langle\dot{w},\mathrm{e}^{\mathrm{i}(\lambda_{k,\gamma}-\lambda_{1,\gamma})t}\right\rangle-\frac{\mathrm{i}}{\lambda_k-\lambda_1}(Z_\gamma-Z)(\dot{w})_k.$$

We conclude applying the previous result on $\dot{w}$, and using the inequality,

$$\left|\frac{1}{\lambda_{k,\gamma}-\lambda_{1,\gamma}}-\frac{1}{\lambda_k-\lambda_1}\right|\leqslant\frac{C\gamma^2}{k^4},$$

which is a consequence of (A.11).

For $(E,F)=(H_0^2((0,T),\mathbb{R}),h^4(\mathbb{N}^*,\mathbb{C}))$, we have:

$$(Z_\gamma-Z)(w)_k=-\left(\frac{1}{(\lambda_{k,\gamma}-\lambda_{1,\gamma})^2}-\frac{1}{(\lambda_k-\lambda_1)^2}\right)\left\langle\ddot{w},\mathrm{e}^{\mathrm{i}(\lambda_{k,\gamma}-\lambda_{1,\gamma})t}\right\rangle$$
$$-\frac{1}{(\lambda_k-\lambda_1)^2}(Z_\gamma-Z)(\ddot{w})_k.$$

We conclude applying the first result on $\ddot{w}$ and the inequality,

$$\left|\frac{1}{(\lambda_{k,\gamma}-\lambda_{1,\gamma})^2}-\frac{1}{(\lambda_k-\lambda_1)^2}\right|\leqslant\frac{C\gamma^2}{k^6},$$

which is a consequence of (A.11).

For $(E,F)=(H_0^3((0,T),\mathbb{R}),h^6(\mathbb{N}^*,\mathbb{C}))$, we have:

$$(Z_\gamma-Z)(w)_k=\mathrm{i}\left(\frac{1}{(\lambda_{k,\gamma}-\lambda_{1,\gamma})^3}+\frac{\mathrm{i}}{(\lambda_k-\lambda_1)^3}\right)\left\langle\frac{\mathrm{d}^3w}{\mathrm{d}t^3},\mathrm{e}^{\mathrm{i}(\lambda_{k,\gamma}-\lambda_{1,\gamma})t}\right\rangle$$
$$-\frac{1}{(\lambda_k-\lambda_1)^3}(Z_\gamma-Z)\left(\frac{\mathrm{d}^3w}{\mathrm{d}t^3}\right)_k.$$

We conclude applying the first result on $\mathrm{d}^3w/\mathrm{d}t^3$ and the inequality,

$$\left|\frac{1}{(\lambda_{k,\gamma}-\lambda_{1,\gamma})^3}-\frac{1}{(\lambda_k-\lambda_1)^3}\right|\leqslant\frac{C\gamma^2}{k^8},$$

which is a consequence of (A.11). □

Proposition 14. *Let $T=4/\pi$. There exists $\gamma_1>0$ such that, for every $\gamma\in[0,\gamma_1]$, there exists a continuous linear map,*

$$Z_\gamma^{-1}:h_r^6(\mathbb{N}^*,\mathbb{C})\to H_0^3\big((0,T),\mathbb{R}\big),$$

such that for every $d\in h_r^6(\mathbb{N}^*,\mathbb{C})$, $Z_\gamma\circ Z_\gamma^{-1}(d)=d$. Moreover, there exists a constant C_2 such that for every $\gamma\in[0,\gamma_1]$ and for every $d\in h_r^6(\mathbb{N},\mathbb{C})$, the function $w:=Z_\gamma^{-1}(d)$ satisfies:

$$\|w\|_E\leqslant C_2\|d\|_F,\quad \textit{for } (E,F)=\big(L^2,l^2\big),\big(H_0^1,h^2\big),\big(H_0^2,h^4\big),\big(H_0^3,h^6\big).\tag{3.51}$$

Proof. Let $d \in h^6_r(\mathbb{N}^*, \mathbb{C})$. Let $(w_n)_{n\in\mathbb{N}}$ be the sequence in $H^3_0((0,T),\mathbb{C})$ defined by induction by:

$$\begin{cases} w_0 = Z^{-1}(d), \\ w_{n+1} = Z^{-1}((Z - Z_\gamma)(w_n)), \quad \forall n \in \mathbb{N}. \end{cases}$$

Then, we have:

$$\|w_n\|_E \leqslant C_0 \big(C_0 C_1 \gamma^2\big)^n \|d\|_F \quad \text{with } (E,F) = \big(L^2, l^2\big), \big(H^1_0, h^2\big), \big(H^2_0, h^4\big), \big(H^3_0, h^6\big).$$

When $C_0 C_1 \gamma^2 \leqslant 1/2$, $\sum w_n$ converges normally in $H^3_0((0,T),\mathbb{R})$ and $w = \sum_{n=0}^{\infty} w_n$ satisfies $Z_\gamma(w) = d$ and (3.51) with $C_2 := 2C_0$. □

Theorem 8. *Let $T = 4/\pi$ and $\gamma_0 \in (0, \gamma_1)$. There exists a constant C and a continuous linear map,*

$$\begin{aligned} \Pi_{(\varphi_{1,\gamma},\gamma)} : \big[T_S\big(\psi_{1,\gamma}(0)\big) \times T_S\big(\psi_{1,\gamma}(T)\big)\big] \cap F^\gamma_9 &\to E^\gamma_7, \\ (\Psi_0, \Psi_T) &\mapsto (\Psi_0, w), \end{aligned}$$

such that, for every $(\Psi_0, \Psi_T) \in F^\gamma_9$ satisfying,

$$\Re\big(\big\langle \Psi_0, \psi_{1,\gamma}(0)\big\rangle\big) = \Re\big(\big\langle \Psi_T, \psi_{1,\gamma}(T)\big\rangle\big) = 0, \tag{3.52}$$

we have:

$$\begin{gathered} \Phi'_\gamma(\varphi_{1,\gamma}, 0).\Pi_{(\varphi_{1,\gamma},\gamma)}(\Psi_0, \Psi_T) = (\Psi_0, \Psi_T), \\ \|w\|_{L^2} \leqslant C \big\|(\Psi_0, \Psi_T)\big\|_{F^\gamma_3}, \qquad \|w\|_{H^1_0} \leqslant C \big\|(\Psi_0, \Psi_T)\big\|_{F^\gamma_5}, \\ \|w\|_{H^2_0} \leqslant C \big\|(\Psi_0, \Psi_T)\big\|_{F^\gamma_7}, \qquad \|w\|_{H^3_0} \leqslant C \big\|(\Psi_0, \Psi_T)\big\|_{F^\gamma_9}. \end{gathered}$$

Proof. We apply the previous proposition. The right-hand side of the moment problem:

$$\int_0^T w(t) e^{i(\lambda_{k,\gamma} - \lambda_{1,\gamma})t} \, dt = \frac{1}{i b_{k,\gamma}} \big(\langle \Psi_0, \varphi_{k,\gamma} \rangle - \langle \Psi_T, \varphi_{k,\gamma} \rangle e^{i\lambda_{k,\gamma} T} \big), \quad \forall k \geqslant 1,$$

belongs to $h^6(\mathbb{N}^*, \mathbb{C})$ because $(\Psi_0, \Psi_T) \in F^\gamma_9$ and $|b_{k,\gamma}| \geqslant c\gamma/k^3$ (see Proposition 1 in Section 3.1). The condition (3.52) implies that the first term of the right-hand side of the moment problem belongs to $\mathbb{R}$. □

3.6. Controllability of the linearized system around $(\psi(t), u(t))$ and bounds (3.19)–(3.22)

3.6.1. Strategy

We use the same idea as in the previous subsection: we associate a linear map $M_{(\psi_0,u)}$ to the controllability of the linearized system around $(\psi(t), u(t))$,and we show this linear map is closed enough to $M_{(\varphi_{1,\gamma},\gamma)}$ to be surjective. More precisely, we use the following proposition:

Proposition 15. *Let $T = 4/\pi$, M and M_γ be bounded linear operators from $L^2((0,T),\mathbb{R})$ to $h^3(\mathbb{N}^*,\mathbb{C})$, from $H_0^1((0,T),\mathbb{R})$ to $h^5(\mathbb{N}^*,\mathbb{C})$, from $H_0^2((0,T),\mathbb{R})$ to $h^7(\mathbb{N}^*,\mathbb{C})$ and from $H_0^3((0,T),\mathbb{R})$ to $h^9(\mathbb{N}^*,\mathbb{C})$. We assume there exist a continuous linear operator $M_\gamma^{-1}: h^9(\mathbb{N}^*,\mathbb{C}) \to H_0^3((0,T),\mathbb{R})$ and a positive constant C_0 such that for every $d \in h^9(\mathbb{N}^*,\mathbb{C})$, $M_\gamma \circ M_\gamma^{-1}(d) = d$ and $\|M_\gamma^{-1}(d)\|_E \leqslant C_0\|d\|_F$ for every $(E,F) \in \{(L^2,h^3), (H_0^1,h^5), (H_0^2,h^7), (H_0^3,h^9)\}$. We also assume there exist positive constants C_1, Δ_3, Δ_5, Δ_7, Δ_9 with $C_0C_1\Delta_3 \leqslant 1/2$, satisfying, for every $w \in H_0^3((0,T),\mathbb{R})$:*

$$\begin{gathered}
\big\|(M - M_\gamma)(w)\big\|_{h^3} \leqslant C_1\Delta_3\|w\|_{L^2},\\
\big\|(M - M_\gamma)(w)\big\|_{h^5} \leqslant C_1\big[\Delta_3\|w\|_{H_0^1} + \Delta_5\|w\|_{L^2}\big],\\
\big\|(M - M_\gamma)(w)\big\|_{h^7} \leqslant C_1\big[\Delta_3\|w\|_{H_0^2} + \Delta_5\|w\|_{H_0^1} + \Delta_7\|w\|_{L^2}\big],\\
\big\|(M - M_\gamma)(w)\big\|_{h^9} \leqslant C_1\big[\Delta_3\|w\|_{H_0^3} + \Delta_5\|w\|_{H_0^2} + \Delta_7\|w\|_{H_0^1} + \Delta_9\|w\|_{L^2}\big].
\end{gathered}$$

Then, there exists a continuous linear operator $M^{-1}: h^9(\mathbb{N}^*,\mathbb{C}) \to H_0^3((0,T),\mathbb{R})$ such that for every $d \in h^9(\mathbb{N}^*,\mathbb{C})$, $M \circ M^{-1}(d) = d$ and the function $w := M^{-1}(d)$ satisfies:

$$\begin{gathered}
\|w\|_{L^2} \leqslant 2C_0\|d\|_{h^3},\\
\|w\|_{H_0^1} \leqslant 2C_0\big[\|d\|_{h^5} + 2C_2\Delta_5\|d\|_{h^3}\big],\\
\|w\|_{H_0^2} \leqslant 2C_0\big[\|d\|_{h^7} + 2C_2\Delta_5\|d\|_{h^5} + \big(2C_2\Delta_7 + 8C_2^2\Delta_5^2\big)\|d\|_{h^3}\big],\\
\|w\|_{H_0^3} \leqslant 2C_0\big[\|d\|_{h^9} + 2C_2\Delta_5\|d\|_{h^7} + \big(2C_2\Delta_7 + 8C_2^2\Delta_5^2\big)\|d\|_{h^5}\\
+ \big(2C_2\Delta_9 + 16C_2^2\Delta_7\Delta_5 + 48C_2^3\Delta_5^3\big)\|d\|_{h^3}\big],
\end{gathered}$$

where $C_2 := C_0C_1$.

Proof. Let $d \in h^9(\mathbb{N}^*,\mathbb{C})$. We construct by induction a sequence $(w_n)_{n\in\mathbb{N}}$ in $H_0^3((0,T),\mathbb{R})$ by:

$$w_0 := M_\gamma^{-1}(d), \qquad w_{n+1} := M_\gamma^{-1}\big((M_\gamma - M)(w_n)\big), \quad \forall n \in \mathbb{N}.$$

Then, we have, for every $n \in \mathbb{N}$:

$$\|w_n\|_{L^2} \leqslant C_0 C_2^n \Delta_3^n \|d\|_{h^3},$$
$$\|w_n\|_{H_0^1} \leqslant C_0 C_2^n \big[\Delta_3^n \|d\|_{h^5} + n\Delta_3^{n-1}\Delta_5 \|d\|_{h^3}\big],$$
$$\|w_n\|_{H_0^2} \leqslant C_0 C_2^n \big[\Delta_3^n \|d\|_{h^7} + n\Delta_5\Delta_3^{n-1}\|d\|_{h^5} + \big(n\Delta_7\Delta_3^{n-1} + n(n-1)\Delta_5^2\Delta_3^{n-2}\big)\|d\|_{h^3}\big],$$
$$\|w_n\|_{H_0^3} \leqslant C_0 C_2^n \big[\Delta_3^n \|d\|_{h^9} + n\Delta_3^{n-1}\Delta_5\|d\|_{h^7} + \big(n\Delta_3^{n-1}\Delta_7 + n(n-1)\Delta_3^{n-2}\Delta_5^2\big)\|d\|_{h^5}$$
$$+ \big(n\Delta_3^{n-1}\Delta_9 + 2n(n-1)\Delta_3^{n-2}\Delta_7\Delta_5 + n(n-1)(n-2)\Delta_3^{n-3}\Delta_5^3\big)\|d\|_{h^3}\big].$$

When $C_2\Delta_3 \leqslant 1/2$, $\sum w_n$ is normally convergent in $H_0^3((0,T),\mathbb{R})$, and $w := \sum_{n=0}^{\infty} w_n$ gives the solution. □

Let $v \in C^\infty([0,T],\mathbb{R})$ be such that $v^{(k)}(0) = v^{(k)}(T) = 0$ for every $k \in \mathbb{N}^*$. Let $u = \gamma + v$, $\psi_0 \in S \cap H^9_{(\gamma)}(I,\mathbb{C})$ and $\psi \in C^0([0,T], S\cap H^9(I,\mathbb{C}))$ be the solution of

$$\begin{cases} \dot\psi = \frac{\mathrm{i}}{2}\psi'' + \mathrm{i}u(t)q\psi, \\ \psi(0) = \psi_0, \\ \psi(t,-\frac12) = \psi(t,\frac12) = 0. \end{cases}$$

Let $\Psi_0 \in H^7_{(\gamma)}(I,\mathbb{C})$ be such that $\Re(\langle \Psi_0, \psi_0\rangle) = 0$. The linearized control system around $(\psi(t), u(t))$ is:

$$\begin{cases} \dot\Psi = \frac{\mathrm{i}}{2}\Psi'' + \mathrm{i}u(t)q\Psi + \mathrm{i}w(t)q\psi, \\ \Psi(0) = \Psi_0, \\ \Psi(t,-\frac12) = \Psi(t,\frac12) = 0, \end{cases}$$

where the state is Ψ and the control is w. To get the controllability of the linearized system around $(\psi_{1,\gamma},\gamma)$ we decomposed the solution on the basis $(\varphi_{k,\gamma})_{k\in\mathbb{N}^*}$. The natural idea in the general case consists in decomposing $\Psi(t)$ on the basis $(\varphi_{k,u(t)})$, for every t: $\Psi(t) = \sum_{k=1}^{\infty} x_k(t)\varphi_{k,u(t)}$. Unfortunately, in this decomposition, the condition $\Re(\langle \Psi(t),\psi(t)\rangle) = 0$ does not give any information, in particular $x_0(0)$, $x_0(T)$ do not belong to $\mathrm{i}\mathbb{R}$. To take the conditions $\Re(\langle\Psi_0,\psi_0\rangle) = \Re(\langle\Psi_T,\psi_T\rangle) = 0$ into account, we decompose $\Psi(t)$ on $(\xi_k(t))_{k\in\mathbb{N}^*}$ defined by:

$$\xi_1(t) = \psi(t), \qquad \xi_k(t) = \varphi_{k,u(t)} - \langle \varphi_{k,u(t)}, \psi(t)\rangle \psi(t), \quad \text{for } k \geqslant 2.$$

Remark. This family is independent when $\psi_0 \in L^2(I,\mathbb{C})$, $v \in H_0^1((0,T),\mathbb{R})$ and (ψ_0, v) is closed enough to $(\varphi_{1,\gamma}, 0)$ in $L^2(I,\mathbb{C}) \times H_0^1((0,T),\mathbb{R})$.

In order to justify this point, it is sufficient to prove that these assumptions imply:

$$\forall t \in [0,T], \quad x_1(t) := \langle \psi(t), \varphi_{1,u(t)}\rangle \neq 0.$$

We have:

$$\dot{x}_1(t) = -\mathrm{i}\lambda_{1,u(t)} x_1(t) + \dot{u}(t)\left\langle \psi(t), \frac{\mathrm{d}\varphi_{1,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(t)}\right\rangle,$$

$$x_1(t) = \left(\langle \psi_0, \varphi_{1,\gamma}\rangle + \int_0^t \dot{u}(\tau)\left\langle \psi(\tau), \frac{\mathrm{d}\varphi_{1,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(\tau)}\right\rangle \mathrm{e}^{\mathrm{i}\int_0^\tau \lambda_{1,u(s)}\,\mathrm{d}s}\,\mathrm{d}\tau\right)\mathrm{e}^{-\mathrm{i}\int_0^t \lambda_{1,u(s)}\,\mathrm{d}s}.$$

Thus,

$$\begin{aligned}\left|x_1(t)\mathrm{e}^{\mathrm{i}\int_0^t \lambda_{1,u(s)}\,\mathrm{d}s} - 1\right| &= \Bigg|\langle \psi_0 - \varphi_{1,\gamma}, \varphi_{1,\gamma}\rangle \\ &\quad + \int_0^t \dot{u}(\tau)\left\langle \psi(\tau), \frac{\mathrm{d}\varphi_{1,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(\tau)}\right\rangle \mathrm{e}^{\mathrm{i}\int_0^\tau \lambda_{1,u(s)}\,\mathrm{d}s}\,\mathrm{d}\tau\Bigg| \\ &\leqslant \|\psi_0 - \varphi_{1,\gamma}\|_{L^2} + \|\dot{u}\|_{L^2}\sqrt{T}C^*,\end{aligned}$$

and $|x_1(t)| > 1/2$, for every $t \in [0,T]$, when (ψ_0, u) is closed enough to $(\varphi_{1,\gamma}, 0)$ in $L^2(I,\mathbb{C}) \times H^1((0,T),\mathbb{R})$.

If we have a decomposition $\Psi(t) = \sum_{k=1}^\infty y_k(t)\xi_k(t)$ then $y_1(t) = \langle \Psi(t), \psi(t)\rangle \in \mathrm{i}\mathbb{R}$. We find such a decomposition starting from the equality $\Psi(t) = \sum_{k=1}^\infty \langle \Psi(t), \varphi_{k,u(t)}\rangle \varphi_{k,u(t)}$ and the coefficients are:

$$y_1(t) = \langle \Psi(t), \psi(t)\rangle,$$

$$y_k(t) = \langle \Psi(t), \varphi_{k,u(t)}\rangle - \frac{\langle \Psi(t), \varphi_{1,u(t)}\rangle}{\langle \psi(t), \varphi_{1,u(t)}\rangle}\langle \psi(t), \varphi_{k,u(t)}\rangle.$$

The function $\gamma \in (\gamma^*, \gamma^*) \mapsto \varphi_{k,\gamma}$ is analytic for every $k \in \mathbb{N}^*$ (see Appendix A) so $y_k \in C^1([0,T],\mathbb{C})$ for every $k \in \mathbb{N}^*$ and these functions satisfy the following ordinary differential equations:

$$\dot{y}_1(t) = \mathrm{i}w(t)\langle q\psi(t), \psi(t)\rangle,$$

$$\begin{aligned}\dot{y}_k(t) &= -\mathrm{i}\lambda_{k,u(t)} y_k(t) + \mathrm{i}w(t)\langle q\psi(t), \varphi_{k,u(t)}\rangle + \dot{u}(t)\left\langle \Psi(t), \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(t)}\right\rangle \\ &\quad - \dot{u}(t)\frac{\langle \Psi(t), \varphi_{1,u(t)}\rangle}{\langle \psi(t), \varphi_{1,u(t)}\rangle}\left\langle \psi(t), \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(t)}\right\rangle - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\langle \Psi(t), \varphi_{1,u(t)}\rangle}{\langle \psi(t), \varphi_{1,u(t)}\rangle}\right)\langle \psi(t), \varphi_{k,u(t)}\rangle,\end{aligned}$$

where $\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}|_{u(t)}$ denotes the derivative of the map $\gamma \in (-\gamma^*, \gamma^*) \mapsto \varphi_{k,\gamma} \in L^2((0,T),\mathbb{C})$ considered at the point $\gamma = u(t)$. We decompose $\Psi = \Psi_1 + \Psi_2$ where Ψ_1 does not depend on w and Ψ_2 depends on w linearly:

$$\begin{cases} \dot{\Psi}_1 = \frac{\mathrm{i}}{2}\Psi_1'' + \mathrm{i}u(t)q\Psi_1, \\ \Psi_1(0) = \Psi_0, \\ \Psi_1(t,-\frac{1}{2}) = \Psi_1(t,\frac{1}{2}) = 0, \end{cases}$$

$$\begin{cases} \dot{\Psi}_2 = \frac{\mathrm{i}}{2}\Psi_2'' + \mathrm{i}u(t)q\Psi_2 + \mathrm{i}w(t)q\psi, \\ \Psi_2(0) = 0, \\ \Psi_2(t,-\frac{1}{2}) = \Psi_2(t,\frac{1}{2}) = 0. \end{cases}$$

If $v(0) = v(T) = 0$, the equality $\Psi(T) = \Psi_T$ is equivalent to

$$M_{(\psi_0,u)}(w) = d(\Psi_0, \Psi_T),$$

where $M_{(\psi_0,u)}(w)$ is the sequence defined by:

$$M_{(\psi_0,u)}(w)_1 := \int_0^T w(t)\big\langle q\psi(t), \psi(t)\big\rangle \,\mathrm{d}t,$$

$$\begin{aligned} M_{(\psi_0,u)}(w)_k := \int_0^T \Bigg[& w(t)\big\langle q\psi(t), \varphi_{k,u(t)}\big\rangle - \mathrm{i}\dot{u}(t)\bigg\langle \Psi_2(t), \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(t)}\bigg\rangle \\ & + \mathrm{i}\dot{u}(t)\frac{\langle \Psi_2(t), \varphi_{1,u(t)}\rangle}{\langle \psi(t), \varphi_{1,u(t)}\rangle}\bigg\langle \psi(t), \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(t)}\bigg\rangle \\ & + \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\frac{\langle \Psi_2(t), \varphi_{1,u(t)}\rangle}{\langle \psi(t), \varphi_{1,u(t)}\rangle}\bigg)\big\langle \psi(t), \varphi_{k,u(t)}\big\rangle\Bigg] \mathrm{e}^{\mathrm{i}\int_0^t \lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t, \quad k \geqslant 2, \end{aligned}$$

and $d(\Psi_0, \Psi_T)$ is the sequence defined by,

$$d(\Psi_0, \Psi_T)_1 := -\mathrm{i}\big(\langle \Psi_T, \psi_T\rangle - \langle \Psi_0, \psi_0\rangle\big),$$

$$\begin{aligned} d(\Psi_0, \Psi_T)_k := & -\mathrm{i}\bigg(\langle \Psi_T, \varphi_{k,\gamma}\rangle - \frac{\langle \Psi_T, \varphi_{1,\gamma}\rangle}{\langle \psi_T, \varphi_{1,\gamma}\rangle}\langle \psi_T, \varphi_{k,\gamma}\rangle\bigg)\mathrm{e}^{\mathrm{i}\int_0^T \lambda_{k,u(s)}\,\mathrm{d}s} \\ & + \mathrm{i}\bigg(\langle \Psi_0, \varphi_{k,\gamma}\rangle - \frac{\langle \Psi_0, \varphi_{1,\gamma}\rangle}{\langle \psi_0, \varphi_{1,\gamma}\rangle}\langle \psi_0, \varphi_{k,\gamma}\rangle\bigg) \\ & + \int_0^T \Bigg[\mathrm{i}\dot{u}(t)\bigg\langle \Psi_1(t), \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(t)}\bigg\rangle - \mathrm{i}\dot{u}(t)\frac{\langle \Psi_1(t), \varphi_{1,u(t)}\rangle}{\langle \psi(t), \varphi_{1,u(t)}\rangle}\bigg\langle \psi(t), \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(t)}\bigg\rangle \\ & - \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\frac{\langle \Psi_1(t), \varphi_{1,u(t)}\rangle}{\langle \psi(t), \varphi_{1,u(t)}\rangle}\bigg)\big\langle \psi(t), \varphi_{k,u(t)}\big\rangle\Bigg]\mathrm{e}^{\mathrm{i}\int_0^t \lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t, \quad k \geqslant 2. \end{aligned}$$

3.6.2. Preliminaries

Every term appearing in the problem $M_{(\psi_0,u)}(w) = d(\Psi_0, \Psi_T)$ are of the general form:

$$S^0 := \left(\int_0^T w(t) \langle f(t), \varphi_{k,u(t)} \rangle e^{i \int_0^t \lambda_{k,u(s)} \, ds} \, dt \right)_{k \in \mathbb{N}^*},$$

$$S^1 := \left(\int_0^T w(t) \left\langle f(t), \frac{d\varphi_{k,\gamma}}{d\gamma} \bigg|_{u(t)} \right\rangle e^{i \int_0^t \lambda_{k,u(s)} \, ds} \, dt \right)_{k \in \mathbb{N}^*}.$$

This is why we dedicate this subsection to the research of bounds for the h^3, h^5, h^7 and h^9-norms of such terms. For technical reasons, we also find bounds on

$$S^2 := \left(\int_0^T w(t) \left\langle f(t), \frac{d^2\varphi_{k,\gamma}}{d\gamma^2} \bigg|_{u(t)} \right\rangle e^{i \int_0^t \lambda_{k,u(s)} \, ds} \, dt \right)_{k \in \mathbb{N}^*}.$$

Lemma 2. *There exists a constant $C > 0$ such that, for every $\gamma \in [-\gamma^*, \gamma^*]$ and for every $f \in L^2(I, \mathbb{C})$,*

$$\sum_{k=1}^{\infty} \left| \left\langle f, k \frac{d\varphi_{k,\gamma}}{d\gamma} \bigg|_{\gamma_1} \right\rangle \right|^2 \leqslant C \|f\|_{L^2}^2.$$

Proof. We first prove the inequality when $\gamma = 0$. We use the explicit formula (A.4) or (A.5):

$$k \frac{d\varphi_{k,\gamma}}{d\gamma} \bigg|_0 = \sum_{j=0}^{\infty} a_{j,k} \varphi_j,$$

where $a_{j,k} = 0$ when j and k have the same parity and

$$a_{j,k} = \frac{16(-1)^{(k+j+1)/2} k^2 j}{\pi^4 (j+k)^3 (j-k)^3},$$

when j and k have different parity. We check there exists a constant C such that

$$\forall k \in \mathbb{N}^*, \ \sum_{j=1}^{\infty} |a_{j,k}| \leqslant C \quad \text{and} \quad \forall j \in \mathbb{N}^*, \ \sum_{k=1}^{\infty} |a_{j,k}| \leqslant C.$$

Therefore, for every $(x_j)_{j \in \mathbb{N}^*} \in l^2(\mathbb{N}^*, \mathbb{C})$,

$$\sum_{k=1}^{\infty} \left| \sum_{j=1}^{\infty} a_{k,j} x_j \right|^2 \leqslant C^2 \sum_{j=1}^{\infty} |x_j|^2.$$

Let $\gamma \in [-\gamma^*, \gamma^*]$ and $f \in L^2(I, \mathbb{C})$. For every $k \in \mathbb{N}^*$, we have:

$$\left|\left\langle f, k\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{\gamma}\right\rangle\right| \leqslant \left|\left\langle f, k\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{0}\right\rangle\right| + \|f\|_{L^2} k \left\| \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_0 - \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{\gamma}\right\|_{L^2}.$$

We conclude thanks to the previous result and the inequality (A.17). □

Lemma 3. *There exists a constant $C > 0$ such that, for every $\gamma \in [-\gamma^*, \gamma^*]$ and for every $f \in H^1(I, \mathbb{C})$,*

$$\big(k\langle f, \varphi_{k,\gamma}\rangle\big)_{k\in\mathbb{N}^*} = \left(\frac{\sqrt{2}(-1)^{k+1+[k/2]}}{\pi}\big(f(1/2) - (-1)^k f(-1/2)\big)\right)_{k\in\mathbb{N}^*}$$
$$+ \textit{terms with an } l^2\textit{-norm bounded by } C\|f\|_{H^1}.$$

Proof. We have:

$$\langle f, \varphi_{k,\gamma}\rangle = \langle f, \varphi_{k,\gamma} - \tilde{\varphi}_{k,\gamma}\rangle + \gamma\left\langle f, \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_0\right\rangle + \langle f, \varphi_k\rangle.$$

The first term of the right-hand side of this equality belongs to $h^1(\mathbb{N}^*, \mathbb{C})$ thanks to (A.8) and

$$\big\|\langle f, \varphi_{k,\gamma} - \tilde{\varphi}_{k,\gamma}\rangle\big\|_{h^1} \leqslant \|f\|_{L^2} C^* \gamma^2 \sqrt{\frac{\pi^2}{6}}.$$

The second one belongs to $h^1(\mathbb{N}^*, \mathbb{C})$ and its h^1-norm can be bounded by $C\gamma\|f\|_{L^2}$ thanks to Lemma 2. Using the explicit expression of φ_k given in Appendix B and an integration by parts, we get:

$$\langle f, \varphi_k\rangle = \frac{(-1)^{k+1+[k/2]}\sqrt{2}}{k\pi}\big(f(1/2) - (-1)^k f(-1/2)\big) + \frac{(-1)^k}{k\pi}\left\langle f', \frac{1}{k\pi}\varphi_k'\right\rangle.$$

The family $((1/k\pi)\varphi_k')$ is orthonormal in $L^2(I, \mathbb{C})$ so the second term of the right-hand side of this equality belongs to $h^1(\mathbb{N}^*, \mathbb{C})$ and its h^1-norm is bounded by $C\|f'\|_{L^2}$. □

Lemma 4. *There exists a constant $C > 0$ such that, for every $\gamma \in [-\gamma^*, \gamma^*]$ and for every $f \in H^3 \cap H_0^1(I, \mathbb{C})$,*

$$\big(k^3\langle f, \varphi_{k,\gamma}\rangle\big)_{k\in\mathbb{N}^*} = \left(\frac{2\sqrt{2}(-1)^{k+1+[k/2]}}{\pi^3}\big(A_\gamma f(1/2) - (-1)^k A_\gamma f(-1/2)\big)\right)_{k\in\mathbb{N}^*}$$
$$+ \textit{terms with an } l^2\textit{-norm bounded by } C\|f\|_{H^3}.$$

Proof. We have:

$$k^3\langle f,\varphi_{k,\gamma}\rangle = k^3\left(\frac{1}{\lambda_{k,\gamma}}-\frac{1}{\lambda_k}\right)\langle A_\gamma f,\varphi_{k,\gamma}\rangle + \frac{2k}{\pi^2}\langle A_\gamma f,\varphi_{k,\gamma}\rangle.$$

The first term of the right-hand side of this equality belongs to $l^2(\mathbb{N}^*,\mathbb{C})$ thanks to (A.13) and its l^2-norm is bounded by $C\|A_\gamma f\|_{L^2}$. We conclude applying the previous lemma to the second term. □

Lemma 5. *There exists a constant $C>0$ such that, for every $\gamma\in[-\gamma^*,\gamma^*]$ and for every $f\in H^2\cap H^1_0(I,\mathbb{C})$,*

$$\sum_{k=1}^{\infty}\left|k^3\left\langle f,\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_\gamma\right\rangle\right|^2 \leqslant C\|f\|^2_{H^2}.$$

Proof. Using Eq. (A.14) and integrations by parts we get:

$$\left\langle f,\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_\gamma\right\rangle = \frac{1}{\lambda_{k,\gamma}}\left\langle A_\gamma f,\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_\gamma\right\rangle - \frac{1}{\lambda_{k,\gamma}^2}\big\langle A_\gamma(qf),\varphi_{k,\gamma}\big\rangle - \frac{\lambda'_{k,\gamma}}{\lambda_{k,\gamma}}\langle f,\varphi_{k,\gamma}\rangle.$$

We use Lemma 2 and (A.12) in the first term of the right-hand side of this equality. We use the Cauchy–Schwarz inequality and (A.12) in the second one. We conclude thanks to (A.15) and (A.12) in the third term of the right-hand side. □

Lemma 6. *There exists a constant C such that, for every $\gamma\in[-\gamma^*,\gamma^*]$ and for every $f\in H^2\cap H^1_0(I,\mathbb{C})$,*

$$\sum_{k=1}^{\infty}\left|k^3\left\langle f,\frac{\mathrm{d}^2\varphi_{k,\gamma}}{\mathrm{d}\gamma^2}\bigg|_\gamma\right\rangle\right|^2 \leqslant C\|f\|^2_{H^2}.$$

Proof. Using Eq. (A.20) and integrations by parts we get:

$$\begin{aligned}\left\langle f,\frac{\mathrm{d}^2\varphi_{k,\gamma}}{\mathrm{d}\gamma^2}\bigg|_\gamma\right\rangle &= \frac{1}{\lambda_{k,\gamma}}\left\langle A_\gamma f,\frac{\mathrm{d}^2\varphi_{k,\gamma}}{\mathrm{d}\gamma^2}\bigg|_\gamma\right\rangle - \frac{2\lambda'_{k,\gamma}}{\lambda_{k,\gamma}}\left\langle f,\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_\gamma\right\rangle\\ &\quad - \frac{2}{\lambda_{k,\gamma}}\left\langle qf,\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_\gamma\right\rangle - \frac{\lambda''_{k,\gamma}}{\lambda_{k,\gamma}}\langle f,\varphi_{k,\gamma}\rangle.\end{aligned}$$

We use (A.21) and (A.12) in the first term of the right-hand side of this equality. We use (A.19), (A.18) and (A.12) in the second one. We apply Lemma 2 on the third term, together with (A.12). We conclude using (A.22), (A.12) and the orthonormality of the family $(\varphi_{k,\gamma})_{k\in\mathbb{N}^*}$ in the last term. □

Lemma 7. *There exists a constant $C > 0$ such that, for every $\gamma \in [-\gamma^*, \gamma^*]$ and for every $f \in H^2 \cap H_0^1(I, \mathbb{C})$,*

$$\sum_{k=1}^{\infty} \left| k^3 \left\langle f, \frac{\mathrm{d}^3 \varphi_{k,\gamma}}{\mathrm{d}\gamma^3} \bigg|_\gamma \right\rangle \right|^2 \leqslant C \|f\|_{H^2}^2.$$

Proof. Using Eq. (A.24) and integrations by parts we get:

$$\left\langle f, \frac{\mathrm{d}^3 \varphi_{k,\gamma}}{\mathrm{d}\gamma^3} \bigg|_\gamma \right\rangle = \frac{1}{\lambda_{k,\gamma}} \left\langle A_\gamma f, \frac{\mathrm{d}^3 \varphi_{k,\gamma}}{\mathrm{d}\gamma^3} \bigg|_\gamma \right\rangle - \frac{3\lambda'_{k,\gamma}}{\lambda_{k,\gamma}} \left\langle f, \frac{\mathrm{d}^2 \varphi_{k,\gamma}}{\mathrm{d}\gamma^2} \bigg|_\gamma \right\rangle - \frac{3}{\lambda_{k,\gamma}} \left\langle q f, \frac{\mathrm{d}^2 \varphi_{k,\gamma}}{\mathrm{d}\gamma^2} \bigg|_\gamma \right\rangle$$
$$- \frac{3\lambda''_{k,\gamma}}{\lambda_{k,\gamma}} \left\langle f, \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma} \bigg|_\gamma \right\rangle - \frac{\lambda'''_{k,\gamma}}{\lambda_{k,\gamma}^2} \langle A_\gamma f, \varphi_{k,\gamma} \rangle.$$

We deal with each term, one by one, using the Cauchy–Schwarz inequality and the bounds (A.25), (A.21), (A.18), (A.19), (A.22), (A.26) and (A.12). □

Lemma 8. *Let $T = 4/\pi$ and $u \in L^\infty((0, T), \mathbb{R})$ be such that $\|u\|_{L^\infty} \leqslant \gamma^*$. There exists a constant $C > 0$ such that, for every $f \in L^2((0, T), \mathbb{C})$,*

$$\left(\int_0^T f(t) \mathrm{e}^{\mathrm{i} \int_0^t \lambda_{k,u(s)} \, \mathrm{d}s} \, \mathrm{d}t \right)_{k \in \mathbb{N}^*}$$

belongs to $l^2(\mathbb{N}^*, \mathbb{C})$ and its l^2-norm is bounded by $C\|f\|_{L^2}$.

Proof. We have:

$$\int_0^T f(t) \mathrm{e}^{\mathrm{i} \int_0^t \lambda_{k,u(s)} \, \mathrm{d}s} \, \mathrm{d}t = \int_0^T f(t) \mathrm{e}^{\mathrm{i}\lambda_k t} \, \mathrm{d}t + \int_0^T f(t) \left(\mathrm{e}^{\mathrm{i} \int_0^t \lambda_{k,u(s)} \, \mathrm{d}s} - \mathrm{e}^{\mathrm{i}\lambda_k t} \right) \mathrm{d}t.$$

The first term of the right-hand side of this equality belongs to $l^2(\mathbb{N}^*, \mathbb{C})$ because it is a Fourier coefficient of an L^2-function. In the second one, we use:

$$\left| \mathrm{e}^{\mathrm{i} \int_0^t \lambda_{k,u(s)} \, \mathrm{d}s} - \mathrm{e}^{\mathrm{i}\lambda_k t} \right| \leqslant \int_0^t |\lambda_{k,u(s)} - \lambda_k| \, \mathrm{d}s \leqslant \frac{C^* \|u\|_{L^2}^2}{k},$$

which is a consequence of (A.11). □

Proposition 16. *There exists a constant $C > 0$ such that, for every $u \in L^\infty((0,T),\mathbb{R})$ satisfying $\|u\|_\infty \leqslant \gamma^*$, for every $f \in C^0([0,T], H^3 \cap H^1_0(I,\mathbb{C}))$ and for every $w \in L^2((0,T),\mathbb{R})$, S^0 belongs to $h^3(\mathbb{N}^*,\mathbb{C})$ and*

$$\|S^0\|_{h^3} \leqslant C\|w\|_{L^2}\|f\|_{C^0([0,T],H^3)}.$$

Proof. Thanks to Lemma 4 and the Cauchy–Schwarz inequality, we have:

$$k^3 S^0_k = \int_0^T w(t)\frac{2\sqrt{2}(-1)^{k+1+[k/2]}}{\pi^3}\big(A_\gamma f(t,1/2) - (-1)^k A_\gamma f(t,-1/2)\big)e^{i\int_0^t \lambda_{k,u(s)}\,ds}\,dt$$
$$+ \text{terms with an } l^2\text{-norm bounded by } C\|f\|_{C^0([0,T],H^3)}\|w\|_{L^2}.$$

Therefore

$$\|k^3 S^0_k\|_{l^2} \leqslant \left\|\int_0^T w(t)\frac{2\sqrt{2}}{\pi^3}\big(A_\gamma f(t,1/2) - A_\gamma f(t,-1/2)\big)e^{i\int_0^t \lambda_{k,u(s)}\,ds}\,dt\right\|_{l^2}$$
$$+ \left\|\int_0^T w(t)\frac{2\sqrt{2}}{\pi^3}\big(A_\gamma f(t,1/2) + A_\gamma f(t,-1/2)\big)e^{i\int_0^t \lambda_{k,u(s)}\,ds}\,dt\right\|_{l^2}$$
$$+ \text{terms with an } l^2\text{-norm bounded by } C\|w\|_{L^2}\|f\|_{C^0([0,T],H^3)}.$$

We conclude applying Lemma 8 on the two first terms of the right-hand side of this inequality. □

Proposition 17. *There exists a constant $C > 0$ such that, for every $u \in L^\infty((0,T),\mathbb{R})$ satisfying $\|u\|_\infty \leqslant \gamma^*$, for every $f \in C^0([0,T], H^2 \cap H^1_0(I,\mathbb{C}))$ and for every $w \in L^2((0,T),\mathbb{R})$, S^1 belongs to $h^3(\mathbb{N}^*,\mathbb{C})$ and*

$$\|S^1\|_{h^3} \leqslant C\|w\|_{L^2}\|f\|_{C^0([0,T],H^2)}.$$

Proof. We use the Cauchy–Schwarz inequality in $L^2((0,T),\mathbb{C})$ and Lemma 5. □

Proposition 18. *There exists a constant $C > 0$ such that, for every $u \in L^\infty((0,T),\mathbb{R})$ satisfying $\|u\|_\infty \leqslant \gamma^*$, for every $f \in C^0([0,T], H^2 \cap H^1_0(I,\mathbb{C}))$ and for every $w \in L^2((0,T),\mathbb{C})$, S^2 belongs to $h^3(\mathbb{N}^*,\mathbb{C})$ and*

$$\|S^2\|_{h^3} \leqslant C\|w\|_{L^2}\|f\|_{C^0([0,T],H^2)}.$$

Proof. We use the Cauchy–Schwarz inequality in $L^2((0,T),\mathbb{C})$ and Lemma 6. □

Proposition 19. *There exists a constant $C>0$ such that, for every $u\in H^1((0,T),\mathbb{R})$ satisfying $\|u\|_{H^1}\leqslant\gamma^*$ and $\|u\|_{L^\infty}\leqslant\gamma^*$, for every $f\in C^1([0,T],H^3\cap H_0^1(I,\mathbb{C}))$ and for every $w\in H_0^1((0,T),\mathbb{C})$, S^0 belongs to $h^5(\mathbb{N}^*,\mathbb{C})$ and*

$$\|S^0\|_{h^5}\leqslant C\big[\|w\|_{L^2}\|f\|_{C^1([0,T],H^3)}+\|w\|_{H^1}\|f\|_{C^0([0,T],H^3)}\big],$$

$$(S_k^0)_{k\geqslant2}=\left(\frac{\mathrm{i}}{\lambda_k}\int_0^T\big[\dot w(t)\langle f(t),\varphi_{k,u(t)}\rangle+w(t)\langle\dot f(t),\varphi_{k,u(t)}\rangle\big]\mathrm{e}^{\mathrm{i}\int_0^t\lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t\right)_{k\geqslant2}$$

$+$ *terms with an h^5-norm bounded by*:

$$C\big[\|w\|_{H^1}\|u\|_{H^1}\|f\|_{C^0([0,T],L^2)}+\|w\|_{L^2}\|u\|_{H^1}\big(\|f\|_{C^0([0,T],H^2)}+\|f\|_{C^1([0,T],L^2)}\big)\big].$$

Proof. We have:

$$\begin{aligned}S_k^0=&-\int_0^T\frac{1}{\mathrm{i}\lambda_{k,u(t)}}\dot w(t)\langle f(t),\varphi_{k,u(t)}\rangle\mathrm{e}^{\mathrm{i}\int_0^t\lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t\\&+\int_0^T\frac{1}{\mathrm{i}\lambda_{k,u(t)}^2}w(t)\dot u(t)\lambda'_{k,u(t)}\langle f(t),\varphi_{k,u(t)}\rangle\mathrm{e}^{\mathrm{i}\int_0^t\lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t\\&-\int_0^T\frac{1}{\mathrm{i}\lambda_{k,u(t)}}w(t)\langle\dot f(t),\varphi_{k,u(t)}\rangle\mathrm{e}^{\mathrm{i}\int_0^t\lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t\\&-\int_0^T\frac{1}{\mathrm{i}\lambda_{k,u(t)}}w(t)\dot u(t)\left\langle f(t),\left.\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\right|_{u(t)}\right\rangle\mathrm{e}^{\mathrm{i}\int_0^t\lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t.\end{aligned}$$

We call this decomposition $S_k^0=\mathcal{A}_k+\mathcal{B}_k+\mathcal{C}_k+\mathcal{D}_k$. Using,

$$\frac{1}{\lambda_{k,u}}=\left(\frac{1}{\lambda_{k,u}}-\frac{1}{\lambda_k}\right)+\frac{1}{\lambda_k},$$

we get the decompositions $\mathcal{A}=\mathcal{A}^{(1)}+\mathcal{A}^{(2)}$, $\mathcal{C}=\mathcal{C}^{(1)}+\mathcal{C}^{(2)}$. We apply Proposition 16 for $\mathcal{A}^{(2)}$ and $\mathcal{C}^{(2)}$, we get:

$$\|\mathcal{A}^{(2)}\|_{h^5}\leqslant C\|w\|_{H^1}\|f\|_{C^0([0,T],H^3)},$$
$$\|\mathcal{C}^{(2)}\|_{h^5}\leqslant C\|w\|_{L^2}\|f\|_{C^1([0,T],H^3)}.$$

Thanks to (A.13), we get similarly,

$$\left\|\mathcal{A}^{(1)}\right\|_{h^5} \leqslant C\|w\|_{H^1}\|u\|_{H^1}\|f\|_{C^0([0,T],L^2)},$$
$$\left\|\mathcal{C}^{(1)}\right\|_{h^5} \leqslant C\|w\|_{L^2}\|u\|_{H^1}\|f\|_{C^1([0,T],L^2)}.$$

Using (A.12), the Cauchy–Schwarz inequality and Lemma 5, we get:

$$|\mathcal{D}_k| \leqslant \|w\|_{L^2}\left(\int_0^T \left|\dot{u}(t)\frac{2}{C_*(k\pi)^2}\left\langle f(t), \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(t)}\right\rangle\right|^2 \mathrm{d}t\right)^{1/2},$$
$$\|\mathcal{D}\|_{h^5} \leqslant C\|w\|_{L^2}\|\dot{u}\|_{L^2}\|f\|_{C^0([0,T],H^2)}.$$

Using (A.19) and the orthonormality of the family $(\varphi_{k,\gamma})_{k\in\mathbb{N}^*}$ in $L^2(I,\mathbb{C})$, we get:

$$\|\mathcal{B}\|_{h^5} \leqslant C\|w\|_{L^2}\|\dot{u}\|_{L^2}\|f\|_{C^0([0,T],L^2)}. \quad \square$$

Proposition 20. *There exists a constant $C > 0$ such that, for every $u \in H^1((0,T),\mathbb{R})$ satisfying $\|u\|_{H^1} \leqslant \gamma^*$ and $\|u\|_{L^\infty} \leqslant \gamma^*$, for every $f \in C^1([0,T], H^2 \cap H^1_0(I,\mathbb{C}))$ and for every $w \in H^1_0((0,T),\mathbb{C})$, S^1 belongs to h^5 and*

$$\left\|S^1\right\|_{h^5} \leqslant C\left[\|w\|_{L^2}\|f\|_{C^1([0,T],H^2)} + \|w\|_{H^1}\|f\|_{C^0([0,T],H^2)}\right].$$

Proof. We have:

$$\begin{aligned}
S^1_k = &-\int_0^T \frac{1}{\mathrm{i}\lambda_{k,u(t)}}\dot{w}(t)\left\langle f(t), \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(t)}\right\rangle \mathrm{e}^{\mathrm{i}\int_0^t \lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t \\
&+ \int_0^T \frac{1}{\mathrm{i}\lambda^2_{k,u(t)}} w(t)\dot{u}(t)\lambda'_{k,u(t)}\left\langle f(t), \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(t)}\right\rangle \mathrm{e}^{\mathrm{i}\int_0^t \lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t \\
&- \int_0^T \frac{1}{\mathrm{i}\lambda_{k,u(t)}} w(t)\left\langle \dot{f}(t), \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(t)}\right\rangle \mathrm{e}^{\mathrm{i}\int_0^t \lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t \\
&- \int_0^T \frac{1}{\mathrm{i}\lambda_{k,u(t)}} w(t)\dot{u}(t)\left\langle f(t), \frac{\mathrm{d}^2\varphi_{k,\gamma}}{\mathrm{d}\gamma^2}\bigg|_{u(t)}\right\rangle \mathrm{e}^{\mathrm{i}\int_0^t \lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t.
\end{aligned}$$

We call this decomposition $S^1_k = \mathcal{E}_k + \mathcal{F}_k + \mathcal{G}_k + \mathcal{H}_k$. Thanks to (A.12) and the Cauchy–Schwarz inequality, we have:

$$|\mathcal{E}_k| \leqslant \|w\|_{H^1}\frac{C}{k^2}\left(\int_0^T \left|\left\langle f(t), \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(t)}\right\rangle\right|^2\right)^{1/2},$$

$$|\mathcal{G}_k| \leqslant \|w\|_{L^2} \frac{C}{k^2} \left(\int_0^T \left| \left\langle \dot{f}(t), \frac{d\varphi_{k,\gamma}}{d\gamma} \Big|_{u(t)} \right\rangle \right|^2 \right)^{1/2},$$

$$|\mathcal{H}_k| \leqslant \|w\|_{L^2} \frac{C}{k^2} \left(\int_0^T \left| \dot{u}(t) \left\langle f(t), \frac{d^2\varphi_{k,\gamma}}{d\gamma^2} \Big|_{u(t)} \right\rangle \right|^2 dt \right)^{1/2}.$$

Using Lemma 5 for $\mathcal{E}$, $\mathcal{G}$ and Lemma 6 for $\mathcal{H}$, we get:

$$\|\mathcal{E}\|_{h^5} \leqslant C\|w\|_{H^1}\|f\|_{C^0([0,T],H^2)},$$
$$\|\mathcal{G}\|_{h^5} \leqslant C\|w\|_{L^2}\|f\|_{C^1([0,T],H^2)},$$
$$\|H\|_{h^5} \leqslant C\|w\|_{L^2}\|u\|_{H^1}\|f\|_{C^0([0,T],H^2)}.$$

Thanks to (A.12), (A.19) and (A.18), we have:

$$|\mathcal{F}_k| \leqslant \frac{C}{k^6}\|w\|_{L^2}\|u\|_{H^1}\|f\|_{C^0([0,T],L^2)},$$
$$\|\mathcal{F}\|_{h^5} \leqslant C\|w\|_{L^2}\|u\|_{H^1}\|f\|_{C^0([0,T],L^2)}. \quad \square$$

Proposition 21. *There exists a constant $C > 0$ such that, for every $u \in H^1((0,T),\mathbb{R})$ satisfying $\|u\|_{H^1} \leqslant \gamma^*$ and $\|u\|_{H^1} \leqslant \gamma^*$, for every $f \in C^1([0,T], H^2 \cap H_0^1(I,\mathbb{C}))$ and for every $w \in H_0^1((0,T),\mathbb{C})$, S^2 belongs to $h^5(\mathbb{N}^*,\mathbb{C})$ and*

$$\|S^2\|_{h^5} \leqslant C\big[\|w\|_{L^2}\|f\|_{C^1([0,T],H^2)} + \|w\|_{H^1}\|f\|_{C^0([0,T],H^2)}\big].$$

Proof. The proof is the same as the one of the previous proposition, using Lemmas 6, 7 instead of Lemmas 5, 6 and (A.21) instead of (A.18). □

Proposition 22. *There exists a constant $C > 0$ such that, for every $u \in H^2((0,T),\mathbb{R})$ satisfying $\|u\|_{H^1} \leqslant \gamma^*$ and $\|u\|_{L^\infty} \leqslant \gamma^*$, for every $f \in C^2([0,T], H^3 \cap H_0^1(I,\mathbb{C}))$ and for every $w \in H_0^2((0,T),\mathbb{R})$, S^0 belongs to $h^7(\mathbb{N}^*,\mathbb{C})$, and*

$$\begin{aligned}
\|S^0\|_{h^7} \leqslant C\big[&\|w\|_{L^2}\|f\|_{C^2([0,T],H^3)} + \|w\|_{H^1}\big(\|f\|_{C^1([0,T],H^3)} + \|u\|_{H^2}\|f\|_{C^0([0,T],H^2)}\big) \\
&+ \|w\|_{H^2}\|f\|_{C^0([0,T],H^3)}\big],
\end{aligned}$$

$$\begin{aligned}
(S^0)_{k\geqslant 2} = \Bigg(&\frac{-1}{\lambda_k^2}\int_0^T \big[\ddot{w}(t)\langle f(t), \varphi_{k,u(t)}\rangle + 2\dot{w}(t)\langle \dot{f}(t), \varphi_{k,u(t)}\rangle \\
&+ w(t)\langle \ddot{f}(t), \varphi_{k,u(t)}\rangle\big] e^{i\int_0^t \lambda_{k,u(s)}\,ds}\,dt\Bigg)_{k\geqslant 2} \\
&+ \textit{terms with an } h^7\textit{-norm bounded by } C\big[\|w\|_{L^2}\|u\|_{H^1}\|f\|_{C^2([0,T],L^2)}
\end{aligned}$$

$$+\|w\|_{H^1}\big[\|u\|_{H^1}\|f\|_{C^1([0,T],H^2)}+\|u\|_{H^2}\|f\|_{C^0([0,T],H^2)}\big]$$
$$+\|w\|_{H^2}\|u\|_{H^1}\|f\|_{C^0([0,T],L^2)}\big].$$

Proof. We use the same decomposition as in the proof of Proposition 19. Using,

$$\frac{1}{\lambda_{k,u}}=\left(\frac{1}{\lambda_{k,u}}-\frac{1}{\lambda_k}\right)+\frac{1}{\lambda_k},$$

we get the decompositions $\mathcal{A}=\mathcal{A}^{(1)}+\mathcal{A}^{(2)}$, $\mathcal{C}=\mathcal{C}^{(1)}+\mathcal{C}^{(2)}$, $\mathcal{D}=\mathcal{D}^{(1)}+\mathcal{D}^{(2)}$. Using (A.13) and (A.12), we get:

$$\big|\mathcal{A}_k^{(1)}\big|\leqslant\frac{C}{k^7}\int_0^T\big|u(t)^2\dot{w}(t)\big\langle A_{u(t)}f(t),\varphi_{k,u(t)}\big\rangle\big|\,\mathrm{d}t.$$

Thanks to the Cauchy–Schwarz inequality and $\|u\|_{L^\infty}\leqslant\gamma^*$, we get:

$$\big|\mathcal{A}_k^{(1)}\big|\leqslant\frac{C}{k^7}\|u\|_{H^1}\|\dot{w}\|_{L^2}\left(\int_0^T\big|\big\langle A_{u(t)}f(t),\varphi_{k,u(t)}\big\rangle\big|^2\,\mathrm{d}t\right)^{1/2}.$$

We conclude using the orthonormality of the family $(\varphi_{k,u})_{k\in\mathbb{N}^*}$. We study $\mathcal{C}^{(1)}$ with the same arguments. Finally, we get:

$$\big\|\mathcal{A}^{(1)}\big\|_{h^7}\leqslant C\|\dot{w}\|_{L^2}\|u\|_{H^1}\|f\|_{C^0([0,T],H^2)},$$
$$\big\|\mathcal{C}^{(1)}\big\|_{h^7}\leqslant C\|w\|_{L^2}\|u\|_{H^1}\|f\|_{C^1([0,T],H^2)}.$$

Using (A.13) and the Cauchy–Schwarz inequality, we get:

$$\big|\mathcal{D}_k^{(1)}\big|\leqslant\frac{C}{k^5}\|u\|_{H^1}\|w\|_{L^2}\left(\int_0^T\left|\left\langle f(t),\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(t)}\right\rangle\right|^2\mathrm{d}t\right)^{1/2}.$$

We conclude thanks to Lemma 5 that

$$\big\|\mathcal{D}^{(1)}\big\|_{h^7}\leqslant C\|w\|_{L^2}\|u\|_{H^1}\|f\|_{C^0([0,T],H^2)}.$$

We use Proposition 19 for $\mathcal{A}^{(2)}$, $\mathcal{B}^{(2)}$ and Proposition 20 for $\mathcal{D}^{(2)}$ and we get:

$$\big\|\mathcal{A}^{(2)}\big\|_{h^7}\leqslant C\big[\|w\|_{H^1}\|f\|_{C^1([0,T],H^3)}+\|w\|_{H^2}\|f\|_{C^0([0,T],H^3)}\big],$$
$$\big\|\mathcal{C}^{(2)}\big\|_{h^7}\leqslant C\big[\|w\|_{L^2}\|f\|_{C^2([0,T],H^3)}+\|w\|_{H^1}\|f\|_{C^1([0,T],H^3)}\big],$$
$$\big\|\mathcal{D}^{(2)}\big\|_{h^7}\leqslant C\big[\|w\|_{H^1}\big(\|u\|_{H^1}\|f\|_{C^1([0,T],H^2)}+\|u\|_{H^2}\|f\|_{C^0([0,T],H^2)}\big)\big].$$

Using (A.12), (A.19) and the Cauchy–Schwarz inequality, we get:

$$|\mathcal{B}_k| \leqslant \frac{C}{k^7}\|w\|_{L^2}\left(\int_0^T \left|\dot{u}(t)\right|^2 \left|\langle A_{u(t)} f(t), \varphi_{k,u(t)}\rangle\right|^2 \mathrm{d}t\right)^{1/2}.$$

We conclude thanks to the orthonormality of $(\varphi_{k,u})_{k\in\mathbb{N}^*}$ that

$$\|\mathcal{B}\|_{h^7} \leqslant C\|w\|_{L^2}\|\dot{u}\|_{L^2}\|f\|_{C^0([0,T],H^2)}.$$

In order to get the second result of this proposition, we apply the second part of Proposition 19 on $\mathcal{A}^{(2)}$ and $\mathcal{C}^{(2)}$. □

Proposition 23. *There exists a constant $C > 0$ such that, for every $u \in H^2((0,T),\mathbb{R})$ satisfying $\|u\|_{H^1} \leqslant \gamma^*$ and $\|u\|_{L^\infty} \leqslant \gamma^*$, for every $f \in C^2([0,T], H^2 \cap H^1_0(I,\mathbb{C}))$ and for every $w \in H^2_0((0,T),\mathbb{R})$, S^1 belongs to $h^7(\mathbb{N}^*,\mathbb{C})$ and*

$$\begin{aligned}\left\|S^1\right\|_{h^7} \leqslant C\big[&\|w\|_{L^2}\|f\|_{C^2([0,T],H^2)} + \|w\|_{H^1}\big(\|u\|_{H^2}\|f\|_{C^0([0,T],H^2)} + \|f\|_{C^1([0,T],H^2)}\big)\\ &+ \|w\|_{H^2}\|f\|_{C^0([0,T],H^2)}\big].\end{aligned}$$

Proof. We use the same decomposition as in the proof of Proposition 20. Using,

$$\frac{1}{\lambda_{k,u}} = \left(\frac{1}{\lambda_{k,u}} - \frac{1}{\lambda_k}\right) + \frac{1}{\lambda_k},$$

we get the decompositions $\mathcal{E} = \mathcal{E}^{(1)} + \mathcal{E}^{(2)}$, $\mathcal{G} = \mathcal{G}^{(1)} + \mathcal{G}^{(2)}$, $\mathcal{H} = \mathcal{H}^{(1)} + \mathcal{H}^{(2)}$. Thanks to (A.11), we get:

$$\left|\mathcal{E}_k^{(1)}\right| \leqslant \int_0^T \frac{C}{k^5}\left|u(t)^2\dot{w}(t)\left\langle f(t), \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(t)}\right\rangle\right|\mathrm{d}t.$$

Using the Cauchy–Schwarz inequality and $\|u\|_{L^\infty} \leqslant \gamma^*$, we get:

$$\left|\mathcal{E}_k^{(1)}\right| \leqslant \frac{C}{k^5}\|u\|_{H^1}\|\dot{w}\|_{L^2}\left(\int_0^T \left|\left\langle f(t), \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(t)}\right\rangle\right|^2 \mathrm{d}t\right)^{1/2}.$$

We conclude thanks to Lemma 5. We study $\mathcal{G}^{(1)}$ with the same arguments. Finally, we get:

$$\left\|\mathcal{E}^{(1)}\right\|_{h^7} \leqslant C\|w\|_{H^1}\|u\|_{H^1}\|f\|_{C^0([0,T],H^2)},$$
$$\left\|\mathcal{G}^{(1)}\right\|_{h^7} \leqslant C\|w\|_{L^2}\|u\|_{H^1}\|f\|_{C^1([0,T],H^2)}.$$

Thanks to (A.11) and the Cauchy–Schwarz inequality, we get:

$$\left|\mathcal{H}_k^{(1)}\right| \leqslant \frac{C}{k^5}\|w\|_{L^2}\left(\int_0^T \left|\dot{u}(t)\left\langle f(t), \frac{\mathrm{d}^2\varphi_{k,\gamma}}{\mathrm{d}\gamma^2}\bigg|_{u(t)}\right\rangle\right|^2 \mathrm{d}t\right)^{1/2}.$$

We conclude thanks to Lemma 6 that

$$\left\|\mathcal{H}^{(1)}\right\|_{h^7} \leqslant C\|w\|_{L^2}\|u\|_{H^1}\|f\|_{C^0([0,T],H^2)}.$$

Applying Proposition 20 for $\mathcal{E}^{(2)}$, $\mathcal{G}^{(2)}$ and Proposition 21 for $\mathcal{H}^{(2)}$, we get:

$$\begin{aligned}
\left\|\mathcal{E}^{(2)}\right\|_{h^7} &\leqslant C\left[\|w\|_{H^1}\|f\|_{C^1([0,T],H^2)} + \|w\|_{H^2}\|f\|_{C^0([0,T],H^2)}\right],\\
\left\|\mathcal{G}^{(2)}\right\|_{h^7} &\leqslant C\left[\|w\|_{L^2}\|f\|_{C^2([0,T],H^2)} + \|w\|_{H^1}\|f\|_{C^1([0,T],H^2)}\right],\\
\left\|\mathcal{H}^{(2)}\right\|_{h^7} &\leqslant C\|w\|_{H^1}\left[\|u\|_{H^1}\|f\|_{C^1([0,T],H^2)} + \|u\|_{H^2}\|f\|_{C^0([0,T],H^2)}\right].
\end{aligned}$$

Thanks to (A.12), (A.19) and the Cauchy–Schwarz inequality, we get:

$$|\mathcal{F}_k| \leqslant \frac{C}{k^5}\|u\|_{H^1}\left(\int_0^T \left|w(t)\left\langle f(t), \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(t)}\right\rangle\right|^2 \mathrm{d}t\right)^{1/2}.$$

Using Lemma 5, we conclude that

$$\|\mathcal{F}\|_{h^7} \leqslant C\|w\|_{L^2}\|u\|_{H^1}\|f\|_{C^0([0,T],H^2)}. \quad \square$$

Proposition 24. *There exists a constant $C > 0$ such that, for every $u \in H^3((0,T),\mathbb{R})$ satisfying $\|u\|_{H^1} \leqslant \gamma^*$ and $\|u\|_{L^\infty} \leqslant \gamma^*$, for every $f \in C^3([0,T], H^3 \cap H_0^1(I,\mathbb{C}))$ and for every $w \in H_0^3((0,T),\mathbb{C})$, S^0 belongs to $h^9(\mathbb{N}^*,\mathbb{C})$, and*

$$\begin{aligned}
\left\|S^0\right\|_{h^9} \leqslant C\Big\{&\|w\|_{L^2}\|f\|_{C^3([0,T],H^3)} + \|w\|_{H^1}\big[\|f\|_{C^2([0,T],H^3)}\\
&+ \|u\|_{H^2}\|f\|_{C^1([0,T],H^2)} + \|u\|_{H^3}\|f\|_{C^0([0,T],H^2)}\big]\\
&+ \|w\|_{H^2}\big[\|f\|_{C^1([0,T],H^3)} + \|u\|_{H^2}\|f\|_{C^0([0,T],H^2)}\big] + \|w\|_{H^3}\|f\|_{C^0([0,T],H^3)}\Big\},
\end{aligned}$$

$$\begin{aligned}
\left(S^0\right)_{k\geqslant 2} = \Bigg(&\frac{-1}{\lambda_k^2}\int_0^T \bigg[\frac{\mathrm{d}^3 w}{\mathrm{d}t^3}(t)\left\langle f(t), \varphi_{k,u(t)}\right\rangle + 3\ddot{w}(t)\left\langle \dot{f}(t), \varphi_{k,u(t)}\right\rangle\\
&+ 3\dot{w}(t)\left\langle \ddot{f}(t), \varphi_{k,u(t)}\right\rangle + w(t)\left\langle \frac{\partial^3 f}{\partial t^3}(t), \varphi_{k,u(t)}\right\rangle\bigg]\mathrm{e}^{\mathrm{i}\int_0^t \lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t\Bigg)_{k\geqslant 2}\\
&+ \textit{terms with an } h^9\textit{-norm bounded by}
\end{aligned}$$

$$C\big\{\|w\|_{L^2}\|u\|_{H^1}\|f\|_{C^3([0,T],L^2)}+\|w\|_{H^1}\big[\|u\|_{H^1}\|f\|_{C^2([0,T],H^2)}$$
$$+\|u\|_{H^2}\|f\|_{C^1([0,T],H^2)}+\|u\|_{H^3}\|f\|_{C^0([0,T],H^2)}\big]$$
$$+\|w\|_{H^2}\big[\|u\|_{H^1}\|f\|_{C^1([0,T],H^2)}+\|u\|_{H^2}\|f\|_{C^0([0,T],H^2)}\big]$$
$$+\|w\|_{H^3}\|u\|_{H^1}\|f\|_{C^0([0,T],L^2)}\big\}.$$

Proof. We use the decomposition $S^0=\mathcal{A}+\mathcal{B}+\mathcal{C}+\mathcal{D}$ introduced in the proof of Proposition 19. We have:

$$\begin{aligned}\mathcal{A}_k={}&-\int_0^T\frac{1}{\lambda_{k,u(t)}^2}\ddot{w}(t)\big\langle f(t),\varphi_{k,u(t)}\big\rangle e^{i\int_0^t\lambda_{k,u(s)}\,ds}\,dt\\&+\int_0^T\frac{2}{\lambda_{k,u(t)}^3}\dot{w}(t)\dot{u}(t)\lambda'_{k,u(t)}\big\langle f(t),\varphi_{k,u(t)}\big\rangle e^{i\int_0^t\lambda_{k,u(s)}\,ds}\,dt\\&-\int_0^T\frac{1}{\lambda_{k,u(t)}^2}\dot{w}(t)\big\langle \dot{f}(t),\varphi_{k,u(t)}\big\rangle e^{i\int_0^t\lambda_{k,u(s)}\,ds}\,dt\\&-\int_0^T\frac{1}{\lambda_{k,u(t)}^2}\dot{w}(t)\dot{u}(t)\left\langle f(t),\frac{d\varphi_{k,\gamma}}{d\gamma}\bigg|_{u(t)}\right\rangle e^{i\int_0^t\lambda_{k,u(s)}\,ds}\,dt.\end{aligned}$$

We call this decomposition $\mathcal{A}=\mathcal{A}^a+\mathcal{A}^b+\mathcal{A}^c+\mathcal{A}^d$. Using,

$$\frac{1}{\lambda_{k,u}^2}=\left(\frac{1}{\lambda_{k,u}^2}-\frac{1}{\lambda_k^2}\right)+\frac{1}{\lambda_k^2},$$

we get the decompositions: $\mathcal{A}^a=\mathcal{A}^{a,1}+\mathcal{A}^{a,2}$, $\mathcal{A}^c=\mathcal{A}^{c,1}+\mathcal{A}^{c,2}$, $\mathcal{A}^d=\mathcal{A}^{d,1}+\mathcal{A}^{d,2}$. We use:

$$\left|\frac{1}{\lambda_{k,u}^2}-\frac{1}{\lambda_k^2}\right|\leqslant\frac{Cu^2}{k^7},$$

which is a consequence of (A.11) and (A.12), and the same kind of arguments as in the previous proof. We get:

$$\begin{aligned}\big\|\mathcal{A}^{a,1}\big\|_{h^9}&\leqslant C\|w\|_{H^2}\|u\|_{H^1}\|f\|_{C^0([0,T],H^2)},\\\big\|\mathcal{A}^{c,1}\big\|_{h^9}&\leqslant C\|w\|_{H^1}\|u\|_{H^1}\|f\|_{C^1([0,T],H^2)},\\\big\|\mathcal{A}^{d,1}\big\|_{h^9}&\leqslant C\|w\|_{H^1}\|u\|_{H^1}\|f\|_{C^0([0,T],H^2)}.\end{aligned}$$

We apply Proposition 19 for $\mathcal{A}^{a,2}$, $\mathcal{C}^{c,2}$ and Proposition 20 for $\mathcal{A}^{d,2}$, we get:

$$\begin{aligned}\|\mathcal{A}^{a,2}\|_{h^9} &\leqslant C\big[\|w\|_{H^2}\|f\|_{C^1([0,T],H^3)}+\|w\|_{H^3}\|f\|_{C^0([0,T],H^3)}\big],\\ \|\mathcal{A}^{c,2}\|_{h^9} &\leqslant C\big[\|w\|_{H^1}\|f\|_{C^2([0,T],H^3)}+\|w\|_{H^2}\|f\|_{C^1([0,T],H^3)}\big],\\ \|\mathcal{A}^{d,2}\|_{h^9} &\leqslant C\|w\|_{H^2}\big[\|u\|_{H^1}\|f\|_{C^1([0,T],H^2)}+\|u\|_{H^2}\|f\|_{C^0([0,T],H^2)}\big].\end{aligned}$$

We have:

$$\|\mathcal{A}^b\|_{h^9}\leqslant C\|w\|_{H^1}\|u\|_{H^1}\|f\|_{C^0([0,T],H^2)},$$

therefore

$$\begin{aligned}\|\mathcal{A}\|_{h^9}\leqslant C\big\{&\|w\|_{H^3}\|f\|_{C^0([0,T],H^3)}+\|w\|_{H^2}\big[\|f\|_{C^1([0,T],H^3)}+\|u\|_{H^2}\|f\|_{C^0([0,T],H^2)}\big]\\ &+\|w\|_{H^1}\|f\|_{C^2([0,T],H^3)}\big\}.\end{aligned}$$

In the same way, we get:

$$\begin{aligned}\|\mathcal{C}\|_{h^9}\leqslant C\big\{&\|w\|_{H^2}\|f\|_{C^1([0,T],H^3)}+\|w\|_{H^1}\big[\|f\|_{C^2([0,T],H^3)}+\|u\|_{H^2}\|f\|_{C^1([0,T],H^2)}\big]\\ &+\|w\|_{L^2}\|f\|_{C^3([0,T],H^3)}\big\},\\ \|\mathcal{D}\|_{h^9}\leqslant C\big\{&\|w\|_{H^2}\|u\|_{H^2}\|f\|_{C^0([0,T],H^2)}+\|w\|_{H^1}\big[\|u\|_{H^1}\|f\|_{C^2([0,T],H^2)}\\ &+\|u\|_{H^2}\|f\|_{C^1([0,T],H^2)}+\|u\|_{H^3}\|f\|_{C^0([0,T],H^2)}\big]\big\},\\ \|\mathcal{B}\|_{h^9}\leqslant C\big\{&\|w\|_{H^1}\|u\|_{H^2}\|f\|_{C^0([0,T],H^2)}+\|w\|_{L^2}\|u\|_{H^1}\|f\|_{C^1([0,T],H^2)}\big\}.\end{aligned}$$

For the second part of the proposition, we apply the second part of Proposition 19 to $\mathcal{A}^{(a,2)}$, $\mathcal{A}^{(c,2)}$, $\mathcal{C}^{(a,2)}$ and $\mathcal{C}^{(c,2)}$. □

Proposition 25. *There exists a constant $C>0$ such that, for every $u\in H^3((0,T),\mathbb{R})$ satisfying $\|u\|_{H^1}\leqslant\gamma^*$ and $\|u\|_{L^\infty}\leqslant\gamma^*$, for every $f\in C^3([0,T],H^2\cap H^1_0(I,\mathbb{C}))$ and for every $w\in H^3_0((0,T),\mathbb{C})$, S^1 belongs to $h^9(\mathbb{N}^*,\mathbb{C})$ and*

$$\begin{aligned}\|S^1\|_{h^9}\leqslant C\big\{&\|w\|_{L^2}\|f\|_{C^3([0,T],H^2)}+\|w\|_{H^1}\big[\|f\|_{C^2([0,T],H^2)}+\|u\|_{H^2}\|f\|_{C^1([0,T],H^2)}\big]\\ &+\|w\|_{H^2}\big[\|f\|_{C^1([0,T],H^2)}+\|u\|_{H^2}\|f\|_{C^0([0,T],H^2)}\big]+\|w\|_{H^3}\|f\|_{C^0([0,T],H^2)}\big\}.\end{aligned}$$

Proof. The strategy is the same as in the proof of the previous proposition. □

3.6.3. Study of $(M_{(\psi_0,u)} - M_{(\varphi_{1,\gamma},\gamma)})(w)$

In this subsection, we get the bounds assumed in Proposition 15. Let $\gamma \in (0,\gamma^*)$ and $T = 4/\pi$. For every $(\psi_0, v) \in E_3^\gamma$, we introduce the quantities:

$$\Delta_3 := \gamma + \delta_3 \quad \text{where } \delta_3 := \big\|(\psi_0, v) - (\varphi_{1,\gamma}, 0)\big\|_{E_3^0}.$$

For every $(\psi_0, v) \in E_9^\gamma$, we introduce the quantities:

$$\Delta_5 := \gamma + \delta_5, \qquad \Delta_7 := \gamma + \delta_7 + \delta_5^2, \qquad \Delta_9 := \gamma + \delta_9 + \delta_7\delta_5 + \delta_5^3,$$

where $\delta_{\mathrm{i}} := \|(\psi_0, u) - (\varphi_{1,\gamma},\gamma)\|_{E_{\mathrm{i}}^0}$, for $i = 5, 7, 9$. We should write $\Delta_{\mathrm{i}}(\psi_0, v)$ and $\delta_{\mathrm{i}}(\psi_0, v)$ because these quantities depend on (ψ_0, v). In order to simplify the notations, we will write Δ_{i} and δ_{i}. There is no confusion possible. Let $\mathcal{V}$ be the E_3^γ-neighbourhood of $(\varphi_{1,\gamma}, 0)$ defined by:

$$\mathcal{V} := \Big\{(\psi_0, v) \in E_3^\gamma;\ \Delta_3 \leqslant 1/4,\ \|u\|_{H^1} \leqslant \gamma^*,\ \|u\|_{L^\infty} \leqslant \gamma^*,\ \|u\|_{L^1} < \sqrt{2}/\sqrt{17} \\ \text{where } u := \gamma + v\Big\}.$$

In this subsection, we prove there exists a constant C_1 such that, for every $(\psi_0, v) \in E_9^\gamma \cap \mathcal{V}$, for every $w \in H_0^3((0,T),\mathbb{R})$, we have:

$$\begin{aligned}
&\big\|(M_{(\psi_0,u)} - M_{(\varphi_{1,\gamma},\gamma)})(w)\big\|_{h^3} \leqslant C_1 \Delta_3 \|w\|_{L^2},\\
&\big\|(M_{(\psi_0,u)} - M_{(\varphi_{1,\gamma},\gamma)})(w)\big\|_{h^5} \leqslant C_1 \big[\Delta_3 \|w\|_{H_0^1} + \Delta_5 \|w\|_{L^2}\big],\\
&\big\|(M_{(\psi_0,u)} - M_{(\varphi_{1,\gamma},\gamma)})(w)\big\|_{h^7} \leqslant C_1 \big[\Delta_3 \|w\|_{H_0^2} + \Delta_5 \|w\|_{H_0^1} + \Delta_7 \|w\|_{L^2}\big],\\
&\big\|(M_{(\psi_0,u)} - M_{(\varphi_{1,\gamma},\gamma)})(w)\big\|_{h^9} \leqslant C_1 \big[\Delta_3 \|w\|_{H_0^3} + \Delta_5 \|w\|_{H_0^2} + \Delta_7 \|w\|_{H_0^1} + \Delta_9 \|w\|_{L^2}\big].
\end{aligned} \tag{3.53}$$

In the next propositions, we deal with each term in $M_{(\psi_0,u)}$ one by one.

Proposition 26. *There exists a constant* $C > 0$ *such that, for every* $(\psi_0, v) \in E_9^\gamma \cap \mathcal{V}$*, for every* $w \in L^2((0,T),\mathbb{C})$*,*

$$\big|(M_{(\psi_0,u)} - M_{(\varphi_{1,\gamma},\gamma)})(w)_1\big| \leqslant C \Delta_3 \|w\|_{L^2}.$$

Proof. We have:

$$(M_{(\psi_0,u)} - M_{(\varphi_{1,\gamma},\gamma)})(w)_1 = \int_0^T \mathrm{i} w(t)\big(\langle q\Lambda(t), \psi(t)\rangle + \langle q\psi_{1,\gamma}, \Lambda(t)\rangle\big)\,\mathrm{d}t,$$

$$\big|(M_{(\psi_0,u)} - M_{(\varphi_{1,\gamma},\gamma)})(w)_1\big| \leqslant 2\sqrt{T}\|\Lambda\|_{C^0([0,T],L^2)}\|w\|_{L^2((0,T),\mathbb{R})},$$

where $\Lambda := \psi - \psi_{1,\gamma}$. The function Λ solves:

$$\begin{cases} \mathrm{i}\dot{\Lambda} = -\frac{1}{2}\Lambda'' - u(t)q\Lambda - (u-\gamma)q\psi_{1,\gamma}, \\ \Lambda(0) = \psi_0 - \varphi_{1,\gamma}, \\ \Lambda(t,-\frac{1}{2}) = \Lambda(t,\frac{1}{2}) = 0, \end{cases}$$

so, using Proposition 45 in Appendix B,

$$\|\Lambda\|_{C^0([0,T],L^2)} \leqslant C\big[\|\psi_0 - \varphi_{1,\gamma}\|_{L^2} + \|u-\gamma\|_{L^2}\big] \leqslant C\Delta_3. \quad \square$$

Proposition 27. *There exists a constant $C > 0$ such that, for every $(\psi_0, v) \in E_9^\gamma \cap \mathcal{V}$, for every $w \in H_0^3((0,T),\mathbb{R})$, the sequence $X(w) = (X_k(w))_{k\geqslant 2}$ defined by,*

$$X_k(w) := \int_0^T w(t)\big[\langle q\psi(t), \varphi_{k,u(t)}\rangle \mathrm{e}^{\mathrm{i}\int_0^t \lambda_{k,u(s)}\,\mathrm{d}s} - \langle q\psi_{1,\gamma}(t), \varphi_{k,\gamma}\rangle \mathrm{e}^{\mathrm{i}\lambda_{k,\gamma}t}\big]\,\mathrm{d}t,$$

belongs to h^9 and satisfies:

$$\begin{gathered} \big\|X(w)\big\|_{h^3} \leqslant C\Delta_3\|w\|_{L^2}, \\ \big\|X(w)\big\|_{h^5} \leqslant C\big[\Delta_3\|w\|_{H_0^1} + \Delta_5\|w\|_{L^2}\big], \\ \big\|X(w)\big\|_{h^7} \leqslant C\big[\Delta_3\|w\|_{H_0^2} + \Delta_5\|w\|_{H_0^1} + \Delta_7\|w\|_{L^2}\big], \\ \big\|X(w)\big\|_{h^9} \leqslant C\big[\Delta_3\|w\|_{H_0^3} + \Delta_5\|w\|_{H_0^2} + \Delta_7\|w\|_{H_0^1} + \Delta_9\|w\|_{L^2}\big]. \end{gathered}$$

Proof. First, we prove:

$$\begin{aligned} \int_0^T w(t)\langle q\psi(t), \varphi_{k,u(t)}\rangle \mathrm{e}^{\mathrm{i}\int_0^t \lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t &= \frac{1}{\lambda_k}\int_0^T w(t)\langle A_{u(t)}\big(q\psi(t)\big), \varphi_k\rangle \mathrm{e}^{\mathrm{i}\lambda_k t}\,\mathrm{d}t \\ &\quad + \text{terms with an } h^3\text{-norm bounded by } C\Delta_3\|w\|_{L^2}. \end{aligned} \tag{3.54}$$

For this, we use the following decomposition:

$$\begin{aligned} &\int_0^T w(t)\langle q\psi(t), \varphi_{k,u(t)}\rangle \mathrm{e}^{\mathrm{i}\int_0^t \lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t \\ &\quad = \int_0^T \frac{w(t)}{\lambda_{k,u(t)}}\langle A_{u(t)}\big(q\psi(t)\big), \varphi_{k,u(t)}\rangle\big(\mathrm{e}^{\mathrm{i}\int_0^t \lambda_{k,u(s)}\,\mathrm{d}s} - \mathrm{e}^{\mathrm{i}\lambda_k t}\big)\,\mathrm{d}t \end{aligned}$$

$$+\int_0^T \frac{w(t)}{\lambda_{k,u(t)}}\big\langle A_{u(t)}\big(q\psi(t)\big), \varphi_{k,u(t)}-\tilde{\varphi}_{k,u(t)}\big\rangle e^{i\lambda_k t}\,dt$$
$$+\int_0^T \frac{w(t)}{\lambda_{k,u(t)}}u(t)\left\langle A_{u(t)}\big(q\psi(t)\big), \frac{d\varphi_{k,\gamma}}{d\gamma}\bigg|_0\right\rangle e^{i\lambda_k t}\,dt$$
$$+\int_0^T w(t)\left(\frac{1}{\lambda_{k,u(t)}}-\frac{1}{\lambda_k}\right)\big\langle A_{u(t)}\big(q\psi(t)\big), \varphi_k\big\rangle e^{i\lambda_k t}\,dt$$
$$+\frac{1}{\lambda_k}\int_0^T w(t)\big\langle A_{u(t)}\big(q\psi(t)\big), \varphi_k\big\rangle e^{i\lambda_k t}\,dt \tag{3.55}$$

and we prove that the h^3-norms of the four first terms of the right-hand side of (3.55) are bounded by $C\Delta_3\|w\|_{L^2}$. In the first term of the right-hand side of (3.55), we use (A.12),

$$\left|e^{i\int_0^t \lambda_{k,u(s)}\,ds}-e^{i\lambda_k t}\right| \leqslant \int_0^t |\lambda_{k,u(s)}-\lambda_k|\,ds \leqslant \frac{C\|u\|_{L^2}^2}{k},$$

which is a consequence of (A.11) and the Cauchy–Schwarz inequality in $L^2((0,T),\mathbb{C})$. We get:

$$\left|\int_0^T \frac{w(t)}{\lambda_{k,u(t)}}\big\langle A_{u(t)}\big(q\psi(t)\big), \varphi_{k,u(t)}\big\rangle\big(e^{i\int_0^t \lambda_{k,u(s)}\,ds}-e^{i\lambda_k t}\big)\,dt\right|$$
$$\leqslant \frac{C}{k^3}\|u\|_{L^2}^2\|w\|_{L^2}\left(\int_0^T \left|\big\langle A_{u(t)}\big(q\psi(t)\big), \varphi_{k,u(t)}\big\rangle\right|^2 dt\right)^{1/2}.$$

Thanks to the orthonormality of the family $(\varphi_{k,u})_{k\in\mathbb{N}^*}$, we get the following bound on the h^3-norm of the first term of the right-hand side of (3.55):

$$\begin{aligned} &C\|w\|_{L^2}\|u\|_{L^2}^2\big\|A_{u(t)}\big(q\psi(t)\big)\big\|_{C^0([0,T],L^2(I,\mathbb{C}))} \\ &\quad\leqslant C\|w\|_{L^2}\big(\gamma^2+\|u-\gamma\|_{L^2}^2\big)\|\psi\|_{C^0([0,T],H^2(I,\mathbb{C}))} \\ &\quad\leqslant C\|w\|_{L^2}\Delta_3\big(\|\varphi_{1,\gamma}\|_{H^2}+\delta_3\big) \\ &\quad\leqslant C\Delta_3\|w\|_{L^2((0,T),\mathbb{R})}. \end{aligned}$$

Now, we deal with the second term of the right-hand side of (3.55). Using (A.12) and (A.8), we get:

$$\left|\int_0^T \frac{w(t)}{\lambda_{k,u(t)}}\langle A_{u(t)}\big(q\psi(t)\big), \varphi_{k,u(t)} - \tilde{\varphi}_{k,u(t)}\rangle e^{i\lambda_k t}\,dt\right|$$
$$\leqslant \frac{C}{k^4}\|u\|_{L^\infty}^2\|w\|_{L^2}\big\|A_{u(t)}\big(q\psi(t)\big)\big\|_{C^0([0,T],L^2)}.$$

This inequality gives the following bound on the h^3-norm of the second term of the right-hand side of (3.55):

$$C\|u\|_{H^1}^2\|w\|_{L^2}\|\psi\|_{C^0([0,T],H^2)} \leqslant C\Delta_3^2\|w\|_{L^2}\big(\|\varphi_{1,\gamma}\|_{L^2}+\delta_3\big) \leqslant C\Delta_3\|w\|_{L^2}.$$

For the third term of the right-hand side of (3.55), we use (A.12) and Cauchy–Schwarz inequality to get:

$$\left|\int_0^T \frac{w(t)}{\lambda_{k,u(t)}}u(t)\left\langle A_{u(t)}\big(q\psi(t)\big), \frac{d\varphi_{k,\gamma}}{d\gamma}\bigg|_0\right\rangle e^{i\lambda_k t}\,dt\right|$$
$$\leqslant \frac{C}{k^2}\|u\|_{L^2}\left(\int_0^T \left|w(t)\left\langle A_{u(t)}\big(q\psi(t)\big), \frac{d\varphi_{k,\gamma}}{d\gamma}\bigg|_0\right\rangle\right|^2 dt\right)^{1/2}.$$

Thanks to Lemma 2, we conclude the following bound on the h^3-norm of the third term of the right-hand side of (3.55):

$$C\|u\|_{L^2}\|w\|_{L^2}\|\psi_0\|_{H^2} \leqslant C\Delta_3\|w\|_{L^2}.$$

For the fourth term of the right-hand side of (3.55), we use (A.13) and we get:

$$\left|\int_0^T w(t)\left(\frac{1}{\lambda_{k,u(t)}} - \frac{1}{\lambda_k}\right)\langle A_{u(t)}\big(q\psi(t)\big), \varphi_k\rangle e^{i\lambda_k t}\,dt\right|$$
$$\leqslant \frac{C}{k^5}\|u\|_{L^\infty}^2\|w\|_{L^2}\|\psi_0\|_{H^2}.$$

This leads to the following bound on the h^3-norm of the fourth term of the right-hand side of (3.55):

$$C\|u\|_{H^1}\|w\|_{L^2}\|\psi_0\|_{H^2} \leqslant C\delta_3\|w\|_{L^2}.$$

This ends the proof of (3.54).

We have:

$$\int_0^T w(t)\big[\langle q\psi(t),\varphi_{k,u(t)}\rangle e^{i\int_0^t \lambda_{k,u(s)}\,ds} - \langle q\psi_{1,\gamma}(t),\varphi_{k,\gamma}\rangle e^{i\lambda_{k,\gamma}t}\big]\,dt$$
$$= \frac{1}{\lambda_k}\int_0^T w(t)\big\langle A_{u(t)}\big(q\psi(t)\big) - A_\gamma\big(q\psi_{1,\gamma}(t)\big),\varphi_k\big\rangle e^{i\lambda_k t}\,dt$$
$$+ \text{terms with an } h^3\text{-norm bounded by } C\Delta_3\|w\|_{L^2}.$$

We define $f(t) := A_{u(t)}(q\psi(t)) - A_\gamma(q\psi_{1,\gamma}(t))$. We have:

$$\frac{1}{\lambda_k}\int_0^T w(t)\langle f(t),\varphi_k\rangle e^{i\lambda_k t}\,dt$$
$$= -\frac{2(-1)^{[(k+1)/2]}}{(k\pi)^3}\int_0^T w(t)\big[f(t,1/2) - (-1)^k f(t,-1/2)\big]e^{i\lambda_k t}\,dt$$
$$+ \frac{2}{(k\pi)^3}\int_0^T w(t)\left\langle f(t)', \frac{1}{k\pi}\varphi_k\right\rangle e^{i\lambda_k t}\,dt.$$

Thanks to Lemma 8 and the orthonormality of the family $((1/k\pi)\varphi_k')_{k\in\mathbb{N}^*}$, this term is also dominated in h^3 by:

$$\|w\|_{L^2}\|f\|_{C^0([0,T],H^1)} \leqslant \|w\|_{L^2}\Delta_3.$$

This ends the proof of $\|X(w)\|_{h^3} \leqslant C\Delta_3\|w\|_{L^2([0,T],\mathbb{R})}$.

Now, we study $X(w)$ in h^5. Using Proposition 19 and similar arguments, we get:

$$\int_0^T w(t)\langle q\psi(t),\varphi_{k,u(t)}\rangle e^{i\int_0^t \lambda_{k,u(s)}\,ds}\,dt$$
$$= -\int_0^T \frac{1}{i\lambda_k}\big(\dot{w}(t)\langle q\psi(t),\varphi_k\rangle + w(t)\langle q\dot{\psi}(t),\varphi_k\rangle\big)e^{i\lambda_k t}\,dt$$
$$+ \text{terms with an } h^5\text{-norm bounded by } C\big[\Delta_3\|w\|_{H^1} + \Delta_5\|w\|_{L^2}\big].$$

We conclude with the same strategy as in h^3.

For the study of $X(w)$ in h^7, we use the following consequence of Proposition 22:

$$\int_0^T w(t)\langle q\psi(t),\varphi_{k,u(t)}\rangle e^{i\int_0^t \lambda_{k,u(s)}\,ds}\,dt$$
$$= -\int_0^T \frac{1}{\lambda_k^2}\big[\ddot{w}(t)\langle q\psi(t),\varphi_k\rangle + 2\dot{w}(t)\langle q\dot{\psi}(t),\varphi_k\rangle + w(t)\langle q\ddot{\psi}(t),\varphi_k\rangle\big]e^{i\lambda_k t}\,dt$$
$$+ \text{terms with an } h^7\text{-norm bounded by } C\big[\Delta_3\|w\|_{H^2} + \Delta_5\|w\|_{H^1} + \Delta_7\|w\|_{L^2}\big].$$

For the study in h^9, we use the following consequence of Proposition 24:

$$\int_0^T w(t)\langle q\psi(t),\varphi_{k,u(t)}\rangle e^{i\int_0^t \lambda_{k,u(s)}\,ds}\,dt$$
$$= i\int_0^T \frac{1}{\lambda_k^3}\bigg[\frac{d^3w}{dt^3}(t)\langle q\psi(t),\varphi_k\rangle + 3\ddot{w}(t)\langle q\dot{\psi}(t),\varphi_k\rangle$$
$$+ 3\dot{w}(t)\langle q\ddot{\psi}(t),\varphi_k\rangle + w(t)\Big\langle q\frac{\partial^3\psi}{\partial t^3}(t),\varphi_k\Big\rangle\bigg]e^{i\lambda_k t}\,dt$$
$$+ \text{terms with an } h^9\text{-norm bounded by}$$
$$C\big[\Delta_3\|w\|_{H^3} + \Delta_5\|w\|_{H^2} + \Delta_7\|w\|_{H^1} + \Delta_9\|w\|_{L^2}\big]. \quad \square$$

Proposition 28. *There exists a constant C such that, for every $(\psi_0, v) \in E_9^\gamma \cap \mathcal{V}$, for every $w \in H_0^3((0,T),\mathbb{R})$, the sequence $X(w) = (X_k(w))_{k\geqslant 2}$ defined by,*

$$X_k(w) := \int_0^T \dot{u}(t)\Big\langle \Psi_2(t), \frac{d\varphi_{k,\gamma}}{d\gamma}\Big|_{u(t)}\Big\rangle e^{i\int_0^t \lambda_{k,u(s)}\,ds}\,dt,$$

belongs to h^9 and satisfies:

$$\big\|X(w)\big\|_{h^3} \leqslant C\Delta_3\|w\|_{L^2},$$
$$\big\|X(w)\big\|_{h^5} \leqslant C\big[\Delta_3\|w\|_{H_0^1} + \Delta_5\|w\|_{L^2}\big],$$
$$\big\|X(w)\big\|_{h^7} \leqslant C\big[\Delta_3\|w\|_{H_0^2} + \Delta_5\|w\|_{H_0^1} + \Delta_7\|w\|_{L^2}\big],$$
$$\big\|X(w)\big\|_{h^9} \leqslant C\big[\Delta_3\|w\|_{H_0^3} + \Delta_5\|w\|_{H_0^2} + \Delta_7\|w\|_{H_0^1} + \Delta_9\|w\|_{L^2}\big].$$

Proof. We apply Propositions 17, 20, 23, 25 together with the following bounds:

$$\begin{aligned}
&\|\Psi_2\|_{C^0([0,T],H^2)} \leqslant C\|w\|_{L^2}\|\psi_0\|_{H^2},\\
&\|\Psi_2\|_{C^1([0,T],H^2)} \leqslant C\big[\|w\|_{H^1}\|\psi_0\|_{H^2}+\|w\|_{L^2}\|\psi_0\|_{H^4}\big],\\
&\|\Psi_2\|_{C^2([0,T],H^2)} \leqslant C\big[\|w\|_{H^2}\|\psi_0\|_{H^2}+\|w\|_{H^1}\|\psi_0\|_{H^4}+\|w\|_{L^2}\|\psi_0\|_{H^6}\big],\\
&\|\Psi_2\|_{C^3([0,T],H^2)} \leqslant C\big[\|w\|_{H^3}\|\psi_0\|_{H^2}+\|w\|_{H^2}\|\psi_0\|_{H^4}+\|w\|_{H^1}\|\psi_0\|_{H^6}\\
&\qquad +\|w\|_{L^2}\|\psi_0\|_{H^8}\big],
\end{aligned}$$

which are consequences of Propositions 45, 47, 49. We have the following bound on the h^3-norm of $X(w)$:

$$C\|u\|_{H^1}\|w\|_{L^2}\|\psi_0\|_{H^2} \leqslant C\Delta_3\|w\|_{L^2}(1+\delta_3) \leqslant C\Delta_3\|w\|_{L^2}.$$

We have the following bound on the h^5-norm of $X(w)$:

$$\begin{aligned}
&C\|u\|_{H^1}\big[\|w\|_{H^1}\|\psi_0\|_{H^2}+\|w\|_{L^2}\|\psi_0\|_{H^4}\big]+C\|u\|_{H^2}\|w\|_{L^2}\|\psi_0\|_{H^2}\\
&\quad\leqslant C\Delta_3\big[\|w\|_{H^1}(1+\delta_2)+\|w\|_{L^2}(1+\delta_5)\big]+C\Delta_5\|w\|_{L^2}(1+\delta_3).
\end{aligned}$$

We have the following bound on the h^7-norm of $X(w)$:

$$\begin{aligned}
&C\big[\|u\|_{H^1}\|\Psi_2\|_{C^2([0,T],H^2)}+\|u\|_{H^2}\|\Psi_2\|_{C^1([0,T],H^2)}+\big(\|u\|_{H^3}+\|u\|_{H^2}^2\big)\|\Psi_2\|_{C^0([0,T],H^2)}\big]\\
&\quad\leqslant C\big\{\Delta_3\big[\|w\|_{H^2}(1+\delta_2)+\|w\|_{H^1}(1+\delta_4)+\|w\|_{L^2}(1+\delta_6)\big]\\
&\qquad+\Delta_5\big[\|w\|_{H^1}(1+\delta_2)+\|w\|_{L^2}(1+\delta_4)\big]+\big[\Delta_7+\Delta_5^2\big]\|w\|_{L^2}(1+\delta_2)\big\}.
\end{aligned}$$

We have the following bound on the h^9-norm of $X(w)$:

$$\begin{aligned}
&C\big\{\|u\|_{H^1}\|\Psi_2\|_{C^3([0,T],H^2)}+\|u\|_{H^2}\big[\|\Psi_2\|_{C^2([0,T],H^2)}+\|u\|_{H^2}\|\Psi_2\|_{C^1([0,T],H^2)}\big]\\
&\qquad+\|u\|_{H^3}\big[\|\Psi_2\|_{C^1([0,T],H^2)}+\|u\|_{H^2}\|\Psi_2\|_{C^0([0,T],H^2)}\big]+\|u\|_{H^4}\|\Psi\|_{C^0([0,T],H^2)}\big\}\\
&\quad\leqslant C\big\{\Delta_3\big[\|w\|_{H^3}(1+\delta_2)+\|w\|_{H^2}(1+\delta_4)+\|w\|_{H^1}(1+\delta_6)+\|w\|_{L^2}(1+\delta_8)\big]\\
&\qquad+\Delta_5\big[\|w\|_{H^2}(1+\delta_2)+\|w\|_{H^1}(1+\delta_4)+\|w\|_{L^2}(1+\delta_6)\big]\\
&\qquad+\Delta_5^2\big[\|w\|_{H^1}(1+\delta_2)+\|w\|_{L^2}(1+\delta_4)\big]\\
&\qquad+\Delta_7\big[\|w\|_{H^1}(1+\delta_2)+\|w\|_{L^2}(1+\delta_4)+\Delta_5\|w\|_{L^2}(1+\delta_2)\big]+\Delta_9\|w\|_{L^2}(1+\delta_2)\big\}.
\end{aligned}$$

□

Proposition 29. *There exists a constant $C>0$ such that, for every $(\psi_0,v)\in E_9^\gamma\cap\mathcal{V}$, for every $w\in H_0^3((0,T),\mathbb{R})$, the sequence $X(w)=(X_k(w))_{k\geqslant 2}$ defined by,*

$$X_k(w) := \int_0^T \dot{u}(t)\frac{\langle\Psi_2(t),\varphi_{1,u(t)}\rangle}{\langle\psi(t),\varphi_{1,u(t)}\rangle}\left\langle\psi(t),\left.\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\right|_{u(t)}\right\rangle \mathrm{e}^{\mathrm{i}\int_0^t\lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t,$$

belongs to h^9 and satisfies:

$$\|X(w)\|_{h^3} \leqslant C\Delta_3\|w\|_{L^2},$$
$$\|X(w)\|_{h^5} \leqslant C\big[\Delta_3\|w\|_{H_0^1} + \Delta_5\|w\|_{L^2}\big],$$
$$\|X(w)\|_{h^7} \leqslant C\big[\Delta_3\|w\|_{H_0^2} + \Delta_5\|w\|_{H_0^1} + \Delta_7\|w\|_{L^2}\big],$$
$$\|X(w)\|_{h^9} \leqslant C\big[\Delta_3\|w\|_{H_0^3} + \Delta_5\|w\|_{H_0^2} + \Delta_7\|w\|_{H_0^1} + \Delta_9\|w\|_{L^2}\big].$$

Proof. We apply again Propositions 17, 20, 23 and 25, with

$$w \leftarrow \tilde{w} := \dot{u}(t)\frac{\langle \Psi_2(t), \varphi_{1,u(t)}\rangle}{\langle \psi(t), \varphi_{1,u(t)}\rangle} \quad \text{and} \quad f \leftarrow \psi.$$

Since $\Delta_3 \leqslant 1/4$ then $|\langle \psi(t), \varphi_{1,u(t)}\rangle| \geqslant 1/2$ for every t. Indeed,

$$\langle \psi(t), \varphi_{1,u(t)}\rangle = \langle \psi(t) - \psi_{1,\gamma}(t), \varphi_{1,u(t)}\rangle + \langle (\varphi_{1,\gamma} - \varphi_{1,u(t)})\mathrm{e}^{-\mathrm{i}\lambda_{1,\gamma}t}, \varphi_{1,u(t)}\rangle + \mathrm{e}^{-\mathrm{i}\lambda_{1,\gamma}t},$$
$$\big|\langle \psi(t), \varphi_{1,u(t)}\rangle\big| \geqslant 1 - \|\psi - \psi_{1,\gamma}\|_{C^0([0,T],L^2)} - |\gamma| - \big|u(t)\big| \geqslant 1 - 2\Delta_3.$$

Therefore, we have, thanks to Proposition 45,

$$\|\widetilde{w}\|_{L^2} \leqslant 2\|u\|_{H^1}\|\Psi_2\|_{C^0([0,T],L^2)} \leqslant C\|u\|_{H^1}\|w\|_{L^2}\|\psi_0\|_{L^2} \leqslant C\Delta_3\|w\|_{L^2}(1+\delta_2).$$

We compute $\dot{\widetilde{w}}$, $\ddot{\widetilde{w}}$, $\frac{\mathrm{d}^3\widetilde{w}}{\mathrm{d}t^3}$ and, we get in the same way:

$$\|\dot{\widetilde{w}}\|_{L^2} \leqslant C\big[\Delta_5\|w\|_{L^2} + \Delta_3\|w\|_{H^1}\big],$$
$$\|\ddot{\widetilde{w}}\|_{L^2} \leqslant C\big[\Delta_7\|w\|_{L^2} + \Delta_5\|w\|_{H^1} + \Delta_3\|w\|_{H^2}\big],$$
$$\left\|\frac{\mathrm{d}^3\widetilde{w}}{\mathrm{d}t^3}\right\|_{L^2} \leqslant C\big[\Delta_9\|w\|_{L^2} + \Delta_7\|w\|_{H^1} + \Delta_5\|w\|_{H^2} + \Delta_3\|w\|_{H^3}\big]. \quad \square$$

Proposition 30. *There exists a constant $C > 0$ such that, for every $(\psi_0, v) \in E_9^\gamma \cap \mathcal{V}$, for every $w \in H_0^3((0,T),\mathbb{R})$, the sequence $X(w) = (X_k(w))_{k\geqslant 2}$ defined by,*

$$X_k(w) := \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\langle \Psi_2(t), \varphi_{1,u(t)}\rangle}{\langle \psi(t), \varphi_{1,u(t)}\rangle}\right)\langle \psi(t), \varphi_{k,u(t)}\rangle \mathrm{e}^{\mathrm{i}\int_0^t \lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t,$$

belongs to h^9 and satisfies:

$$\|X(w)\|_{h^3} \leqslant C\Delta_3\|w\|_{L^2},$$
$$\|X(w)\|_{h^5} \leqslant C\big[\Delta_3\|w\|_{H_0^1} + \Delta_5\|w\|_{L^2}\big],$$
$$\|X(w)\|_{h^7} \leqslant C\big[\Delta_3\|w\|_{H_0^2} + \Delta_5\|w\|_{H_0^1} + \Delta_7\|w\|_{L^2}\big],$$
$$\|X(w)\|_{h^9} \leqslant C\big[\Delta_3\|w\|_{H_0^3} + \Delta_5\|w\|_{H_0^2} + \Delta_7\|w\|_{H_0^1} + \Delta_9\|w\|_{L^2}\big].$$

Proof. Let $\widetilde{w} = \frac{\mathrm{d}}{\mathrm{d}t}\big(\frac{\langle \Psi_2(t), \varphi_{1,u(t)}\rangle}{\langle \psi(t), \varphi_{1,u(t)}\rangle}\big)$. We have:

$$\widetilde{w} = \frac{\langle \dot{\Psi}_2, \varphi_{1,u}\rangle + \dot{u}\langle \Psi_2, \frac{\mathrm{d}\varphi_{1,\gamma}}{\mathrm{d}\gamma}|_u\rangle}{\langle \psi, \varphi_{1,u}\rangle} - \big\langle \Psi_2, \varphi_{1,u}\big\rangle \frac{\langle \dot{\psi}, \varphi_{1,u}\rangle + \dot{u}\langle \psi, \frac{\mathrm{d}\varphi_{1,\gamma}}{\mathrm{d}\gamma}|_u\rangle}{\langle \psi, \varphi_{1,u}\rangle^2},$$
$$\|\widetilde{w}\|_{L^2} \leqslant 2\|\dot{\Psi}_2\|_{L^2([0,T],L^2)} + 4\|\Psi_2\|_{C^0([0,T],L^2)}\big[\|\dot{u}\|_{L^2} + \|\psi_0\|_{H^2} + \|\dot{u}\|_{L^2}\|\psi_0\|_{L^2}\big],$$

so there exists a constant C such that $\|\widetilde{w}\|_{L^2} \leqslant C(1+\delta_3)\|w\|_{L^2}$. In the same way:

$$\|\widetilde{w}\|_{H^1} \leqslant C\big[(1+\delta_3)\|w\|_{H^1} + (1+\delta_5)\|w\|_{L^2}\big],$$
$$\|\widetilde{w}\|_{H^2} \leqslant C\big[(1+\delta_3)\|w\|_{H^2} + (1+\delta_5)\|w\|_{H^1} + (1+\delta_7)\|w\|_{L^2}\big],$$
$$\|\widetilde{w}\|_{H^3} \leqslant C\big[(1+\delta_3)\|w\|_{H^3} + (1+\delta_5)\|w\|_{H^2} + (1+\delta_7)\|w\|_{H^1} + (1+\delta_9)\|w\|_{L^2}\big].$$

In order to have a small factor in front of $\|w\|$ we use the following decomposition, for $k \geqslant 2$:

$$\big\langle \psi(t), \varphi_{k,u(t)}\big\rangle = \big\langle (\psi - \psi_{1,\gamma})(t), \varphi_{k,u(t)}\big\rangle + \big\langle \psi_{1,\gamma}(t), \varphi_{k,u(t)}\big\rangle,$$

which split the sequence $X(w)$ into two sequences:

$$X(w) = (J_k)_{k\geqslant 2} + (K_k)_{k\geqslant 2}.$$

We study $(J_k)_{k\geqslant 2}$ thanks to Propositions 16, 19, 22 and 24. The function $\Lambda := \psi - \psi_{1,\gamma}$ satisfies:

$$\begin{cases} \dot{\Lambda} = \frac{\mathrm{i}}{2}\Lambda'' + \mathrm{i}uq\Lambda + \mathrm{i}(u-\gamma)q\psi_{1,\gamma}, \\ \Lambda_0 = \psi_0 - \varphi_{1,\gamma} \\ \Lambda(t, -\frac{1}{2}) = \Lambda(t, \frac{1}{2}) = 0. \end{cases}$$

Therefore, using Propositions 45, 47, 49 and 51, we get:

$$\|\Lambda\|_{C^k([0,T],H^s)} \leqslant C\delta_{s+2k}.$$

For the study of $(K_k)_{k\geqslant 2}$, we have:

$$\begin{aligned}\langle\varphi_{1,\gamma},\varphi_{k,u}\rangle &= \frac{\lambda_{1,\gamma}}{\lambda_{k,u}}\langle\varphi_{1,\gamma},\varphi_{k,u}\rangle - \frac{u-\gamma}{\lambda_{k,u}}\langle q\varphi_{1,\gamma},\varphi_{k,u}\rangle \\ &= \frac{\lambda_{1,\gamma}^2}{\lambda_{k,u}^2}\langle\varphi_{1,\gamma},\varphi_{k,u}-\varphi_{k,\gamma}\rangle - \frac{(u-\gamma)\lambda_{1,\gamma}}{\lambda_{k,u}^2}\langle q\varphi_{1,\gamma},\varphi_{k,u}\rangle \\ &\quad - \frac{u-\gamma}{\lambda_{k,u}^2}\big\langle A_u(q\varphi_{1,\gamma}),\varphi_{k,u}\big\rangle,\end{aligned}$$

therefore

$$\big|\langle\varphi_{1,\gamma},\varphi_{k,u}\rangle\big| \leqslant C\frac{\Delta_3}{k^4}. \tag{3.56}$$

This inequality gives $\|K\|_{h^3} \leqslant C\|w\|_{L^2}\Delta_3$. For the bound on the h^5-norm of $(K_k)_{k\geqslant 2}$, we use an integration by parts:

$$\begin{aligned}K_k = &-\int_0^T \frac{1}{\mathrm{i}\lambda_{k,u(t)}}\dot{\widetilde{w}}(t)\big\langle\psi_{1,\gamma}(t),\varphi_{k,u(t)}\big\rangle e^{\mathrm{i}\int_0^t \lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t \\ &+\int_0^T \frac{1}{\mathrm{i}\lambda_{k,u(t)}^2}\widetilde{w}(t)\dot{u}(t)\lambda'_{k,u(t)}\big\langle\psi_{1,\gamma}(t),\varphi_{k,u(t)}\big\rangle e^{\mathrm{i}\int_0^t \lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t \\ &-\int_0^T \frac{1}{\mathrm{i}\lambda_{k,u(t)}}\widetilde{w}(t)(-\mathrm{i}\lambda_{k,\gamma})\big\langle\psi_{1,\gamma}(t),\varphi_{k,u(t)}\big\rangle e^{\mathrm{i}\int_0^t \lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t \\ &-\int_0^T \frac{1}{\mathrm{i}\lambda_{k,u(t)}}\widetilde{w}(t)\dot{u}(t)\left\langle\psi_{1,\gamma}(t),\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(t)}\right\rangle e^{\mathrm{i}\int_0^t \lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t.\end{aligned}$$

We give a bound of the h^5-norm of the first, the second and the third term thanks to (3.56). For the fourth term, we use Lemma 5. We get:

$$\|K\|_{h^5} \leqslant C\big[\Delta_3\|w\|_{H^1} + \Delta_5\|w\|_{L^2}\big].$$

For the study of $(K_k)_{k\in\mathbb{N}^*}$ in h^7, we work as in the proof of the h^7-bound for L^0. Considering the previous integration by parts, the second term of the right-hand side can be directly bounded in h^7 by $\Delta_3\|w\|_{L^2}$, in the first, third and fourth terms of the right-hand side, we decompose:

$$\frac{1}{\lambda_{k,u}} = \left(\frac{1}{\lambda_{k,u}} - \frac{1}{\lambda_k}\right) + \frac{1}{\lambda_k}.$$

For the parts containing $(1/\lambda_{k,u} - 1/\lambda_k)$, we use (A.13) and (3.56). For the parts containing $(1/\lambda_k)$, we apply the previous result.

For the study of $X(w)$ in h^9, we use an other integration by parts with respect to t. □

3.6.4. Study of the right-hand side $d(\Psi_0, \Psi_T)$

We recall $\gamma \in (0, \gamma^*)$ and $T = 4/\pi$. We use the same notations as in the previous subsection. This subsection is dedicated to the proof of the following proposition.

Proposition 31. *There exists a constant C such that, for every $(\psi_0, v) \in E_9^\gamma \cap \mathcal{V}$, for every $(\Psi_0, \Psi_T) \in F_9^\gamma$ satisfying,*

$$\Re\big(\langle \Psi_0, \psi_0\rangle\big) = \Re\big(\langle \Psi_T, \psi_T\rangle\big) = 0,$$

the sequence $d(\Psi_0, \Psi_T)$ belongs to $h_r^9(\mathbb{N}^*, \mathbb{C})$, and satisfies:

$$\big\|d(\Psi_0, \Psi_T)\big\|_{h^3} \leqslant C\big\|(\Psi_0, \Psi_T)\big\|_{F_3^\gamma},$$
$$\big\|d(\Psi_0, \Psi_T)\big\|_{h^5} \leqslant C\big[\big\|(\Psi_0, \Psi_T)\big\|_{F_5^\gamma} + \Delta_5\big\|(\Psi_0, \Psi_T)\big\|_{F_3^\gamma}\big],$$
$$\big\|d(\Psi_0, \Psi_T)\big\|_{h^7} \leqslant C\big[\big\|(\Psi_0, \Psi_T)\big\|_{F_7^\gamma} + \Delta_5\big\|(\Psi_0, \Psi_T)\big\|_{F_5^\gamma} + \Delta_7\big\|(\Psi_0, \Psi_T)\big\|_{F_3^\gamma}\big],$$
$$\begin{aligned}\big\|d(\Psi_0, \Psi_T)\big\|_{h^9} \leqslant C\big[&\big\|(\Psi_0, \Psi_T)\big\|_{F_9^\gamma} + \Delta_5\big\|(\Psi_0, \Psi_T)\big\|_{F_7^\gamma} + \Delta_7\big\|(\Psi_0, \Psi_T)\big\|_{F_5^\gamma} \\ &+ \Delta_9\big\|(\Psi_0, \Psi_T)\big\|_{F_3^\gamma}\big].\end{aligned}$$

In the next propositions, we prove these bounds on each term in $d(\Psi_0, \Psi_T)$.

Proposition 32. *There exists a constant C such that, for every $(\psi_0, v) \in E_9^\gamma \cap \mathcal{V}$, for every $(\Psi_0, \Psi_T) \in F_9^\gamma(I, \mathbb{C})$ satisfying,*

$$\Re\big(\langle \Psi_0, \psi_0\rangle\big) = \Re\big(\langle \Psi_T, \psi_T\rangle\big) = 0,$$

the sequence $Y = (Y_k)_{k\geqslant 2}$ defined by,

$$Y_k := \langle \Psi_0, \varphi_{k,\gamma}\rangle - \frac{\langle \Psi_0, \varphi_{1,\gamma}\rangle}{\langle \psi_0, \varphi_{1,\gamma}\rangle}\langle \psi_0, \varphi_{k,\gamma}\rangle$$
$$\left(\textit{respectively, } Y_k := \langle \Psi_T, \varphi_{k,\gamma}\rangle - \frac{\langle \Psi_T, \varphi_{1,\gamma}\rangle}{\langle \psi_T, \varphi_{1,\gamma}\rangle}\langle \psi_T, \varphi_{k,\gamma}\rangle\right)$$

belongs to h^9 and satisfies:

$$\|Y\|_{h^3} \leqslant C\big\|(\Psi_0, \Psi_T)\big\|_{F_3^\gamma},$$
$$\|Y\|_{h^5} \leqslant C\big[\big\|(\Psi_0, \Psi_T)\big\|_{F_5^\gamma} + \Delta_5\big\|(\Psi_0, \Psi_T)\big\|_{F_3^\gamma}\big],$$

$$\|Y\|_{h^7} \leqslant C\big[\big\|(\Psi_0,\Psi_T)\big\|_{F_7^\gamma} + \Delta_5\big\|(\Psi_0,\Psi_T)\big\|_{F_5^\gamma} + \Delta_7\big\|(\Psi_0,\Psi_T)\big\|_{F_3^\gamma}\big],$$

$$\|Y\|_{h^9} \leqslant C\big[\big\|(\Psi_0,\Psi_T)\big\|_{F_9^\gamma} + \Delta_5\big\|(\Psi_0,\Psi_T)\big\|_{F_7^\gamma} + \Delta_7\big\|(\Psi_0,\Psi_T)\big\|_{F_5^\gamma} + \Delta_9\big\|(\Psi_0,\Psi_T)\big\|_{F_3^\gamma}\big].$$

Proof. First, we study the sequence $(\langle \Psi_0, \varphi_{k,\gamma}\rangle)_{k\geqslant 2}$ in h^3, h^5, h^7 and h^9. The function Ψ_0 satisfies the boundary conditions, we need to write:

$$\langle \Psi_0, \varphi_{k,\gamma}\rangle = \frac{1}{\lambda_{k,\gamma}^a}\langle A_\gamma^a \Psi_0, \varphi_{k,\gamma}\rangle \quad \text{for } a = 1,2,3,4.$$

The function $A_\gamma^a\Psi_0$ belongs to $H_0^1(I,\mathbb{C})$ for $a = 1,2,3,4$ so Lemma 3 gives:

$$\big\|\langle A_\gamma^a \Psi_0, \varphi_{k,\gamma}\rangle\big\|_{h^1} \leqslant C\|\Psi_0\|_{H^{2a+1}}.$$

In conclusion, we have:

$$\big\|\langle \Psi_0, \varphi_{k,\gamma}\rangle\big\|_{h^s} \leqslant C\|\Psi_0\|_{H^s} \quad \text{for } s = 3,5,7,9.$$

Now, we study the sequence,

$$\left(\frac{\langle \Psi_0, \varphi_{1,\gamma}\rangle}{\langle \psi_0, \varphi_{1,\gamma}\rangle}\langle \psi_0, \varphi_{k,\gamma}\rangle\right)_{k\geqslant 2},$$

in h^3, h^5, h^7 and h^9. We have, for every $k \geqslant 2$,

$$\langle \psi_0, \varphi_{k,\gamma}\rangle = \langle \psi_0 - \varphi_{1,\gamma}, \varphi_{k,\gamma}\rangle.$$

The function $(\psi_0 - \varphi_{1,\gamma})$ satisfies the boundary conditions we need to write:

$$\langle \psi_0 - \varphi_{1,\gamma}, \varphi_{k,\gamma}\rangle = \frac{1}{\lambda_{k,\gamma}^a}\big\langle A_\gamma^a(\psi_0 - \varphi_{k,\gamma}), \varphi_{k,\gamma}\big\rangle \quad \text{for } a = 1,2,3,4.$$

The function $A_\gamma^a(\psi_0 - \varphi_{1,\gamma})$ belongs to $H_0^1(I,\mathbb{C})$ for $a = 1,2,3,4$ so Lemma 3 gives:

$$\big\|\langle \psi_0, \varphi_{k,\gamma}\rangle\big\|_{h^s} \leqslant C\|\psi_0 - \varphi_{1,\gamma}\|_{H^s} \leqslant C\delta_s \quad \text{for } s = 3,5,7,9.$$

We have:

$$\big|\langle \psi_0, \varphi_{1,\gamma}\rangle\big| \geqslant 1 - \big|\langle \psi_0 - \varphi_{1,\gamma}, \varphi_{1,\gamma}\rangle\big| \geqslant 3/4 \quad \text{because } \Delta_3 \leqslant 1/4.$$

Therefore

$$\left\|\frac{\langle \Psi_0, \varphi_{1,\gamma}\rangle}{\langle \psi_0, \varphi_{1,\gamma}\rangle}\langle \psi_0, \varphi_{1,\gamma}\rangle\right\|_{h^s} \leqslant C\Delta_s\|\Psi_0\|_{L^2} \quad \text{for } s = 3,5,7,9. \quad \square$$

Proposition 33. *There exists a constant C such that, for every $(\psi_0, v) \in E_9^\gamma \cap \mathcal{V}$, for every $(\Psi_0, \Psi_T) \in F_9^\gamma$ satisfying,*

$$\Re\big(\langle \Psi_0, \psi_0\rangle\big) = \Re\big(\langle \Psi_T, \psi_T\rangle\big) = 0,$$

the sequence $Y = (Y_k)_{k\geqslant 2}$ defined by,

$$Y_k := \int_0^T \dot{u}(t)\left\langle \Psi_1(t), \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(t)}\right\rangle \mathrm{e}^{\mathrm{i}\int_0^t \lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t,$$

belongs to $h^9(\mathbb{N}^*, \mathbb{C})$ and satisfies:

$$\|Y\|_{h^3} \leqslant C\big\|(\Psi_0, \Psi_T)\big\|_{F_3^\gamma},$$

$$\|Y\|_{h^5} \leqslant C\big[\big\|(\Psi_0, \Psi_T)\big\|_{F_5^\gamma} + \Delta_5\big\|(\Psi_0, \Psi_T)\big\|_{F_3^\gamma}\big],$$

$$\|Y\|_{h^7} \leqslant C\big[\big\|(\Psi_0, \Psi_T)\big\|_{F_7^\gamma} + \Delta_5\big\|(\Psi_0, \Psi_T)\big\|_{F_5^\gamma} + \Delta_7\big\|(\Psi_0, \Psi_T)\big\|_{F_3^\gamma}\big],$$

$$\|Y\|_{h^9} \leqslant C\big[\big\|(\Psi_0, \Psi_T)\big\|_{F_9^\gamma} + \Delta_5\big\|(\Psi_0, \Psi_T)\big\|_{F_7^\gamma} + \Delta_7\big\|(\Psi_0, \Psi_T)\big\|_{F_5^\gamma} + \Delta_9\big\|(\Psi_0, \Psi_T)\big\|_{F_3^\gamma}\big].$$

Proof. We apply Propositions 17, 20, 23, 25, using the following consequence of Propositions 45, 47, 49, 51:

$$\|\Psi_1\|_{C^k([0,T],H^s)} \leqslant \|\Psi_0\|_{H^{s+2k}}. \quad \square$$

Proposition 34. *There exists a constant C such that, for every $(\psi_0, v) \in E_9^\gamma \cap \mathcal{V}$, for every $(\Psi_0, \Psi_T) \in F_9^\gamma$ satisfying,*

$$\Re\big(\langle \Psi_0, \psi_0\rangle\big) = \Re\big(\langle \Psi_T, \psi_T\rangle\big) = 0,$$

the sequence $Y = (Y_k)_{k\geqslant 2}$ defined by,

$$Y_k := \int_0^T \dot{u}(t)\frac{\langle \Psi_1(t), \varphi_{1,u(t)}\rangle}{\langle \psi(t), \varphi_{1,u(t)}\rangle}\left\langle \psi(t), \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(t)}\right\rangle \mathrm{e}^{\mathrm{i}\int_0^t \lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t,$$

belongs to h^9 and satisfies:

$$\|Y\|_{h^3} \leqslant C\big\|\big(\Psi_0, \Psi_T\big)\big\|_{F_3^\gamma},$$

$$\|Y\|_{h^5} \leqslant C\big[\big\|(\Psi_0, \Psi_T)\big\|_{F_5^\gamma} + \Delta_5\big\|(\Psi_0, \Psi_T)\big\|_{F_3^\gamma}\big],$$

$$\|Y\|_{h^7} \leqslant C\big[\big\|(\Psi_0, \Psi_T)\big\|_{F_7^\gamma} + \Delta_5\big\|(\Psi_0, \Psi_T)\big\|_{F_5^\gamma} + \Delta_7\big\|(\Psi_0, \Psi_T)\big\|_{F_3^\gamma}\big],$$

$$\|Y\|_{h^9} \leqslant C\big[\big\|(\Psi_0, \Psi_T)\big\|_{F_9^\gamma} + \Delta_5\big\|(\Psi_0, \Psi_T)\big\|_{F_7^\gamma} + \Delta_7\big\|(\Psi_0, \Psi_T)\big\|_{F_5^\gamma} + \Delta_9\big\|(\Psi_0, \Psi_T)\big\|_{F_3^\gamma}\big].$$

Proof. Let

$$\widetilde{w}(t) := \dot{u}(t) \frac{\langle \Psi_1(t), \varphi_{1,u(t)} \rangle}{\langle \psi(t), \varphi_{1,u(t)} \rangle}.$$

Computing the derivatives and using,

$$\|\Psi_1\|_{C^k([0,T],H^s)} \leqslant \|\Psi_0\|_{H^{s+2k}},$$

we get:

$$\begin{gathered}
\|\widetilde{w}\|_{L^2} \leqslant C \Delta_3 \|\Psi_0\|_{L^2}, \\
\|\widetilde{w}\|_{H^1} \leqslant C\big[\Delta_3 \|\Psi_0\|_{H^2} + \Delta_5 \|\Psi_0\|_{L^2}\big], \\
\|\widetilde{w}\|_{H^2} \leqslant C\big[\Delta_3 \|\Psi_0\|_{H^4} + \Delta_5 \|\Psi_0\|_{H^2} + \Delta_7 \|\Psi_0\|_{L^2}\big], \\
\|\widetilde{w}\|_{H^3} \leqslant C\big[\Delta_3 \|\Psi_0\|_{H^6} + \Delta_5 \|\Psi_0\|_{H^4} + \Delta_7 \|\Psi_0\|_{H^2} + \Delta_9 \|\Psi_0\|_{L^2}\big].
\end{gathered}$$

Now, we just apply Propositions 17, 20, 23 and 25. □

Proposition 35. *There exists a constant C such that, for every $(\psi_0, v) \in E_9^\gamma \cap \mathcal{V}$, for every $(\Psi_0, \Psi_T) \in F_9^\gamma$ satisfying,*

$$\Re\big(\langle \Psi_0, \psi_0 \rangle\big) = \Re\big(\langle \Psi_T, \psi_T \rangle\big) = 0,$$

the sequence $Y = (Y_k)_{k \geqslant 2}$ defined by,

$$Y_k := \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\langle \Psi_1(t), \varphi_{1,u(t)} \rangle}{\langle \psi(t), \varphi_{1,u(t)} \rangle} \right) \big\langle \psi(t), \varphi_{k,u(t)} \big\rangle \mathrm{e}^{\mathrm{i} \int_0^t \lambda_{k,u(s)}\, \mathrm{d}s}\, \mathrm{d}t,$$

belongs to h^9 and satisfies:

$$\begin{gathered}
\|Y\|_{h^3} \leqslant C \big\| (\Psi_0, \Psi_T) \big\|_{F_3^\gamma}, \\
\|Y\|_{h^5} \leqslant C\big[\big\|(\Psi_0, \Psi_T)\big\|_{F_5^\gamma} + \Delta_5 \big\|(\Psi_0, \Psi_T)\big\|_{F_3^\gamma}\big], \\
\|Y\|_{h^7} \leqslant C\big[\big\|(\Psi_0, \Psi_T)\big\|_{F_7^\gamma} + \Delta_5 \big\|(\Psi_0, \Psi_T)\big\|_{F_5^\gamma} + \Delta_7 \big\|(\Psi_0, \Psi_T)\big\|_{F_3^\gamma}\big], \\
\|Y\|_{h^9} \leqslant C\big[\big\|(\Psi_0, \Psi_T)\big\|_{F_9^\gamma} + \Delta_5 \big\|(\Psi_0, \Psi_T)\big\|_{F_7^\gamma} + \Delta_7 \big\|(\Psi_0, \Psi_T)\big\|_{F_5^\gamma} + \Delta_9 \big\|(\Psi_0, \Psi_T)\big\|_{F_3^\gamma}\big].
\end{gathered}$$

Proof. Let

$$\widetilde{w} := \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\langle \Psi_1(t), \varphi_{1,u(t)} \rangle}{\langle \psi(t), \varphi_{1,u(t)} \rangle} \right).$$

We have:

$$\|\widetilde{w}\|_{L^2} \leqslant \|\dot{\Psi}_1\|_{L^2([0,T],L^2)} + \|\Psi_1\|_{C^0([0,T],L^2)}\big[\|\dot{u}\|_{L^2} + \|\psi_0\|_{H^2} + |\dot{u}\|_{L^2}\|\psi_0\|_{L^2}\big],$$

$$\begin{aligned}\|\dot{\Psi}_1\|_{L^2([0,T],L^2)} &\leqslant \|\Psi_1\|_{C^0([0,T],H^2)} + \|u\|_{H^1}\|\Psi_1\|_{C^0([0,T],L^2)}\\ &\leqslant \|\Psi_0\|_{H^2} + C\Delta_3\|\Psi_0\|_{L^2},\end{aligned}$$

so $\|\widetilde{w}\|_{L^2} \leqslant C\|\Psi_0\|_{H^2}$. In the same way, we get

$$\begin{aligned}\|\widetilde{w}\|_{H^1} &\leqslant C\big[\|\Psi_0\|_{H^4} + \Delta_5\|\Psi_0\|_{L^2}\big],\\ \|\widetilde{w}\|_{H^2} &\leqslant C\big[\|\Psi_0\|_{H^6} + \Delta_5\|\Psi_0\|_{H^2} + \Delta_7\|\Psi_0\|_{L^2}\big],\\ \|\widetilde{w}\|_{H^3} &\leqslant C\big[\|\Psi_0\|_{H^8} + \Delta_5\|\Psi_0\|_{H^4} + \Delta_7\|\Psi_0\|_{H^2} + \Delta_9\|\Psi_0\|_{L^2}\big].\end{aligned}$$

Now, we apply Propositions 16, 19, 22 and 24. □

3.6.5. Controllability of the linearized system around $(\psi(t), u(t))$ and bounds (3.19)–(3.22)

Theorem 9. *Let $T = 4/\pi$, $\gamma \in (0, \gamma^*)$, $(\psi_0, v) \in E_9^\gamma$ and ψ the associated solution of (Σ) with $u = \gamma + v$. We assume $\|u\|_{H^1((0,T),\mathbb{R})} \leqslant \gamma^*$, $\|u\|_{L^\infty((0,T),\mathbb{R})} \leqslant \gamma^*$ and $\|u\|_{L^2((0,T),\mathbb{R})} < \sqrt{2}/\sqrt{17}$. If $\Delta_3 := \gamma + \|(\psi_0, v) - (\varphi_{1,\gamma}, 0)\|_{E_3^0}$ is small enough, then there exist a constant C and a continuous linear map,*

$$\begin{aligned}\Pi_{(\psi_0,v)} : \big[T_S(\psi_0) \times T_S(\psi_T)\big] \cap F_9^\gamma &\to E_7^\gamma,\\ (\Psi_0, \Psi_T) &\mapsto (\Psi_0, w),\end{aligned}$$

such that for every $(\Psi_0, \Psi_T) \in F_9^\gamma$ satisfying,

$$\Re\big(\langle \Psi_0, \psi_0\rangle\big) = \Re\big(\langle \Psi_T, \psi_T\rangle\big) = 0,$$

we have:

$$\Phi'_\gamma(\psi_0, v).\Pi_{(\psi_0,v)}(\Psi_0, \Psi_T) = (\Psi_0, \Psi_T),$$

$$\begin{aligned}\|w\|_{L^2} &\leqslant C\big\|(\Psi_0, \Psi_T)\big\|_{F_3^\gamma},\\ \|w\|_{H^1} &\leqslant C\big[\big\|(\Psi_0, \Psi_T)\big\|_{F_5^\gamma} + \Delta_3\big\|(\Psi_0, \Psi_T)\big\|_{F_3^\gamma}\big],\\ \|w\|_{H^2} &\leqslant C\big[\big\|(\Psi_0, \Psi_T)\big\|_{F_7^\gamma} + \Delta_3\big\|(\Psi_0, \Psi_T)\big\|_{F_5^\gamma} + \Delta_5\big\|(\Psi_0, \Psi_T)\big\|_{F_3^\gamma}\big],\\ \|w\|_{H^3} &\leqslant C\big[\big\|(\Psi_0, \Psi_T)\big\|_{F_9^\gamma} + \Delta_3\big\|(\Psi_0, \Psi_T)\big\|_{F_7^\gamma} + \Delta_5\big\|(\Psi_0, \Psi_T)\big\|_{F_5^\gamma}\\ &\quad + \Delta_7\big\|(\Psi_0, \Psi_T)\big\|_{F_3^\gamma}\big].\end{aligned} \tag{3.57}$$

Proof. Notice that, for every $k \in \mathbb{N}^*$, we have:

$$M_{(\varphi_{1,\gamma},\gamma)}(w)_k = b_{k,\gamma} Z_\gamma(w)_k,$$

where the coefficients $b_{k,\gamma} = \langle q\varphi_{k,\gamma}, \varphi_{1,\gamma}\rangle$ are studied in Proposition 1, and

$$Z_\gamma : L^2\big((0,T),\mathbb{R}\big) \to l^2(\mathbb{N}^*,\mathbb{C})$$

is defined in Proposition 11. Thanks to the behaviour of the coefficients $b_{k,\gamma}$ and Proposition 14, the map $M_{(\varphi_{1,\gamma},\gamma)}$ admits a right inverse,

$$M^{-1}_{(\varphi_{1,\gamma},\gamma)} : h^9_r(\mathbb{N}^*,\mathbb{C}) \to H^3_0\big((0,T),\mathbb{R}\big),$$
$$d \mapsto w,$$

and there exists a constant C_0 such that, for every $d \in h^9_r(\mathbb{N}^*,\mathbb{C})$, the function $w := M_{(\varphi_{1,\gamma},\gamma)}(d)$ satisfies:

$$\|w\|_{L^2} \leqslant C_0\|d\|_{h^3}, \quad \|w\|_{H^1} \leqslant C_0\|d\|_{h^5}, \quad \|w\|_{H^2} \leqslant C_0\|d\|_{h^7}, \quad \|w\|_{H^3} \leqslant C_0\|d\|_{h^9}.$$

Let C_1 be the constant used in (3.53). We assume:

$$\Delta_3 \leqslant \frac{1}{2C_0C_1}.$$

Then, thanks to Proposition 15, $M_{(\psi_0,u)}$ admits a right inverse,

$$M^{-1}_{(\psi_0,u)} : h^9_r(\mathbb{N}^*,\mathbb{C}) \to H^3_0\big((0,T),\mathbb{R}\big),$$

such that, for every $d \in h^9_r(\mathbb{N}^*,\mathbb{C})$, the function $w := M^{-1}_{(\psi_0,u)}(d)$ satisfies:

$$\|w\|_{L^2} \leqslant 2C_0\|d\|_{h^3},$$
$$\|w\|_{H^1_0} \leqslant 2C_0\big[\|d\|_{h^5} + 2C_2\Delta_5\|d\|_{h^3}\big],$$
$$\|w\|_{H^2_0} \leqslant 2C_0\big[\|d\|_{h^7} + 2C_2\Delta_5\|d\|_{h^5} + \big(2C_2\Delta_7 + 8C_2^2\Delta_5^2\big)\|d\|_{h^3}\big],$$
$$\|w\|_{H^3_0} \leqslant 2C_0\big[\|d\|_{h^9} + 2C_2\Delta_5\|d\|_{h^7} + \big(2C_2\Delta_7 + 8C_2^2\Delta_5^2\big)\|d\|_{h^5} + \big(2C_2\Delta_9 + 16C_2^2\Delta_7\Delta_5 + 48C_2^3\Delta_5^3\big)\|d\|_{h^3}\big],$$

where $C_2 := C_0C_1$. For $(\Psi_0,\Psi_T) \in F_9^\gamma$ satisfying,

$$\Re\big(\langle\Psi_0,\psi_0\rangle\big) = \Re\big(\langle\Psi_T,\psi_T\rangle\big) = 0,$$

we define:

$$\Pi_{(\psi_0,u)}(\Psi_0,\Psi_T) := M^{-1}_{(\psi_0,u)}\big(d(\Psi_0,\Psi_T)\big).$$

We check the bounds (3.57) thanks to the previous bound on $M^{-1}_{(\psi_0,u)}(d)$ and the bounds on $d(\Psi_0,\Psi_T)$ given in Proposition 31. □

3.6.6. The local controllability result around $\psi_{1,\gamma}$

The application of the Nash–Moser theorem leads to the following result:

Theorem 10. *Let $T := 4/\pi$. There exists γ_0 such that, for every $\gamma \in (0,\gamma_0)$, there exists $\delta > 0$ such that, for every $\psi_0, \psi_f \in S \cap H^7_{(\gamma)}(I,\mathbb{C})$, satisfying,*

$$\big\|\psi_0 - \psi_{1,\gamma}(0)\big\|_{H^7} < \delta, \qquad \big\|\psi_f - \psi_{1,\gamma}(T)\big\|_{H^7} < \delta,$$

there exists $v \in H^1_0((0,T),\mathbb{R})$ such that the solution of (Σ) with control $u := \gamma + v$ such that $\psi(0) = \psi_0$ satisfies $\psi(T) = \psi_f$.

4. Quasi-static transformations

In this section, we fix $\gamma_0 \in (0,\gamma^*]$. For $\varepsilon > 0$ and $\phi_0 \in [0,2\pi)$, we consider:

$$\begin{cases} \mathrm{i}\dot{\psi}_\varepsilon = -\frac{1}{2}\psi''_\varepsilon - \gamma_0 f(\varepsilon t) q \psi_\varepsilon, & 0 \leqslant t \leqslant 1/\varepsilon,\ q \in I, \\ \psi_\varepsilon(0) = \varphi_1 \mathrm{e}^{\mathrm{i}\phi_0}, \\ \psi_\varepsilon(t,-1/2) = \psi_\varepsilon(t,1/2) = 0, \end{cases}$$

where $f \in C^\infty([0,1],\mathbb{R})$ satisfies $f^{(k)}(0) = 0$, for every $k \in \mathbb{N}$, $f(1) = 1$ and $0 \leqslant f \leqslant 1$. The aim of this section is the proof of the following theorem:

Theorem 11. *Let $(\varepsilon_n)_{n\in\mathbb{N}^*}$ be defined by*:

$$\frac{1}{\varepsilon_n}\int_0^1 \lambda_{1,\gamma_0 f(t)}\,\mathrm{d}t = \phi_0 + 2n\pi.$$

For every $s \in \mathbb{N}$, $(\psi_{\varepsilon_n}(1/\varepsilon_n))_{n\in\mathbb{N}}$ converges to φ_{1,γ_0} in $H^s(I,\mathbb{C})$.

For this, we prove the convergence in $L^2(I,\mathbb{C})$, we find a bound M_s for this sequence in H^s: $\|\psi_{\varepsilon_n}(1/\varepsilon_n)\|_{H^s} \leqslant M_s$ for every $n \in \mathbb{N}^*$ and for every $s \in \mathbb{N}$. We conclude using the convexity of the H^s-norms:

$$\big\|\psi_{\varepsilon_n}(1/\varepsilon_n) - \varphi_{1,\gamma_0}\big\|_{H^s} \leqslant C\big\|\psi_{\varepsilon_n}(1/\varepsilon_n) - \varphi_{1,\gamma_0}\big\|^\theta_{L^2} M^{1-\theta}_{s+1},$$

where $\theta = 1/(s+1)$. With the same arguments we get the following theorem:

Theorem 12. *Let $T = 4/\pi$. For $\varepsilon > 0$ and $\phi_1 \in (-2\pi, 0]$, we consider*:

$$\begin{cases} \mathrm{i}\dot{\xi}_\varepsilon = -\frac{1}{2}\xi_\varepsilon'' - \gamma_0 f(1-\varepsilon t) q \xi_\varepsilon, \\ \xi_\varepsilon(1/\varepsilon) = \varphi_1 \mathrm{e}^{\mathrm{i}\phi_1}, \\ \xi_\varepsilon(t,-1/2) = \xi_\varepsilon(t,1/2) = 0. \end{cases}$$

Let $(\varepsilon_n)_{n\in\mathbb{N}^*}$ be defined by:

$$\frac{1}{\varepsilon_n}\int_0^1 \lambda_{1,\gamma_0 f(t)}\,\mathrm{d}t = -\lambda_{1,\gamma_0}T + 2(n+1)\pi - \phi_1.$$

For every $s \in \mathbb{N}$, $(\xi_{\varepsilon_n}(0))_{n\in\mathbb{N}^*}$ converges to $\varphi_{1,\gamma_0}\mathrm{e}^{-\mathrm{i}\lambda_{1,\gamma_0}T}$ in $H^s(I,\mathbb{C})$.

In order to prove Theorem 11, we define:

$$\Lambda_\varepsilon(t,q) := \psi_\varepsilon(t,q)\mathrm{e}^{\mathrm{i}\int_0^t \lambda_{1,\gamma_0 f(\varepsilon s)}\,\mathrm{d}s - \mathrm{i}\phi_0} - \varphi_{1,\gamma_0 f(\varepsilon t)}(q).$$

Then, we have:

$$\begin{cases} \dot{\Lambda}_\varepsilon = \frac{\mathrm{i}}{2}\Lambda_\varepsilon'' + \mathrm{i}\gamma_0 f(\varepsilon t) q \Lambda_\varepsilon + \mathrm{i}\lambda_{1,\gamma_0 f(\varepsilon t)}\Lambda_\varepsilon - \varepsilon g(\varepsilon t), \\ \Lambda_\varepsilon(0) = 0, \\ \Lambda_\varepsilon(t,-1/2) = \Lambda_\varepsilon(t,\frac{1}{2}) = 0, \end{cases}$$

where

$$g(s) := \gamma_0 \dot{f}(s) \frac{\mathrm{d}\varphi_{1,\gamma}}{\mathrm{d}\gamma}\bigg|_{\gamma_0 f(s)}.$$

In the next propositions, we prove the H^s bound on $(\Lambda_{\varepsilon_n})_{n\in\mathbb{N}^*}$.

Proposition 36. *For every $k \in \mathbb{N}$, there exists a constant C_k such that, for every $\varepsilon \in (0,1]$ and for every $t \in [0, 1/\varepsilon]$,*

$$\left\| \frac{\partial^k}{\partial t^k}\Lambda_\varepsilon(t) \right\|_{L^2(I)} \leqslant C_k.$$

Remark. When $\varepsilon > 0$ is fixed, the function,

$$\Delta(t,q) := \Lambda_\varepsilon(t,q)\mathrm{e}^{-\mathrm{i}\int_0^t \lambda_{1,\gamma_0 f(\varepsilon s)}\,\mathrm{d}s}$$

satisfies the following equations:

$$\begin{cases} \mathrm{i}\dot{\Delta} = -\frac{1}{2}\Delta'' - \gamma_0 f(\varepsilon t) q \Delta - \varepsilon g(\varepsilon t)\mathrm{e}^{-\mathrm{i}\int_0^t \lambda_{1,\gamma_0 f(\varepsilon s)}\,\mathrm{d}s}, \\ \Delta(0) = 0, \\ \Delta(t,-1/2) = \Delta(t,1/2) = 0, \end{cases}$$

which have the general form studied in Appendix B. Thanks to Propositions 45, 47, 49, 51, it is easy to prove that Δ belongs to $C^3([0,T],L^2(I,\mathbb{C}))$ and for $k=1,2,3$, $\partial^k\Delta/\partial t^k$ solves the equation we get by deriving k times with respect to t the equation on Δ. In fact, the functions $\Delta(0)=0$ and $t\mapsto \varepsilon g(\varepsilon t)\exp(-\mathrm{i}\int_0^t \lambda_{1,\gamma_0 f(\varepsilon s)}\,\mathrm{d}s)$ satisfy the conditions we need to derive Δ more than 3 times: $\Delta\in C^\infty([0,T],L^2(I,\mathbb{C}))$ and for every $k\in\mathbb{N}$, the function $\partial^k\Delta/\partial t^k$ solves the equation, we get by deriving k times with respect to t the equation on Δ. Of course, we have the same result for Λ.

Proof. We prove it by induction. Let us first introduce the notation $\Lambda_{k,\varepsilon}:=\frac{\partial^k}{\partial t^k}\Lambda_\varepsilon$. To simplify, we write Λ instead of Λ_ε and Λ_k instead of $\Lambda_{k,\varepsilon}$.

For $k=0$, we use:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda\|_{L^2}^2=\langle\dot\Lambda,\Lambda\rangle+\langle\Lambda,\dot\Lambda\rangle.$$

Thanks to the equation and one integration by parts we get:

$$\langle\dot\Lambda,\Lambda\rangle=-\frac{\mathrm{i}}{2}\langle\Lambda',\Lambda'\rangle+\mathrm{i}\gamma_0 f(\varepsilon t)\langle q\Lambda,\Lambda\rangle+\mathrm{i}\lambda_{1,\gamma_0 f(\varepsilon t)}\langle\Lambda,\Lambda\rangle-\varepsilon\big\langle g(\varepsilon t),\Lambda\big\rangle.$$

Summing the complex conjugate number, and integrating from 0 to t, we get:

$$\big\|\Lambda(t)\big\|_{L^2}^2\leqslant\int_0^t\varepsilon\big\|g(\varepsilon\tau)\big\|_{L^2}\big(1+\big\|\Lambda(\tau)\big\|_{L^2}^2\big)\,\mathrm{d}\tau.$$

Using Gronwall's lemma, we conclude:

$$\big\|\Lambda(t)\big\|_{L^2}^2\leqslant\left(\int_0^1\big\|g(s)\big\|_{L^2}\,\mathrm{d}s\right)\mathrm{e}^{\int_0^1\|g(s)\|_{L^2}\,\mathrm{d}s}.$$

Let $k\in\mathbb{N}^*$. We assume there exist constants C_j, $j=0,\dots,k-1$, such that

$$\big\|\Lambda_j(t)\big\|_{L^2}\leqslant C_j,$$

for $j=0,\dots,k-1$, for every $\varepsilon\in(0,1]$ and for every $t\in[0,1/\varepsilon]$. Since $f^{(j)}(0)=0$ for $j=0,\dots,k-1$ then $\Lambda_j(0)=0$ for $j=1,\dots,k$ and we have:

$$\begin{cases}\dot\Lambda_k=\frac{\mathrm{i}}{2}\Lambda_k''+\mathrm{i}\gamma_0 f(\varepsilon t)\Lambda_k+\mathrm{i}\lambda_{1,\gamma_0 f(\varepsilon t)}\Lambda_k+\mathrm{i}\gamma_0\sum_{j=1}^k\binom{k}{j}\varepsilon^j f^{(j)}(\varepsilon t)q\Lambda_{k-j}\\ \quad+\mathrm{i}\sum_{j=1}^k\binom{k}{j}\varepsilon^j\frac{\mathrm{d}^j}{\mathrm{d}\tau^j}[\lambda_{1,\gamma_0 f(\tau)}]_{\tau=\varepsilon t}\Lambda_{k-j}-\varepsilon^{k+1}g^{(k)}(\varepsilon t),\\ \Lambda_k(0)=0,\\ \Lambda_k(t,-1/2)=\Lambda_k(t,1/2)=0.\end{cases}$$

In the same way as in the case $k=0$, we get:

$$\begin{aligned}\|\Lambda_k(t)\|_{L^2}^2 \leqslant \gamma_0 \int_0^t \sum_{j=1}^{k} \binom{k}{j} \varepsilon^j \left|f^j(\varepsilon s)\right| C_{k-j}\left(1+\|\Lambda_k(s)\|_{L^2}^2\right) \mathrm{d}s \\ \times \int_0^t \sum_{j=1}^{j=k} \binom{k}{j} \varepsilon^j \left|\frac{\mathrm{d}^j}{\mathrm{d}\tau^j}[\lambda_{1,\gamma_0 f(\tau)}]_{\tau=\varepsilon t}\right| C_{k-j}\left(1+\|\Lambda_k(s)\|_{L^2}^2\right) \mathrm{d}s \\ + \int_0^t \varepsilon^{k+1} \left\|g^{(k)}(\varepsilon s)\right\|_{L^2}\left(1+\|\Lambda_k(s)\|_{L^2}^2\right) \mathrm{d}s.\end{aligned}$$

Gronwall's lemma gives:

$$\|\Lambda_{k,\varepsilon}(t)\|_{L^2}^2 \leqslant A_{k,\varepsilon} \exp(A_{k,\varepsilon}),$$

where

$$A_{k,\varepsilon} := \int_0^1 \left(\sum_{j=1}^{k} \binom{k}{j} \varepsilon^{j-1} C_{k-j} \left(\gamma_0 \left|f^{(j)}(t)\right| + \left| \frac{d^j}{\mathrm{d}t^j}[\lambda_{1,\gamma_0 f(t)}] \right| \right) + \varepsilon^k \left\|g^{(k)}(t)\right\|_{L^2} \right) \mathrm{d}t. \quad \square$$

Proposition 37. *For every $s \in \mathbb{N}^*$ and for every $k \in \mathbb{N}$, there exists a constant $D_k^{(s)}$ such that, for every $\varepsilon \in (0,1]$,*

$$\left\| \frac{\partial^k}{\partial t^k} \Lambda_\varepsilon \right\|_{C^0([0,1/\varepsilon],H^{2s})} \leqslant C_k^{(s)}.$$

Corollary 1. *For every $s \in \mathbb{N}$, there exists a constant D_s such that, for every $\varepsilon \in (0,1]$, $\|\Lambda_\varepsilon(1/\varepsilon)\|_{H^s} \leqslant D_s$.*

Proof of Proposition 37. We prove by induction on $s \in \mathbb{N}^*$ the following property P_s:

$$\forall k \in \mathbb{N},\ \exists D_k^{(s)} > 0\colon \quad \forall \varepsilon \in (0,1],\ \left\| \frac{\partial^k}{\partial t^k} \Lambda_\varepsilon \right\|_{C^0([0,1/\varepsilon],H^{2s})} \leqslant D_k^{(s)}.$$

Let $k \in \mathbb{N}$. For $\varepsilon \in (0,1]$, we have:

$$\begin{aligned}\Lambda_{k+1,\varepsilon} = \frac{\mathrm{i}}{2} \Lambda_{k,\varepsilon}'' + \mathrm{i} \sum_{j=0}^{k} \binom{k}{j} \left(\gamma_0 \varepsilon^j f^{(j)}(\varepsilon t) q + \frac{\mathrm{d}^j}{\mathrm{d}t^j}[\lambda_{1,\gamma_0 f(\varepsilon t)}] \right) \Lambda_{k-j,\varepsilon} \\ - \varepsilon^{k+1} g^{(k)}(\varepsilon t).\end{aligned} \tag{4.1}$$

Therefore, there exists a constant $\widetilde{D}_k^{(1)}$ such that, for every $\varepsilon \in (0,1]$,

$$\|\Lambda_{k,\varepsilon}''\|_{C^0([0,1/\varepsilon],L^2)} \leqslant \widetilde{D}_k^{(1)}.$$

Since $\Lambda_{k,\varepsilon}(t,1/2) = \Lambda_{k,\varepsilon}(t,-1/2) = 0$, there exists a constant $C > 0$, which does not depend on k and ε, such that $\|\Lambda_{k,\varepsilon}(t)\|_{H^2} \leqslant C\|\Lambda''_{k,\varepsilon}(t)\|_{L^2}$ and we can take $D_k^{(1)} = C\widetilde{D}_k^{(1)}$. We have proved P_1.

Let $s \in \mathbb{N}^*$. Assume P_{s-1} is true. Let $k \in \mathbb{N}$. Using Eq. (4.1) and P_{s-1}, we get the existence of a constant $\widetilde{D}_k^{(s)}$ such that for every $\varepsilon \in]0,1]$,

$$\|\Lambda''_{k,\varepsilon}\|_{C^0([0,1/\varepsilon],H^{2(s-1)})} \leqslant \widetilde{D}_k^{(s)}.$$

We can take $D_k^{(s)} = D_k^{(1)} + \widetilde{D}_k^{(s)}$. □

Now, we prove the convergence in $L^2(I,\mathbb{C})$, more precisely, we prove the following theorem.

Theorem 13. *There exist constants $\varepsilon_0 > 0$ and $C > 0$ such that, for every $\varepsilon \in (0,\varepsilon_0]$ $\|\Lambda_\varepsilon(1/\varepsilon)\|_{L^2} \leqslant C\gamma_0\varepsilon^{1/4}$.*

For $\varepsilon > 0$, we write:

$$\Lambda_\varepsilon(t) = \sum_{k=1}^{\infty} x_{k,\varepsilon}(t)\varphi_{k,\gamma_0 f(\varepsilon t)},$$

where $x_{k,\varepsilon}(t) := \langle \Lambda_\varepsilon(t), \varphi_{k,\gamma_0 f(\varepsilon t)}\rangle$ belongs to $C^1([0,1/\varepsilon],\mathbb{C})$.

Lemma 9. *There exists a constant C such that, for every $N \in \mathbb{N}^*$, for every $\varepsilon \in (0,1]$ and for every $t \in [0,1/\varepsilon]$,*

$$\sum_{k=N+1}^{\infty} \left|x_{k,\varepsilon}(t)\right|^2 \leqslant \frac{C}{N^4}.$$

Proof. Since $\Lambda_\varepsilon(-1/2,t) = \Lambda_\varepsilon(1/2,t) = 0$, we have:

$$x_{k,\varepsilon}(t) = \frac{1}{\lambda_{k,\gamma_0 f(\varepsilon t)}}\left\langle A_{\gamma_0 f(\varepsilon t)}\Lambda_\varepsilon(t), \varphi_{k,\gamma_0 f(\varepsilon t)}\right\rangle.$$

Thanks to (A.12) and the orthonormality of the family $(\varphi_{k,\gamma_0 f(\varepsilon t)})_{k\in\mathbb{N}^*}$, we get:

$$\sum_{k=N+1}^{\infty} \left|x_{k,\varepsilon}(t)\right|^2 \leqslant \frac{c_1}{N^4}\sum_{k=N+1}^{\infty}\left|\left\langle A_{\gamma_0 f(\varepsilon t)}\Lambda_\varepsilon(t), \varphi_{k,\gamma_0 f(\varepsilon t)}\right\rangle\right|^2 \leqslant \frac{c_2}{N^4}\left\|\Lambda_\varepsilon(t)\right\|_{H^2}^2. \quad \square$$

The coefficient $x_{k,\varepsilon}$ satisfies, for every $k \geqslant 1$, the equations:

$$\begin{cases} \dot{x}_{k,\varepsilon}(t) = \mathrm{i}(\lambda_{1,\gamma_0 f(\varepsilon t)} - \lambda_{k,\gamma_0 f(\varepsilon t)})x_{k,\varepsilon}(t) - \gamma_0\varepsilon\dot{f}(\varepsilon t)\langle \frac{\mathrm{d}\varphi_{1,\gamma}}{\mathrm{d}\gamma}|_{\gamma_0 f(\varepsilon t)}, \varphi_{k,\gamma_0 f(\varepsilon t)}\rangle \\ \quad + \gamma_0\varepsilon\dot{f}(\varepsilon t)\sum_{j=1}^{\infty} x_{j,\varepsilon}(t)\langle\varphi_{j,\gamma_0 f(\varepsilon t)}, \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}|_{\gamma_0 f(\varepsilon t)}\rangle, \\ x_{k,\varepsilon}(0) = 0. \end{cases}$$

For $\varepsilon > 0$, let $N_\varepsilon := [\varepsilon^{-1/8}]$ and $\widetilde{X}_\varepsilon := (\tilde{x}_{2,\varepsilon}, \dots, \tilde{x}_{N_\varepsilon,\varepsilon})$ be the solution of

$$\begin{cases} \dot{\tilde{x}}_{k,\varepsilon}(t) = \mathrm{i}(\lambda_{1,\gamma_0 f(\varepsilon t)} - \lambda_{k,\gamma_0 f(\varepsilon t)})\tilde{x}_{k,\varepsilon}(t) - \gamma_0 \varepsilon \dot{f}(\varepsilon t)\langle \frac{\mathrm{d}\varphi_{1,\gamma}}{\mathrm{d}\gamma}|_{\gamma_0 f(\varepsilon t)}, \varphi_{k,\gamma_0 f(\varepsilon t)}\rangle \\ \quad + \gamma_0 \varepsilon \dot{f}(\varepsilon t)\big(x_{1,\varepsilon}(t)\langle \varphi_{1,\gamma_0 f(\varepsilon t)}, \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}|_{\gamma_0 f(\varepsilon t)}\rangle + \sum_{j=2}^{N_\varepsilon} \tilde{x}_{j,\varepsilon}(t)\langle \varphi_{j,\gamma_0 f(\varepsilon t)}, \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}|_{\gamma_0 f(\varepsilon t)}\rangle\big), \\ \tilde{x}_{k,\varepsilon}(0) = 0, \end{cases}$$

for $k = 2, \dots, N_\varepsilon$.

Proposition 38. *There exists a constant $C > 0$ and $\varepsilon_0 \in (0,1]$ such that, for every $\varepsilon \in (0, \varepsilon_0]$ and for every $t \in [0, 1/\varepsilon]$, we have*:

$$\big\| \widetilde{X}_\varepsilon(t) \big\|_2 \leqslant C\gamma_0\sqrt{\varepsilon}.$$

Here $\|.\|_2$ is the hermitian norm on $\mathbb{C}^n$ for every integer n.

Proof. We have:

$$\begin{cases} \dot{\widetilde{X}}_\varepsilon(t) = C_\varepsilon(t)\widetilde{X}_\varepsilon(t) - \gamma_0\varepsilon \dot{f}(\varepsilon t)(1 + x_{1,\varepsilon}(t))a_\varepsilon(\varepsilon t), \\ \widetilde{X}_\varepsilon(0) = 0, \end{cases}$$

where

$$\begin{aligned} C_\varepsilon(t) &:= D_\varepsilon(\varepsilon t) + \gamma_0\varepsilon\dot{f}(\varepsilon t)A_\varepsilon(\varepsilon t), \\ D_\varepsilon(s) &:= \mathrm{diag}\big(\mathrm{i}(\lambda_{1,\gamma_0 f(s)} - \lambda_{k,\gamma_0 f(s)});\ k = 2, \dots, N_\varepsilon\big), \\ A_\varepsilon(s) &:= \left(\left\langle \varphi_{j,\gamma_0 f(s)}, \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{\gamma_0 f(s)} \right\rangle\right)_{2\leqslant k\leqslant N_\varepsilon,\ 2\leqslant j\leqslant N_\varepsilon}, \\ a_\varepsilon(s) &:= \left(\left\langle \frac{\mathrm{d}\varphi_{1,\gamma}}{\mathrm{d}\gamma}\bigg|_{\gamma_0 f(s)}, \varphi_{k,\gamma_0 f(s)} \right\rangle\right)_{2\leqslant k\leqslant N_\varepsilon}. \end{aligned}$$

We introduce the resolvent $R_\varepsilon(t,s)$ associated to $C_\varepsilon(t)$:

$$\frac{\partial R_\varepsilon}{\partial t}(t,s) = C_\varepsilon(t)R_\varepsilon(t,s), \qquad R_\varepsilon(t,s)R_\varepsilon(s,t) = \mathrm{Id}_{\mathbb{R}^{N_\varepsilon - 1}}.$$

Deriving the second equality with respect to s and using the first one, we get:

$$\frac{\partial R_\varepsilon}{\partial s}(t,s) = -R_\varepsilon(t,s)C_\varepsilon(s). \tag{4.2}$$

We have:

$$\widetilde{X}_\varepsilon(t) = \int_0^t R_\varepsilon(t,s)\big(-\gamma_0\varepsilon\dot{f}(\varepsilon s)\big(1 + x_{1,\varepsilon}(s)\big)a(\varepsilon s)\big)\,\mathrm{d}s.$$

Using (4.2) and an integration by parts, we get:

$$\begin{aligned}\widetilde{X}_\varepsilon(t) &= C_\varepsilon(t)^{-1}\gamma_0\varepsilon\dot f(\varepsilon t)\big(1+x_{1,\varepsilon}(t)\big)a_\varepsilon(\varepsilon t)\\ &\quad+\int_0^t R_\varepsilon(t,s)C_\varepsilon(s)^{-1}\dot C_\varepsilon(s)C_\varepsilon(s)^{-1}\big(\gamma_0\varepsilon\dot f(\varepsilon s)\big(1+x_{1,\varepsilon}(s)\big)a_\varepsilon(\varepsilon s)\big)\,\mathrm{d}s\\ &\quad-\int_0^t R_\varepsilon(t,s)C_\varepsilon(s)^{-1}\gamma_0\varepsilon^2\big(1+x_{1,\varepsilon}(s)\big)\big(\ddot f(\varepsilon s)a_\varepsilon(\varepsilon s)+\dot f(\varepsilon s)\dot a_\varepsilon(\varepsilon s)\big)\,\mathrm{d}s\\ &\quad-\int_0^t R_\varepsilon(t,s)C_\varepsilon(s)^{-1}\big(\gamma_0\varepsilon\dot f(\varepsilon s)\dot x_{1,\varepsilon}(s)a_\varepsilon(\varepsilon s)\big)\,\mathrm{d}s.\end{aligned}$$

To be able to write this equality, we have to check that $C_\varepsilon(s)$ is invertible for every $s\in[0,1/\varepsilon]$. □

Lemma 10. *There exists $\varepsilon_0\in(0,1]$ such that for every $\varepsilon\in(0,\varepsilon_0]$ and for every $s\in[0,1/\varepsilon]$, $C_\varepsilon(s)$ is invertible.*

To be able to exploit the previous expression of $\widetilde{X}_\varepsilon(t)$ we need bounds on the different quantities inside. When A is an $N\times N$ matrix, we write:

$$\|A\|_2=\sup\big\{\|Ax\|_2;\ x\in\mathbb{C}^N,\ \|x\|_2\leqslant 1\big\},$$
$$\|A\|_\infty=\sup\big\{\|Ax\|_\infty;\ x\in\mathbb{C}^N,\ \|x\|_\infty\leqslant 1\big\}.$$

Lemma 11. *There exists a constant $C>0$ such that, for every $\varepsilon\in(0,1]$ and for every $(t,s)\in[0,1/\varepsilon]\times[0,1/\varepsilon]$,*

$$\begin{gathered}\big\|R_\varepsilon(t,s)\big\|_2=1,\\ \big\|C_\varepsilon(s)^{-1}\big\|_2\leqslant C\varepsilon^{-1/4},\qquad \big\|\dot C_\varepsilon(s)\big\|_2\leqslant C\gamma_0\varepsilon,\\ \big|\dot x_{1,\varepsilon}(s)\big|\leqslant C\gamma_0\varepsilon,\qquad \big|x_{1,\varepsilon}(s)\big|\leqslant C\gamma_0,\\ \big\|a_\varepsilon(s)\big\|_2\leqslant C\quad\textit{and}\quad \big\|\dot a_\varepsilon(s)\big\|_2\leqslant C\gamma_0.\end{gathered}$$

Now it is easy to get, for every $\varepsilon\in(0,\varepsilon_0]$, $\widetilde{X}(t)\leqslant C\gamma_0\sqrt{\varepsilon}$. □

Proof of Lemma 10. *Invertibility of $D_\varepsilon(s)$ for $s\in[0,1]$*: We can assume the positive real number γ_0 is small enough so that $\inf\{\lambda_{2,\gamma}-\lambda_{1,\gamma};\ \gamma\in[0,\gamma_0]\}>1$. Indeed, thanks to (A.11), we have:

$$\lim_{\gamma\to 0}(\lambda_{2,\gamma}-\lambda_{1,\gamma})=\lambda_2-\lambda_1=3/2\pi^2>1.$$

Then, for every $s \in [0,1]$, $D_\varepsilon(s)$ is invertible and $\|D_\varepsilon(s)^{-1}\|_2 = 1/(\lambda_{2,\gamma_0 f(s)} - \lambda_{1,\gamma_0 f(s)})$.

Bound on $\|A_\varepsilon(s)\|_2$ *for* $s \in [0,1]$: For $s \in [0,1]$, we have:

$$\|A_\varepsilon(s)\|_2 \leqslant \sqrt{N_\varepsilon}\|A_\varepsilon(s)\|_\infty \leqslant \sqrt{N_\varepsilon}\sup\left\{\sum_{j=2}^{N_\varepsilon}\left|\left\langle \varphi_{j,\gamma_0 f(s)}, \left.\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\right|_{\gamma_0 f(s)}\right\rangle\right|;\ 2 \leqslant k \leqslant N_\varepsilon\right\}.$$

For $k \in \mathbb{N}^*$, we have:

$$\begin{aligned}\sum_{j=2}^{N_\varepsilon}\left|\left\langle \varphi_{j,\gamma_0 f(s)}, \left.\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\right|_{\gamma_0 f(s)}\right\rangle\right| &\leqslant \sqrt{N_\varepsilon}\left(\sum_{j=2}^{N_\varepsilon}\left|\left\langle \varphi_{j,\gamma_0 f(s)}, \left.\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\right|_{\gamma_0 f(s)}\right\rangle\right|^2\right)^{1/2}\\ &\leqslant \sqrt{N_\varepsilon}\left\|\left.\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\right|_{\gamma_0 f(s)}\right\|_{L^2}.\end{aligned}$$

Thanks to (A.18), there exists a constant $C > 0$, which does not depend on ε, such that, for every $s \in [0,1]$,

$$\|A_\varepsilon(s)\|_2 \leqslant C N_\varepsilon \leqslant C\varepsilon^{-1/8}.$$

Invertibility of $C_\varepsilon(s)$ *for* $s \in [0,1]$: Let $\varepsilon_0 \in (0,1]$ be such that $C\gamma_0\|\dot f\|_\infty \varepsilon_0^{7/8} < 1$. Then for every $\varepsilon \in (0,\varepsilon_0]$ and for every $s \in [0,1]$,

$$\|\gamma_0 \varepsilon \dot f(s) A_\varepsilon(s)\|_2 < 1 < \frac{1}{\|D_\varepsilon(s)^{-1}\|_2},$$

so $C_\varepsilon(s)$ is invertible. □

Proof of Lemma 11. *Bound on* $\|R_\varepsilon(t,s)\|_2$: Since $C_\varepsilon(t)^* = -C_\varepsilon(t)$ then $\|R_\varepsilon(t,s)\|_2 = 1$, for every $(t,s) \in [0,1/\varepsilon] \times [0,1/\varepsilon]$.

Bound on $C_\varepsilon(s)^{-1}$: We have $\|C_\varepsilon(s)^{-1}\|_2 = \|C_\varepsilon(s)\|_2$ because $C_\varepsilon(s)^* = -C_\varepsilon(s)$. Moreover, using (A.12), we get:

$$\|C_\varepsilon(s)\|_2 \leqslant (\lambda_{N_\varepsilon,\gamma_0 f(s)} - \lambda_{1,\gamma_0 f(s)}) + C\gamma_0\varepsilon^{7/8} \leqslant C\varepsilon^{-1/4}.$$

Bound on $\dot C_\varepsilon(s)$: We have:

$$\dot C_\varepsilon(s) = \varepsilon \dot D_\varepsilon(\varepsilon s) + \gamma_0\varepsilon^2 \ddot f(\varepsilon s) A_\varepsilon(\varepsilon s) + \gamma_0 \varepsilon^2 \dot f(\varepsilon s)\dot A_\varepsilon(\varepsilon s),$$

where

$$\dot D(\tau) = \mathrm{diag}\big(\mathrm{i}\gamma_0 \dot f(\tau)(\lambda'_{1,\gamma_0 f(\tau)} - \lambda'_{k,\gamma_0 f(\tau)});\ k = 2,\dots,N_\varepsilon\big).$$

Thanks to (A.19), we get:

$$\|\dot D_\varepsilon(\tau)\|_2 \leqslant C\gamma_0.$$

Using (A.18), the orthonormality of the family $(\varphi_{j,\gamma_0 f(\varepsilon t)})_{j\in\mathbb{N}^*}$ and (A.21), we get:

$$\begin{aligned}\big\|\dot{A}_\varepsilon(\tau)\big\|_2 &\leqslant \sqrt{N_\varepsilon}\sup\left\{\sum_{j=2}^{N_\varepsilon}\left|\left\langle \frac{\mathrm{d}\varphi_{j,\gamma}}{\mathrm{d}\gamma}\bigg|_{\gamma_0 f(\tau)}, \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{\gamma_0 f(\tau)}\right\rangle + \left\langle \varphi_{j,\gamma_0 f(\tau)}, \frac{\mathrm{d}^2\varphi_{k,\gamma}}{\mathrm{d}\gamma^2}\bigg|_{\gamma_0 f(\tau)}\right\rangle\right|;\right.\\ &\qquad\qquad \left. 2\leqslant k\leqslant N_\varepsilon\right\}\\ &\leqslant \sqrt{N_\varepsilon}\sup\left\{\sum_{j=2}^{N_\varepsilon}\frac{C}{jk} + \sqrt{N_\varepsilon}\left\|\frac{\mathrm{d}^2\varphi_{k,\gamma}}{\mathrm{d}\gamma^2}\bigg|_{\gamma_0 f(\tau)}\right\|_{L^2};\ 2\leqslant k\leqslant N_\varepsilon\right\}\\ &\leqslant C\big(N_\varepsilon + \sqrt{N_\varepsilon}\ln(N_\varepsilon)\big).\end{aligned}$$

Therefore,

$$\big\|\dot{C}_\varepsilon(s)\big\|_2 \leqslant C\gamma_0\varepsilon.$$

Bound on $x_{1,\varepsilon}(s)$ *and* $\dot{x}_{1,\varepsilon}(s)$: From the equation

$$\begin{cases}\dot{x}_{1,\varepsilon}(t) = \gamma_0\varepsilon\dot{f}(\varepsilon t)\langle \varLambda_\varepsilon(t), \frac{\mathrm{d}\varphi_{1,\gamma}}{\mathrm{d}\gamma}|_{\gamma_0 f(\varepsilon t)}\rangle,\\ x_{1,\varepsilon}(0) = 0,\end{cases}$$

we get, for every $s\in[0,1/\varepsilon]$,

$$\big|\dot{x}_{1,\varepsilon}(s)\big| \leqslant C\gamma_0\varepsilon \quad \text{and} \quad \big|x_{1,\varepsilon}(s)\big| \leqslant C\gamma_0.$$

Bound on $a_\varepsilon(s)$ *and* $\dot{a}_\varepsilon(s)$: Thanks to (A.18), we have:

$$\big\|a_\varepsilon(s)\big\|_2 = \left(\sum_{k=2}^{N_\varepsilon}\left|\left\langle \frac{\mathrm{d}\varphi_{1,\gamma}}{\mathrm{d}\gamma}\bigg|_{\gamma_0 f(s)}, \varphi_{k,\gamma_0 f(s)}\right\rangle\right|^2\right)^{1/2} \leqslant \left\|\frac{\mathrm{d}\varphi_{1,\gamma}}{\mathrm{d}\gamma}\bigg|_{\gamma_0 f(s)}\right\|_{L^2} \leqslant C.$$

Using (A.18) and (A.21), we get:

$$\begin{aligned}\big\|\dot{a}_\varepsilon(s)\big\|_2 &\leqslant \gamma_0\dot{f}(s)\left(\left(\sum_{k=2}^{N_\varepsilon}\left|\left\langle \frac{\mathrm{d}^2\varphi_{1,\gamma}}{\mathrm{d}\gamma^2}\bigg|_{\gamma_0 f(s)}, \varphi_{k,\gamma_0 f(s)}\right\rangle\right|^2\right)^{1/2}\right.\\ &\qquad \left. + \left(\sum_{k=2}^{N_\varepsilon}\left|\left\langle \frac{\mathrm{d}\varphi_{1,\gamma}}{\mathrm{d}\gamma}\bigg|_{\gamma_0 f(s)}, \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{\gamma_0 f(s)}\right\rangle\right|^2\right)^{1/2}\right)\\ &\leqslant C\gamma_0. \quad \square\end{aligned}$$

For $\varepsilon\in(0,1]$, we define $X_\varepsilon(t) := (x_{2,\varepsilon}(t),\dots,x_{N_\varepsilon,\varepsilon}(t))$.

Proposition 39. *There exists a constant $C > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0]$ and for every $t \in [0, 1/\varepsilon]$, $\|X_\varepsilon(t)\|_2 \leqslant C\gamma_0\varepsilon^{1/4}$.*

Proof. Let us write $Y_\varepsilon(t) := (X_\varepsilon - \widetilde{X}_\varepsilon)(t)$. Then,

$$\begin{cases} \dot{Y}_\varepsilon(t) = C_\varepsilon(t)Y_\varepsilon(t) + \gamma_0\varepsilon\dot{f}(\varepsilon t)b_\varepsilon(t), \\ Y_\varepsilon(0) = 0, \end{cases}$$

where

$$b_\varepsilon(t) := \left(\sum_{j=N_\varepsilon+1}^{\infty} x_{j,\varepsilon}(t)\left\langle\varphi_{j,\gamma_0 f(\varepsilon t)}, \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{\gamma_0 f(\varepsilon t)}\right\rangle\right)_{2\leqslant k\leqslant N_\varepsilon}.$$

Hence

$$Y_\varepsilon(t) = \int_0^t R_\varepsilon(t,s)\big(\gamma_0\varepsilon\dot{f}(\varepsilon s)b_\varepsilon(s)\big)\,\mathrm{d}s.$$

Let $s \in [0, 1/\varepsilon]$. Using the orthonormality of the family $(\varphi_{j,\gamma_0 f(\varepsilon t)})_{j\in\mathbb{N}^*}$ and (A.18), we get:

$$\begin{aligned} \|b_\varepsilon(s)\|_2^2 &= \sum_{k=2}^{N_\varepsilon}\left|\sum_{j=N_\varepsilon+1}^{\infty} x_{j,\varepsilon}(t)\left\langle\varphi_{j,\gamma_0 f(\varepsilon t)}, \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{\gamma_0 f(\varepsilon t)}\right\rangle\right|^2 \\ &\leqslant \sum_{k=2}^{N_\varepsilon}\left(\sum_{j=N_\varepsilon+1}^{\infty}\big|x_{j,\varepsilon}(t)\big|^2\right)\left(\sum_{j=N_\varepsilon+1}^{\infty}\left|\left\langle\varphi_{j,\gamma_0 f(\varepsilon t)}, \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{\gamma_0 f(\varepsilon t)}\right\rangle\right|^2\right) \\ &\leqslant \frac{C}{N_\varepsilon^4}\sum_{k=2}^{N_\varepsilon}\frac{C}{k^2}. \end{aligned}$$

Therefore,

$$\big\|Y_\varepsilon(t)\big\|_2 \leqslant C\gamma_0\varepsilon^{1/4}.$$

The inequality $\|X_\varepsilon(t)\|_2 \leqslant \|\widetilde{X}_\varepsilon(t)\|_2 + \|Y_\varepsilon(t)\|_2$ gives the conclusion. □

Proof of Theorem 13. We have:

$$\begin{aligned} \big\|\Lambda_\varepsilon(1/\varepsilon)\big\|_2 &= \left(\big|x_{1,\varepsilon}(1/\varepsilon)\big|^2 + \sum_{j=2}^{N_\varepsilon}\big|x_{j,\varepsilon}(1/\varepsilon)\big|^2 + \sum_{j=N_\varepsilon+1}^{\infty}\big|x_{j,\varepsilon}(1/\varepsilon)\big|^2\right) \\ &\leqslant \left(\big|x_{1,\varepsilon}(1/\varepsilon)\big|^2 + \big\|X_\varepsilon(1/\varepsilon)\big\|_2^2 + \frac{C}{N_\varepsilon^4}\right)^{1/2}. \end{aligned}$$

The function $x_{1,\varepsilon}(t)$ is solution of

$$\begin{cases} \dot{x}_{1,\varepsilon}(t) = \gamma_0 \varepsilon \dot{f}(\varepsilon t) \sum_{j=2}^{\infty} x_{j,\varepsilon}(t) \langle \varphi_{j,\gamma_0 f(\varepsilon t)}, \frac{\mathrm{d}\varphi_{1,\gamma}}{\mathrm{d}\gamma}|_{\gamma_0 f(\varepsilon t)} \rangle, \\ x_{1,\varepsilon}(0) = 0, \end{cases}$$

so

$$\left|x_{1,\varepsilon}(1/\varepsilon)\right| \leqslant \int_0^{1/\varepsilon} \gamma_0 \varepsilon \dot{f}(\varepsilon s) \left(\sum_{j=2}^{\infty} \left| x_{j,\varepsilon}(1/\varepsilon) \right|^2 \right)^{1/2} \left\| \frac{\mathrm{d}\varphi_{1,\gamma}}{\mathrm{d}\gamma} \bigg|_{\gamma_0 f(\varepsilon t)} \right\|_{L^2} \mathrm{d}s,$$

which gives the conclusion. □

Remark. The quasi-static deformation works because the trajectory $(\psi_{1,\gamma}, \gamma)$ is stable. If this trajectory had not been stable, we could have tried to stabilize it first with a suitable feedback as in [8].

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Appendix A. Study of $\varphi_{k,\gamma}$ and $\lambda_{k,\gamma}$

In this appendix, we state some results on the eigenvalues and the eigenfunctions of the operators $A_\gamma : D(A_\gamma) \to L^2(I, \mathbb{C})$ defined by:

$$D(A_\gamma) = H^2 \cap H_0^1(I, \mathbb{C}), \quad A_\gamma \varphi = -\frac{1}{2}\varphi'' - \gamma q \varphi.$$

The operator A_γ has an increasing sequence of eigenvalues $(\lambda_{k,\gamma})_{k \in \mathbb{N}^*}$. We call $\varphi_{k,\gamma}$ the associated eigenfunctions:

$$A_\gamma \varphi_{k,\gamma} = \lambda_{k,\gamma} \varphi_{k,\gamma}, \quad k \geqslant 1. \tag{A.1}$$

We know from [17, Chapter 7, Example 2.14] that $\varphi_{k,\gamma}$ and $\lambda_{k,\gamma}$ are analytic functions of γ:

$$\varphi_{k,\gamma} = \varphi_k + \gamma \varphi_k^{(1)} + \gamma^2 \varphi_k^{(2)} + \cdots,$$
$$\lambda_{k,\gamma} = \lambda_k + \gamma \lambda_k^{(1)} + \gamma^2 \lambda_k^{(2)} + \cdots.$$

When $\gamma = 0$, we have:

$$\lambda_k = \frac{1}{2}(k\pi)^2, \quad \varphi_k = \begin{cases} \sqrt{2}\sin(k\pi q), & \text{when } k \text{ is even}, \\ \sqrt{2}\cos(k\pi q), & \text{when } k \text{ is odd}. \end{cases}$$

The following formula is very useful is this article,

$$\langle q\varphi_{2n+1}, \varphi_{2m}\rangle = -\frac{8(-1)^{m+n}(2m)(2n+1)}{\pi^2(2n+1+2m)^2(2n+1-2m)^2}, \tag{A.2}$$

where $\langle . \rangle$ denotes the usual scalar product in $L^2(I, \mathbb{C})$. With calculations of order 1 with respect to γ, we find the following explicit expressions:

Proposition 40. *For every integer $k \geqslant 1$, $\lambda_k^{(1)} = 0$ and*

$$-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}q^2}\varphi_k^{(1)} - \gamma q\varphi_k = \lambda_k\varphi_k^{(1)}. \tag{A.3}$$

If k is an even integer, then

$$\varphi_k^{(1)} = \frac{16(-1)^{k/2}k}{\pi^4}\sum_{j=0}^{+\infty}\frac{(-1)^j(2j+1)}{(k+2j+1)^3(k-2j-1)^3}\varphi_{2j+1}. \tag{A.4}$$

If k is an odd integer, then

$$\varphi_k^{(1)} = \frac{16(-1)^{(k-1)/2}k}{\pi^4}\sum_{j=1}^{+\infty}\frac{(-1)^j(2j)}{(k+2j)^3(k-2j)^3}\varphi_{2j}. \tag{A.5}$$

We introduce, for every integer $k \geqslant 1$, the functions:

$$\widetilde{\varphi}_{k,\gamma} := \varphi_k + \gamma\varphi_k^{(1)}.$$

Eqs. (A.1) and (A.3) give:

$$A_\gamma\widetilde{\varphi}_{k,\gamma} + \gamma^2 q\varphi_k^{(1)} = \lambda_k\widetilde{\varphi}_{k,\gamma}. \tag{A.6}$$

We recall in the next proposition bounds given in [17, Chapter 7 Example 2.14, Chapter 2 Problem 3.7].

Proposition 41. *There exist positive constants γ^*, C^* and C_* such that, for every $\gamma \neq 0$ satisfying $|\gamma| < \gamma^*$ and for every $k \in \mathbb{N}^*$,*

$$\|\varphi_{k,\gamma}-\varphi_k\|_{L^2(I)}\leqslant\frac{C^*\gamma}{k},\tag{A.7}$$

$$\|\varphi_{k,\gamma}-\widetilde{\varphi}_{k,\gamma}\|_{L^2(I)}\leqslant\frac{C^*\gamma^2}{k^2},\tag{A.8}$$

$$\left\|\frac{\mathrm{d}}{\mathrm{d}x}(\varphi_{k,\gamma}-\varphi_k)\right\|_{L^2(I)}\leqslant C^*\gamma,\tag{A.9}$$

$$\left\|\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\varphi_{k,\gamma}-\varphi_k)\right\|_{L^2(I)}\leqslant C^*\gamma(1+k),\tag{A.10}$$

$$|\lambda_{k,\gamma}-\lambda_k|\leqslant\frac{C^*\gamma^2}{k},\tag{A.11}$$

$$C_*\lambda_k\leqslant\lambda_{k,\gamma}\leqslant C^*\lambda_k,\tag{A.12}$$

$$\left|\frac{1}{\lambda_{k,\gamma}}-\frac{1}{\lambda_k}\right|\leqslant\frac{C^*\gamma^2}{k^5}.\tag{A.13}$$

The vectors $\varphi_{k,\gamma}$ and the complex numbers $\lambda_{k,\gamma}$ are analytic functions of the parameter γ, so we can consider their derivatives with respect to γ. We introduce the notations:

$$\left.\frac{\mathrm{d}^k\varphi_{k,\gamma}}{\mathrm{d}\gamma^k}\right|_{\gamma_0},$$

for the kth derivative of the function $\gamma\mapsto\varphi_{k,\gamma}$ evaluated at the point $\gamma=\gamma_0$ and

$$\lambda'_{k,\gamma_0},\ \lambda''_{k,\gamma_0},\ \lambda'''_{k,\gamma_0}$$

for the first, second and third derivative of the function $\gamma\mapsto\lambda_{k,\gamma}$ evaluated at the point $\gamma=\gamma_0$.

Proposition 42. *We have*:

$$A_{\gamma_0}\left.\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\right|_{\gamma_0}=(q+\lambda'_{k,\gamma_0})\varphi_{k,\gamma_0}+\lambda_{k,\gamma_0}\left.\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\right|_{\gamma_0},\tag{A.14}$$

$$\lambda'_{k,\gamma_0}=-\langle q\varphi_{k,\gamma_0},\varphi_{k,\gamma_0}\rangle,\tag{A.15}$$

$$\left.\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\right|_{\gamma_0}=\sum_{j=1,\ j\neq k}^{\infty}\frac{\langle q\varphi_{k,\gamma_0},\varphi_{j,\gamma_0}\rangle}{\lambda_{j,\gamma_0}-\lambda_{k,\gamma_0}}\varphi_{j,\gamma_0}.\tag{A.16}$$

In particular,

$$\left.\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\right|_0=\varphi_k^{(1)}=\frac{16k}{\pi^4}\sum_{P(j)\neq P(k)}\frac{(-1)^{(k+j+1)/2}j}{(j+k)^3(j-k)^3}\varphi_j,$$

where the sum is taken over all integers j such that j and k have different parity. There exists a constant $C^* > 0$ such that for every $\gamma_0 \in [-\gamma^*, \gamma^*]$,

$$\left\| \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{\gamma_0} - \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_0 \right\|_{L^2} \leqslant \frac{C^*\gamma_0}{k^2}, \tag{A.17}$$

$$\left\| \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{\gamma_0} \right\|_{L^2} \leqslant \frac{C^*}{k}, \tag{A.18}$$

$$|\lambda'_{k,\gamma_0}| \leqslant \frac{C^*\gamma_0}{k}. \tag{A.19}$$

Proof. To get Eq. (A.14), we derive the equation on $\varphi_{k,\gamma}$ with respect to γ. Considering the scalar product of Eq. (A.14) with φ_{k,γ_0} we get (A.15). We compute the decomposition (A.16) using Eq. (A.14). In the case $\gamma_0 = 0$ the formulas (A.4) and (A.5) give the result. We first prove the bound (A.18) for $\gamma_0 = 0$. In this case, we have:

$$\left\| \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_0 \right\|_{L^2}^2 = \left(\frac{16k}{\pi^4}\right)^2 \sum_{P(j)\neq P(k)} \frac{j^2}{(j+k)^6(j-k)^6}.$$

In order to compute this sum, we decompose the fraction,

$$F_k(X) = \frac{X^2}{(X+k)^6(X-k)^6},$$

in the following way:

$$\begin{aligned} F_k(X) = &-\frac{7}{512k^9}\left(\frac{1}{X+k} - \frac{1}{X-k}\right) - \frac{7}{512k^8}\left(\frac{1}{(X+k)^2} + \frac{1}{(X-k)^2}\right) \\ &- \frac{1}{128k^7}\left(\frac{1}{(X+k)^3} - \frac{1}{(X-k)^3}\right) + \frac{1}{256k^6}\left(\frac{1}{(X+k)^4} + \frac{1}{(X-k)^4}\right) \\ &+ \frac{1}{64k^5}\left(\frac{1}{(X+k)^5} - \frac{1}{(X-k)^5}\right) + \frac{1}{64k^4}\left(\frac{1}{(X+k)^6} + \frac{1}{(X-k)^6}\right) \end{aligned}$$

and we sum each term. We find:

$$\begin{aligned} \sum_{P(j)\neq P(k)} F_k(j) = &-\frac{7}{512k^9}\frac{1}{k} - \frac{7}{512k^9}\left(2S_2 - \frac{1}{k^2}\right) - \frac{1}{128k^7}\frac{1}{k^3} + \frac{1}{256k^6}\left(2S_4 - \frac{1}{k^4}\right) \\ &+ \frac{1}{64k^5}\frac{1}{k^5} + \frac{1}{64k^4}\left(2S_6 - \frac{1}{k^6}\right), \end{aligned}$$

where $S_a = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^a}$ for $a = 2, 4, 6$.

Thanks to the expression (A.16), we have:

$$\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\Big|_{\gamma_0}-\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\Big|_{0}=\sum_{j=1,\ j\neq k}^{\infty}\left(\frac{1}{\lambda_{j,\gamma_0}-\lambda_{k,\gamma_0}}-\frac{1}{\lambda_j-\lambda_k}\right)\langle q\varphi_{k,\gamma_0},\varphi_{j,\gamma_0}\rangle\varphi_{j,\gamma_0}$$
$$+\sum_{j=1,\ j\neq k}^{\infty}\frac{1}{\lambda_j-\lambda_k}\big(\langle q\varphi_{k,\gamma_0},\varphi_{j,\gamma_0}\rangle-\langle q\varphi_k,\varphi_j\rangle\big)\varphi_{j,\gamma_0}$$
$$+\sum_{j=1,\ j\neq k}^{\infty}\frac{\langle q\varphi_k,\varphi_j\rangle}{\lambda_j-\lambda_k}(\varphi_{j,\gamma_0}-\varphi_j),$$

$$\left\|\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\Big|_{\gamma_0}-\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\Big|_{0}\right\|_{L^2}\leqslant\left(\sum_{j=1,\ j\neq k}^{\infty}\left(\frac{1}{\lambda_{j,\gamma_0}-\lambda_{k,\gamma_0}}-\frac{1}{\lambda_j-\lambda_k}\right)^2\right)^{1/2}$$
$$+\left(\sum_{j=1,\ j\neq k}^{\infty}\left(\frac{\langle q\varphi_{k,\gamma_0},\varphi_{j,\gamma_0}\rangle-\langle q\varphi_k,\varphi_j\rangle}{\lambda_j-\lambda_k}\right)^2\right)^{1/2}$$
$$+\sum_{j=1,\ j\neq k}^{\infty}\left|\frac{\langle q\varphi_k,\varphi_j\rangle}{\lambda_j-\lambda_k}\right|\|\varphi_{j,\gamma_0}-\varphi_j\|_{L^2}.$$

For the study of the first term of the right-hand side, we have:

$$\left|\frac{1}{\lambda_{j,\gamma_0}-\lambda_{k,\gamma_0}}-\frac{1}{\lambda_j-\lambda_k}\right|\leqslant\left(\frac{C\gamma_0}{j}+\frac{C\gamma_0}{k}\right)\frac{1}{(j^2-k^2)^2},$$
$$\sum_{j=1,\ j\neq k}^{\infty}\left(\frac{1}{\lambda_{j,\gamma_0}-\lambda_{k,\gamma_0}}-\frac{1}{\lambda_j-\lambda_k}\right)^2\leqslant\gamma_0\sum_{j=1,\ j\neq k}^{\infty}\frac{1}{j^2(j+k)^4(j-k)^4}$$
$$+\frac{\gamma_0}{k^2}\sum_{j=1,\ j\neq k}^{\infty}\frac{1}{(j+k)^4(j-k)^4}.$$

We compute explicitly the two sums and we get:

$$\left(\sum_{j=1,\ j\neq k}^{\infty}\left(\frac{1}{\lambda_{j,\gamma_0}-\lambda_{k,\gamma_0}}-\frac{1}{\lambda_j-\lambda_k}\right)^2\right)^{1/2}\leqslant\frac{C\gamma_0}{k^3}.$$

For the study of the second term of the right-hand side, using $\varphi_{l,\gamma_0}=(\varphi_{l,\gamma_0}-\varphi_l)+\varphi_l$ and (A.7), we get:

$$\big|\langle q\varphi_{k,\gamma_0},\varphi_{j,\gamma_0}\rangle-\langle q\varphi_k,\varphi_j\rangle\big|\leqslant\frac{C^*\gamma_0}{k}+\frac{C^*\gamma_0}{j},$$

$$\sum_{j=1,\ j\neq k}^{\infty}\left(\frac{\langle q\varphi_{k,\gamma_0},\varphi_{j,\gamma_0}\rangle-\langle q\varphi_k,\varphi_j\rangle}{\lambda_j-\lambda_k}\right)^2\leqslant\left(\frac{C^*\gamma_0}{k}\right)^2\sum_{j=1,\ j\neq k}^{\infty}\frac{1}{(j+k)^2(j-k)^2}+(C^*\gamma_0)^2\sum_{j=1,\ j\neq k}^{\infty}\frac{1}{j^2(j+k)^2(j-k)^2}.$$

We compute explicitly the two sums and, we get:

$$\left(\sum_{j=1,\ j\neq k}^{\infty}\left(\frac{\langle q\varphi_{k,\gamma_0},\varphi_{j,\gamma_0}\rangle-\langle q\varphi_k,\varphi_j\rangle}{\lambda_j-\lambda_k}\right)^2\right)^{1/2}\leqslant\frac{C\gamma_0}{k^2}.$$

The third term of the right-hand side is bounded by:

$$\sum_{j=1,\ j\neq k}^{\infty}\frac{Ckj}{(k+j)^2(k-j)^2}\frac{1}{|j^2-k^2|}\frac{C^*\gamma_0}{j}\leqslant CC^*\gamma_0 k\sum_{j=1,\ j\neq k}^{\infty}\frac{1}{(k+j)^3|k-j|^3}.$$

We compute explicitly the last sum and we get:

$$\sum_{j=1,\ j\neq k}^{\infty}\left|\frac{\langle q\varphi_k,\varphi_j\rangle}{\lambda_j-\lambda_k}\right|\|\varphi_{j,\gamma_0}-\varphi_j\|_{L^2}\leqslant\frac{C'\gamma_0}{k^2}.\quad\square$$

Proposition 43. *We have*:

$$A_{\gamma_0}\frac{\mathrm{d}^2\varphi_{k,\gamma}}{\mathrm{d}\gamma^2}\bigg|_{\gamma_0}=\lambda_{k,\gamma_0}\frac{\mathrm{d}^2\varphi_{k,\gamma}}{\mathrm{d}\gamma^2}\bigg|_{\gamma_0}+2(q+\lambda'_{k,\gamma_0})\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{\gamma_0}+\lambda''_{k,\gamma_0}\varphi_{k,\gamma_0}.\tag{A.20}$$

There exists a constant $C^*>0$ such that for every $\gamma\in[-\gamma^*,\gamma^*]$,

$$\left\|\frac{\mathrm{d}^2\varphi_{k,\gamma}}{\mathrm{d}\gamma^2}\bigg|_{\gamma_0}\right\|_{L^2}\leqslant\frac{C^*}{k^2},\tag{A.21}$$

$$|\lambda''_{k,\gamma_0}|\leqslant\frac{C^*}{k}.\tag{A.22}$$

Proof. The explicit expression,

$$\lambda''_{k,\gamma_0}=-2\left\langle q\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{\gamma_0},\varphi_{k,\gamma_0}\right\rangle,\tag{A.23}$$

together with (A.18) give the bound (A.22). Using Eq. (A.20), we compute the coefficients z_j in the decomposition:

$$\frac{\mathrm{d}^2\varphi_{k,\gamma}}{\mathrm{d}\gamma^2}\bigg|_{\gamma_0}=\sum_{j=1}^{\infty}z_j\varphi_{j,\gamma_0},$$

$$z_k = -\left\| \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\Big|_{\gamma_0} \right\|_{L^2}^2,$$

$$z_j = 2\lambda'_{k,\gamma_0} \frac{\langle q\varphi_{k,\gamma_0}, \varphi_{j,\gamma_0}\rangle}{(\lambda_{j,\gamma_0} - \lambda_{k,\gamma_0})^2} + 2\frac{\langle q \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}|_{\gamma_0}, \varphi_{j,\gamma_0}\rangle}{\lambda_{j,\gamma_0} - \lambda_{k,\gamma_0}}, \quad \text{for } j \neq k.$$

Thanks to these expressions and (A.19), (A.18), (A.11), we get:

$$\left\| \frac{\mathrm{d}^2\varphi_{k,\gamma}}{\mathrm{d}\gamma^2}\Big|_{\gamma_0} \right\|_{L^2}^2 \leqslant \frac{C}{k^2} \sum_{j=1,\ j\neq k}^{\infty} \frac{1}{(j+k)^2(j-k)^2} + \frac{C^*}{k^4},$$

and we conclude computing the infinite sum. □

Proposition 44. *We have*:

$$A_{\gamma_0} \frac{\mathrm{d}^3\varphi_{k,\gamma}}{\mathrm{d}\gamma^3}\Big|_{\gamma_0} = \lambda_{k,\gamma_0} \frac{\mathrm{d}^3\varphi_{k,\gamma}}{\mathrm{d}\gamma^3}\Big|_{\gamma_0} + 3(q + \lambda'_{k,\gamma_0}) \frac{\mathrm{d}^2\varphi_{k,\gamma}}{\mathrm{d}\gamma^2}\Big|_{\gamma_0} + 3\lambda''_{k,\gamma_0} \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\Big|_{\gamma_0} + \lambda'''_{k,\gamma_0}\varphi_{k,\gamma_0}. \tag{A.24}$$

There exists a constant $C^* > 0$ such that for every $\gamma \in [-\gamma^*, \gamma^*]$,

$$\left\| \frac{\mathrm{d}^3\varphi_{k,\gamma}}{\mathrm{d}\gamma^3}\Big|_{\gamma_0} \right\|_{L^2} \leqslant \frac{C^*}{k^3}, \tag{A.25}$$

$$|\lambda'''_{k,\gamma_0}| \leqslant \frac{C^*}{k^2}. \tag{A.26}$$

Proof. Considering the scalar product of (A.24) with φ_{k,γ_0} we get the explicit expression:

$$\lambda'''_{k,\gamma_0} = -3(q + \lambda'_{k,\gamma_0}) \left\langle \frac{\mathrm{d}^2\varphi_{k,\gamma}}{\mathrm{d}\gamma^2}\Big|_{\gamma_0}, \varphi_{k,\gamma_0} \right\rangle - 3\lambda''_{k,\gamma_0} \left\langle \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\Big|_{\gamma_0}, \varphi_{k,\gamma_0} \right\rangle. \tag{A.27}$$

Then using (A.19), (A.21), (A.22), (A.18), we get the bound (A.26). Using Eq. (A.24), we can compute the coefficients w_j in the decomposition:

$$\frac{\mathrm{d}^3\varphi_{k,\gamma}}{\mathrm{d}\gamma^3}\Big|_{\gamma_0} = \sum_{j=1}^{\infty} w_j \varphi_{j,\gamma_0},$$

$$w_k = -3\left\langle \frac{\mathrm{d}^2\varphi_{k,\gamma}}{\mathrm{d}\gamma^2}\Big|_{\gamma_0}, \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\Big|_{\gamma_0} \right\rangle,$$

$$w_j = \frac{3}{\lambda_{j,\gamma_0} - \lambda_{k,\gamma_0}} \left(\lambda'_{k,\gamma_0} \left\langle \frac{\mathrm{d}^2\varphi_{k,\gamma}}{\mathrm{d}\gamma^2}\Big|_{\gamma_0}, \varphi_{j,\gamma_0} \right\rangle + \left\langle q \frac{\mathrm{d}^2\varphi_{k,\gamma}}{\mathrm{d}\gamma^2}\Big|_{\gamma_0}, \varphi_{j,\gamma_0} \right\rangle \right), \quad j \neq k.$$

Therefore

$$\left\| \frac{\mathrm{d}^3\varphi_{k,\gamma}}{\mathrm{d}\gamma^3}\bigg|_{\gamma_0} \right\|_{L^2}^2 = \sum_{j=1}^{\infty} |w_j|^2 \leqslant \left(\frac{C}{k^6}+\frac{C}{k^4}\right)\sum_{j=1}^{\infty}\frac{1}{(j+k)^2(j-k)^2} + \frac{(C^*)^4}{k^6}$$

and we compute the infinite sum. □

Appendix B. Existence and bounds for the solutions

This appendix is dedicated to existence and regularity results for the solutions of the following system:

$$\begin{cases} \mathrm{i}\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial q^2} - u(t)q\psi + f(t), \quad q\in I,\ t\in[0,T],\\ \psi(0)=\psi_0,\\ \psi(t,-1/2)=\psi(t,1/2)=0. \end{cases} \tag{B.1}$$

We also give some bounds on the solution in spaces $C^0([0,T],H^s(I,\mathbb{C}))$ for $s=0,\dots,7$, useful in the application of Nash–Moser theorem.

Let us recall A_γ is the operator $A_\gamma : D(A_\gamma)\to L^2(I,\mathbb{C})$ defined by:

$$D(A_\gamma) = H^2\cap H_0^1(I,\mathbb{C}), \quad A_\gamma\varphi = -\frac{1}{2}\varphi'' - \gamma q\varphi,$$

and $(T_\gamma(t))_{t\in\mathbb{R}}$ is the group of isometries of $L^2(I,\mathbb{C})$ with infinitesimal generator $-\mathrm{i}A_\gamma$, more precisely, for $\varphi\in L^2(I,\mathbb{C})$ and for every $t\in\mathbb{R}$,

$$T_\gamma(t)\varphi = \sum_{k=1}^{+\infty}\langle\varphi,\varphi_{k,\gamma}\rangle \mathrm{e}^{-\mathrm{i}\lambda_{k,\gamma}t}\varphi_{k,\gamma}.$$

Proposition 45. *Let $T>0$, $\gamma\in\mathbb{R}$ and $u\in L^1((0,T),\mathbb{R})$ be such that $\|u-\gamma\|_{L^1}<\sqrt{2}/\sqrt{17}$. Let $E\in\{L^2(I,\mathbb{C}), H_0^1(I,\mathbb{C}), H^2\cap H_0^1(I,\mathbb{C})\}$, $\psi_0\in E$ and $f\in L^1((0,T),E)$. There exists a unique solution ψ in $C^0([0,T],E)$ of*

$$\psi(t) = T_\gamma(t)\psi_0 + \int_0^t T_\gamma(t-s)\big[\mathrm{i}\big(u(s)-\gamma\big)q\psi(s)+f(s)\big]\,\mathrm{d}s, \tag{B.2}$$

in $L^2(I,\mathbb{C})$, for every $t\in[0,T]$. Moreover,

$$\|\psi\|_{C^0([0,T],E)} \leqslant \mathrm{e}\big(\|\psi_0\|_E + \|f\|_{L^1((0,T),E)}\big).$$

Proof. For the existence, we use Banach fix point theorem:

$$\Omega : C^0\big([0,T],E\big) \to C^0\big([0,T],E\big),$$
$$\psi \to T_\gamma(t)\psi_0 + \int_0^t T_\gamma(t-s)\big[\mathrm{i}\big(u(s)-\gamma\big)q\psi(s) + f(s)\big]\,\mathrm{d}s,$$

Let $\psi_1, \psi_2 \in C^0([0,T],E)$. Since $T_\gamma(\tau)$ is an isometry of E, for every $\tau \in \mathbb{R}$, we have:

$$\big\|\Omega(\psi_1)(t) - \Omega(\psi_2)(t)\big\|_E \leqslant \int_0^t \big|u(s)-\gamma\big|\frac{\sqrt{17}}{\sqrt{2}}\big\|(\psi_1-\psi_2)(s)\big\|_E\,\mathrm{d}s,$$

so $\|\Omega(\psi_1) - \Omega(\psi_2)\|_{C^0([0,T],E)} \leqslant \|u-\gamma\|_{L^1}\frac{\sqrt{17}}{\sqrt{2}}\|\psi_1-\psi_2\|_{C^0([0,T],E)}$.

For the bound, we apply Gronwall's lemma to the inequality:

$$\big\|\psi(t)\big\|_E \leqslant \|\psi_0\|_E + \|f\|_{L^1([0,T],E)} + \int_0^t \big|u(s)-\gamma\big|\frac{\sqrt{17}}{\sqrt{2}}\big\|\psi(s)\big\|_E\,\mathrm{d}s. \quad \square$$

Remark. An existence result can be proved for every $u \in L^1((0,T),\mathbb{R})$, considering a partition of $[0,T]$:

$$[0,T] = \bigsqcup_{i=1}^{i=N}[T_{i-1},T_{\mathrm{i}}]$$

such that $\|u-\gamma\|_{L^1((T_{i-1},T_{\mathrm{i}}),\mathbb{R})} < \sqrt{2}/\sqrt{17}$ for $i = 1,\dots,N$.

Proposition 46. *Let $T>0$, $\gamma \in \mathbb{R}$ and $u \in L^1((0,T),\mathbb{R})$ be such that $\|u-\gamma\|_{L^1} < \sqrt{2}/\sqrt{17}$. Let $\psi_0 \in H^2 \cap H_0^1(I,\mathbb{C})$ and $f \in L^1((0,T),H^2 \cap H_0^1(I,\mathbb{C}))$. If $u \in C^0([0,T],\mathbb{R})$ and $f \in C^0([0,T],L^2(I,\mathbb{C}))$ then the function ψ defined by* (B.2) *belongs to $C^1([0,T],L^2(I,\mathbb{C}))$. It is the unique solution in $C^0([0,T],H^2\cap H_0^1(I,\mathbb{C})) \cap C^1([0,T],L^2(I,\mathbb{C}))$ of*

$$\begin{cases} \frac{\partial\psi}{\partial t} = \frac{\mathrm{i}}{2}\frac{\partial^2\psi}{\partial q^2} + \mathrm{i}u(t)q\psi + f(t), \quad q \in I,\ t\in[0,T],\\ \psi(0) = \psi_0,\\ \psi(t,-1/2) = \psi(t,1/2) = 0. \end{cases} \tag{B.3}$$

Proof. Clearly, ψ satisfies Eqs. (B.3). Let us prove the uniqueness of the solution of (B.3). Let $\psi_1,\psi_2 \in C^0([0,T],H^2\cap H_0^1(I,\mathbb{C}))\cap C^1([0,T],L^2(I,\mathbb{C}))$ be solutions of this system. Then $\Lambda := \psi_1 - \psi_2$ solves:

$$\begin{cases} \mathrm{i}\frac{\partial \Lambda}{\partial t} = -\frac{1}{2}\frac{\partial^2 \Lambda}{\partial q^2} - u(t)q\Lambda, \quad q \in I,\ t \in [0,T], \\ \Lambda(0) = 0, \\ \Lambda(t,-1/2) = \Lambda(t,1/2) = 0. \end{cases}$$

The first equation of this system gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda(t)\|_{L^2}^2 = \langle \dot{\Lambda}(t), \Lambda(t)\rangle + \langle \Lambda(t), \dot{\Lambda}(t)\rangle = 0,$$

so $\Lambda \equiv 0$. □

Corollary 2. *Let $T > 0$, $\gamma \in \mathbb{R}$ and $u \in L^1((0,T),\mathbb{R})$ be such that $\|u-\gamma\|_{L^1((0,T),\mathbb{R})} < \sqrt{2}/\sqrt{17}$. Let $\psi_0 \in H^2 \cap H_0^1(I,\mathbb{C})$ and $f \in L^1((0,T), H^2 \cap H_0^1(I,\mathbb{C})) \cap C^0([0,T], L^2(I,\mathbb{C}))$. The solution ψ in $C^0([0,T], H^2 \cap H_0^1(I,\mathbb{C}))$ of*

$$\psi(t) = T_\gamma(t)\psi_0 + \int_0^t T_\gamma(t-s)\big[\mathrm{i}\big(u(s)-\gamma\big)q\psi(s) + f(s)\big]\,\mathrm{d}s$$

in $L^2(I,\mathbb{C})$ for every $t \in [0,T]$ also solves:

$$\psi(t) = T_{\gamma_1}(t)\psi_0 + \int_0^t T_{\gamma_1}(t-s)\big[\mathrm{i}\big(u(s)-\gamma_1\big)q\psi(s) + f(s)\big]\,\mathrm{d}s$$

in $L^2(I,\mathbb{C})$, for every $t \in [0,T]$, for every $\gamma_1 \in \mathbb{R}$ such that $\|u-\gamma_1\|_{L^1((0,T),\mathbb{R})} < \sqrt{2}/\sqrt{17}$.

Proof. We introduce the notations:

$$B_{L^1}\left(\gamma, \frac{\sqrt{2}}{\sqrt{17}}\right) := \left\{u \in L^1\big((0,T),\mathbb{R}\big);\ \|u\|_{L^1} < \frac{\sqrt{2}}{\sqrt{17}}\right\},$$

$$D(\Omega_\gamma) := B_{L^1}\left(\gamma, \frac{\sqrt{2}}{\sqrt{17}}\right) \times H_0^1(I,\mathbb{C}) \times L^1\big((0,T), H_0^1(I,\mathbb{C})\big),$$

$$\Omega_\gamma : D(\Omega_\gamma) \to C^0\big([0,T], H^2 \cap H_0^1(I,\mathbb{C})\big),$$

$$(u,\psi_0,f) \mapsto \psi \text{ solution of (B.2).}$$

The previous result shows $\Omega_\gamma = \Omega_{\gamma_1}$ on

$$D(\Omega_\gamma) \cap D(\Omega_{\gamma_1}) \cap \big\{C^0\big([0,T],\mathbb{R}\big) \times \big(H^2 \cap H_0^1\big)(I,\mathbb{C}) \\ \times \big[C^0\big([0,T], L^2(I,\mathbb{C})\big) \cap L^1\big((0,T), H^2(I,\mathbb{C})\big)\big]\big\},$$

which is dense in $D(\Omega_\gamma) \cap D(\Omega_{\gamma_1})$. We just need to prove that Ω_γ and Ω_{γ_1} are continuous to conclude. Gronwall's lemma gives, when $\Omega_\gamma(u_1, \psi_{1,0}, f_1) = \psi_1$ and $\Omega_\gamma(u_2, \psi_{2,0}, f_2) = \psi_2$,

$$\|\psi_1 - \psi_2\|_{C^0([0,T],H_0^1)} \leqslant C\big[\|\psi_{1,0} - \psi_{2,0}\|_{H_0^1} + \|u_1 - u_2\|_{L^1([0,T],\mathbb{R})}\|\psi_2\|_{C^0([0,T],H_0^1)} + \|f_1 - f_2\|_{L^1([0,T],H_0^1)}\big].$$

So Ω_γ is continuous and $\Omega_\gamma = \Omega_{\gamma_1}$ on $D(\Omega_\gamma) \cap D(\Omega_{\gamma_1})$. □

This corollary allows us to give the following definition:

Definition 2. Let $\psi_0 \in H_0^1(I,\mathbb{C})$, $u \in L^1((0,T),\mathbb{R})$, $f \in L^1((0,T),H_0^1(I))$ with $\|u\|_{L^1} < \sqrt{2}/\sqrt{17}$. The generalized solution of (B.1) is the unique function $\psi \in C^0([0,T], H_0^1(I,\mathbb{C}))$ solution of

$$\psi(t) = T(t)\psi_0 + \int_0^t T(t-s)\big[\mathrm{i}u(s)q\psi(s) + f(s)\big]\,\mathrm{d}s,$$

in $L^2(I,\mathbb{C})$ for every $t \in [0,T]$. Then, for every γ such that $\|u - \gamma\|_{L^1} \leqslant \sqrt{2}/\sqrt{17}$, we have:

$$\psi(t) = T_\gamma(t)\psi_0 + \int_0^t T_\gamma(t-s)\big[\mathrm{i}\big(u(s) - \gamma\big)q\psi(s) + f(s)\big]\,\mathrm{d}s,$$

in $L^2(I,\mathbb{C})$ for every $t \in [0,T]$.

Proposition 47. *Let $u \in W^{1,1}((0,T),\mathbb{R})$ be such that $\|u\|_{L^1} < \sqrt{2}/\sqrt{17}$, $f \in L^1((0,T), H^2 \cap H_0^1(I,\mathbb{C})) \cap W^{1,1}((0,T), L^2(I,\mathbb{C}))$ and ψ be the solution of* (B.1)*. Let $\gamma := u(0)$. Then the function $\varphi := \frac{\partial\psi}{\partial t}$ is the solution in $C^0([0,T], L^2(I,\mathbb{C}))$ of*

$$\begin{cases} \varphi(t) = T(t)\varphi_0 + \int_0^t T(t-s)l(\mathrm{i}u(s)(q\varphi)(s) + g(s))\,\mathrm{d}s, \\ \varphi_0 = -\mathrm{i}A_{u(0)}\psi_0 + f(0), \\ g(s) = \mathrm{i}\dot{u}(s)(q\psi)(s) + \dot{f}(s). \end{cases}$$

- *If $\psi_0 \in H^3_{(\gamma)}(I,\mathbb{C})$ and $f \in W^{1,1}((0,T), H_0^1(I,\mathbb{C}))$ then, $\varphi \in C^0([0,T], H_0^1(I,\mathbb{C}))$, $\psi \in C^0([0,T], H^3(I,\mathbb{C}))$, $A_{u(t)}\psi(t) \in C^0([0,T], H_0^1(I,\mathbb{C}))$ and we have the following upper bounds when $\|u\|_{H^1} \leqslant 1$:*

$$\|\varphi\|_{C^0([0,T],H^1)} \leqslant C\big(\|\psi_0\|_{H^3} + \|f\|_{W^{1,1}((0,T),H^1)}\big),$$
$$\|\psi\|_{C^0([0,T),H^3)} \leqslant C\big(\|\psi_0\|_{H^3} + \|f\|_{W^{1,1}((0,T),H^1)} + \|f\|_{L^1((0,T),H^2)}\big).$$

- *If* $\psi_0 \in H^4_{(\gamma)}(I,\mathbb{C})$ *and* $f \in W^{1,1}([0,T], H^2 \cap H^1_0(I,\mathbb{C}))$ *then,* $\varphi \in C^0([0,T], H^2 \cap H^1_0(I,\mathbb{C}))$, $\psi \in C^0([0,T], H^4(I,\mathbb{C}))$, $A_{u(t)}\psi(t) \in C^0([0,T], H^2 \cap H^1_0(I,\mathbb{C}))$ *and we have the following bounds when* $\|u\|_{W^{1,1}} \leqslant 1$:

$$\|\varphi\|_{C^0([0,T],H^2)} \leqslant C\big(\|\psi_0\|_{H^4} + \|f\|_{W^{1,1}((0,T),H^2)}\big),$$
$$\|\psi\|_{C^0([0,T),H^4)} \leqslant C\big(\|\psi_0\|_{H^4} + \|f\|_{W^{1,1}((0,T),H^2)}\big).$$

Proof. Deriving the relation on ψ we get the relation on φ. For the sequel, we apply the previous results on φ. Gronwall's lemma gives constants $C = C(u)$ which are uniformly bounded with respect to u in a bounded subset of $W^{1,1}((0,T),\mathbb{R})$, we chose the constant 1 arbitrarily. □

Proposition 48. *Under the same assumptions as in the previous proposition, if* $u \in C^1([0,T],\mathbb{R})$ *and* $f \in C^1([0,T], L^2(I,\mathbb{C}))$*, then* $\varphi \in C^1([0,T], L^2(I,\mathbb{C}))$*. It is the unique solution in* $C^0([0,T], H^2 \cap H^1_0(I,\mathbb{C})) \cap C^1([0,T], L^2(I,\mathbb{C}))$ *of*

$$\begin{cases} \frac{\partial\varphi}{\partial t} = \frac{\mathrm{i}}{2}\frac{\partial^2\varphi}{\partial q^2} + \mathrm{i}u(t)q\varphi + g(t), \quad q \in I,\ t \in [0,T],\\ \varphi(0) = \varphi_0,\\ \varphi(t,-1/2) = \varphi(t,1/2) = 0. \end{cases}$$

Proposition 49. *Let* $u \in W^{2,1}((0,T),\mathbb{R})$ *be such that* $\|u\|_{L^1} < \sqrt{2}/\sqrt{17}$, $\dot{u}(0) = \dot{u}(T) = 0$*. Let* $f \in W^{1,1}((0,T), H^2 \cap H^1_0(I,\mathbb{C})) \cap W^{2,1}((0,T), L^2(I,\mathbb{C}))$, ψ *be the solution of* (B.1) *and* $\varphi := \frac{\partial\psi}{\partial t}$*. Let* $\gamma := u(0)$*. Then the function* $\xi := \frac{\partial\varphi}{\partial t}$ *is the solution in* $C^0([0,T], L^2(I,\mathbb{C}))$ *of*

$$\begin{cases} \xi(t) = T(t)\varphi_0 + \int_0^t T(t-s)[\mathrm{i}u(s)(q\xi)(s) + h(s)]\,\mathrm{d}s,\\ \xi_0 = -\mathrm{i}A_{u(0)}\varphi_0 + g(0),\\ h(s) = 2\mathrm{i}\dot{u}(s)(q\varphi)(s) + \mathrm{i}\ddot{u}(s)(q\psi)(s) + \ddot{f}(s). \end{cases}$$

- *If* $\psi_0 \in H^5(I,\mathbb{C})$, $f \in W^{2,1}((0,T), H^1_0(I,\mathbb{C}))$ *and* $A_\gamma\psi_0 \in H^2 \cap H^1_0(I,\mathbb{C})$, $A_\gamma^2\psi_0 + A_\gamma f(0) \in H^1_0(I,\mathbb{C})$*, then,* $\xi \in C^0([0,T], H^1_0(I,\mathbb{C}))$ *and* $\varphi \in C^0([0,T], H^3(I,\mathbb{C}))$*. If* $f \in C^0([0,T], H^3(I,\mathbb{C}))$ *then* $\psi \in C^0([0,T], H^5(I,\mathbb{C}))$, $A_{u(t)}\psi(t) \in C^0([0,T], H^2 \cap H^1_0(I,\mathbb{C}))$, $A^2_{u(t)}\psi(t) + A_{u(t)}f(t) \in C^0([0,T], H^1_0(I,\mathbb{C}))$ *and we have the following bounds*:

$$\|\xi\|_{C^0([0,T],H^1)} \leqslant C\big\{\|\psi_0\|_{H^5} + \big\|f(0)\big\|_{H^3} + \|f\|_{W^{2,1}((0,T),H^1)} + \|u\|_{W^{2,1}}\big(\|\psi_0\|_{H^1} + \|f\|_{L^1((0,T),H^1)}\big)\big\},$$

$$\|\varphi\|_{C^0([0,T),H^3)} \leqslant C\big\{\|\psi_0\|_{H^5} + \big\|f(0)\big\|_{H^3} + \|f\|_{W^{1,1}((0,T),H^2)} + \|f\|_{W^{2,1}((0,T),H^1)} + \|u\|_{W^{2,1}}\big(\|\psi_0\|_{H^1} + \|f\|_{L^1((0,T),H^1)}\big)\big\},$$

$$\|\psi\|_{C^0([0,T],H^5)} \leqslant C\big\{\|\psi_0\|_{H^5} + \|f\|_{C^0([0,T],H^3)} + \|f\|_{W^{1,1}((0,T),H^2)} + \|f\|_{W^{2,1}((0,T),H^1)} + \|u\|_{W^{2,1}}\big(\|\psi_0\|_{H^1} + \|f\|_{L^1((0,T),H^1)}\big)\big\}.$$

- *If $\psi_0 \in H^6(I,\mathbb{C})$, $f \in W^{2,1}((0,T),H^2 \cap H_0^1(I,\mathbb{C}))$ and $A_\gamma\psi_0 \in H^2 \cap H_0^1(I,\mathbb{C})$, $A_\gamma^2\psi_0 + A_\gamma f(0) \in H^2 \cap H_0^1(I,\mathbb{C})$, then, $\xi \in C^0([0,T],H^2 \cap H_0^1(I,\mathbb{C}))$ and $\varphi \in C^0([0,T],H^4(I,\mathbb{C}))$. If $f \in C^0([0,T],H^4(I,\mathbb{C}))$ then $\psi \in C^0([0,T],H^6(I,\mathbb{C}))$, $A_{u(t)}\psi(t) \in C^0([0,T],H^2 \cap H_0^1(I,\mathbb{C}))$, $A_{u(t)}^2\psi(t) + A_{u(t)}f(t) \in C^0([0,T],H^2 \cap H_0^1(I,\mathbb{C}))$ and we have the following bounds*:

$$\|\xi\|_{C^0([0,T],H^2)} \leqslant C\big\{\|\psi_0\|_{H^6} + \|f(0)\|_{H^4} + \|f\|_{W^{2,1}((0,T),H^2)} + \|u\|_{W^{2,1}}\big(\|\psi_0\|_{H^2} + \|f\|_{L^1((0,T),H^2)}\big)\big\},$$
$$\|\varphi\|_{C^0([0,T],H^4)} \leqslant C\big\{\|\psi_0\|_{H^6} + \big\|f(0)\big\|_{H^4} + \|f\|_{W^{2,1}((0,T),H^2)} + \|u\|_{W^{2,1}}\big(\|\psi_0\|_{H^2} + \|f\|_{L^1((0,T),H^2)}\big)\big\},$$
$$\|\psi\|_{C^0([0,T],H^6)} \leqslant C\big\{\|\psi_0\|_{H^6} + \|f\|_{C^0([0,T],H^4)} + \|f\|_{W^{2,1}((0,T),H^2)} + \|u\|_{W^{2,1}}\big(\|\psi_0\|_{H^2} + \|f\|_{L^1((0,T),H^2)}\big)\big\}.$$

Proposition 50. *Under the same assumptions as in the previous proposition, if $u \in C^2([0,T],\mathbb{R})$ and $f \in C^2([0,T],L^2(I,\mathbb{C}))$, then $\xi \in C^1([0,T],L^2(I,\mathbb{C}))$. It is the unique solution in $C^0([0,T],H^2 \cap H_0^1(I,\mathbb{C})) \cap C^1([0,T],L^2(I,\mathbb{C}))$ of*

$$\begin{cases} \frac{\partial\xi}{\partial t} = \frac{\mathrm{i}}{2}\frac{\partial^2\xi}{\partial q^2} + \mathrm{i}u(t)q\xi + h(t), \\ \xi(0) = \xi_0, \\ \xi(t,-1/2) = \xi(t,1/2) = 0. \end{cases}$$

Proposition 51. *Let $u \in W^{3,1}((0,T),\mathbb{R})$ be such that $\|u\|_{L^1} < \sqrt{2}/\sqrt{17}$, $\dot u(0) = \dot u(T) = \ddot u(0) = \ddot u(T) = 0$. Let $f \in W^{2,1}((0,T),H^2 \cap H_0^1(I,\mathbb{C})) \cap W^{3,1}((0,T),L^2(I,\mathbb{C}))$, ψ be the solution of* (B.1), *$\varphi = \frac{\partial\psi}{\partial t}$ and $\xi = \frac{\partial\varphi}{\partial t}$. Let $\gamma := u(0)$. Then the function $\zeta := \frac{\partial\xi}{\partial t}$ is the solution in $C^0([0,T],L^2(I,\mathbb{C}))$ of*

$$\begin{cases} \zeta(t) = T(t)\zeta_0 + \int_0^t T(t-s)\big[\mathrm{i}u(s)(q\zeta)(s) + k(s)\big]\,\mathrm{d}s, \\ \zeta_0 = -\mathrm{i}A_{u(0)}\xi_0 + h(0), \\ k(s) = 3\mathrm{i}\dot u(s)(q\xi)(s) + 3\mathrm{i}\ddot u(s)(q\varphi)(s) + \mathrm{i}\frac{\mathrm{d}^3u}{\mathrm{d}t^3}(s)(q\psi)(s) + \frac{\partial f}{\partial t^3}(s). \end{cases}$$

If $\psi_0 \in H^7(I,\mathbb{C})$, $f \in W^{3,1}((0,T),H_0^1(I,\mathbb{C}))$ and $A_\gamma\psi_0 \in H^2 \cap H_0^1(I,\mathbb{C})$, $A_\gamma^2\psi_0 + A_\gamma f(0) \in H^2 \cap H_0^1(I,\mathbb{C})$, $\mathrm{i}A_\gamma^3\psi_0 - A_\gamma^2 f(0) - \mathrm{i}A_\gamma \dot f(0) \in H_0^1(I,\mathbb{C})$ then, $\zeta \in C^0([0,T], H_0^1(I,\mathbb{C}))$ and $\xi \in C^0([0,T],H^3(I,\mathbb{C}))$. If $f \in C^1([0,T],H^3(I,\mathbb{C}))$ then $\varphi \in C^0([0,T], H^5(I,\mathbb{C}))$. If $f \in C^0([0,T],H^5(I,\mathbb{C}))$ then $\psi \in C^0([0,T],H^7(I,\mathbb{C}))$, $A_{u(t)}\psi(t) \in C^0([0,T],H^2 \cap H_0^1(I,\mathbb{C}))$, $A_{u(t)}^2\psi(t) + A_{u(t)}f(t) \in C^0([0,T],H^2 \cap H_0^1(I,\mathbb{C}))$, $A_{u(t)}^3\psi(t) + A_{u(t)}^2 f(t) + \dot u(t)A_{u(t)}(q\psi(t)) + \mathrm{i}A_{u(t)}\dot f(t) \in C^0([0,T],H^2 \cap H_0^1(I,\mathbb{C}))$ and we have the following bounds:

$$\begin{aligned}
\|\zeta\|_{C^0([0,T],H_0^1)} \leqslant C\big[&\|\psi_0\|_{H^7} + \|f(0)\|_{H^5} + \|\dot f(0)\|_{H^3} + \|f\|_{W^{3,1}((0,T),H^1)} \\
&+ \|u\|_{W^{2,1}}\big\{\|\psi_0\|_{H^3} + \|f\|_{W^{1,1}((0,T),H^1)}\big\} \\
&+ \|u\|_{W^{3,1}}\big\{\|\psi_0\|_{H^1} + \|f\|_{L^1((0,T),H^1)}\big\}\big], \\
\|\xi\|_{C^0([0,T],H^3)} \leqslant C\big[&\|\psi_0\|_{H^7} + \|f(0)\|_{H^5} + \|\dot f(0)\|_{H^3} \\
&+ \|f\|_{W^{3,1}((0,T),H^1)} + \|f\|_{W^{2,1}((0,T),H^2)} \\
&+ \|u\|_{W^{2,1}}\big\{\|\psi_0\|_{H^3} + \|f\|_{W^{1,1}((0,T),H^1)}\big\} \\
&+ \|u\|_{W^{3,1}}\big\{\|\psi_0\|_{H^1} + \|f\|_{L^1((0,T),H^1)}\big\}\big], \\
\|\varphi\|_{C^0([0,T),H^5)} \leqslant C\big[&\|\psi_0\|_{H^7} + \|f(0)\|_{H^5} + \|f\|_{C^1([0,T],H^3)} \\
&+ \|f\|_{W^{3,1}((0,T),H^1)} + \|f\|_{W^{2,1}((0,T),H^2)} \\
&+ \|u\|_{W^{2,1}}\big\{\|\psi_0\|_{H^3} + \|f\|_{W^{1,1}((0,T),H^1)}\big\} \\
&+ \|u\|_{W^{3,1}}\big\{\|\psi_0\|_{H^1} + \|f\|_{L^1((0,T),H^1)}\big\}\big], \\
\|\psi\|_{C^0([0,T],H^7)} \leqslant C\big[&\|\psi_0\|_{H^7} + \|f\|_{C^0([0,T],H^5)} + \|f\|_{C^1([0,T],H^3)} \\
&+ \|f\|_{W^{3,1}((0,T),H^1)} + \|f\|_{W^{2,1}((0,T),H^2)} \\
&+ \|u\|_{W^{2,1}}\big\{\|\psi_0\|_{H^3} + \|f\|_{W^{1,1}((0,T),H^1)}\big\} \\
&+ \|u\|_{W^{3,1}}\big\{\|\psi_0\|_{H^1} + \|f\|_{L^1((0,T),H^1)}\big\}\big].
\end{aligned}$$

The proofs of Propositions 48–51 are straightforward, we omit them.

Appendix C. An other version of the Nash–Moser theorem and its application

C.1. An other version of the Nash–Moser theorem

Proposition 52. *Let us consider the same assumptions as in Theorem* 6*. We assume moreover that, for every $u, \tilde u \in V \cap E_7$,*

$$\big\|\Phi''(u; v, w) - \Phi''(\tilde u; v, w)\big\|_7 \leqslant C \sum \big(1 + \|u - \tilde u\|_{n'_j}\big)\|v\|_{n''_j}\|w\|_{n'''_j}, \tag{C.1}$$

where the sum is finite, all the subscripts belong to $\{1, 3, 5, 7\}$ *and satisfy* (3.46) *with $m_j \leftarrow n_j$. We also assume that, for every $v, \tilde v \in V \cap E_9$,*

$$\big\|\big(\psi(v) - \psi(\tilde v)\big)g\big\|_1 \leqslant C\|v - \tilde v\|_3\|g\|_3, \tag{C.2}$$

$$\big\|\big(\psi(v) - \psi(\tilde v)\big)g\big\|_3 \leqslant C\big[\|v - \tilde v\|_3\|g\|_5 + \|v - \tilde v\|_5\|g\|_3\big], \tag{C.3}$$

$$\begin{aligned}
\big\|\big(\psi(v) - \psi(\tilde v)\big)g\big\|_5 \leqslant C\big[&\|v - \tilde v\|_3\|g\|_7 + \|v - \tilde v\|_5\|g\|_5 \\
&+ \big(\|v - \tilde v\|_7 + \|v - \tilde v\|_5^2\big)\|g\|_3\big],
\end{aligned} \tag{C.4}$$

$$\begin{aligned}\big\|\big(\psi(v)-\psi(\tilde{v})\big)g\big\|_7 \leqslant C\big[&\|v-\tilde{v}\|_3\|g\|_9+\|v-\tilde{v}\|_5\|g\|_7\\ &+\big(\|v-\tilde{v}\|_7+\|v-\tilde{v}\|_5^2\big)\|g\|_5\\ &+\big(\|v-\tilde{v}\|_9+\|v-\tilde{v}\|_7\|v-\tilde{v}\|_5+\|v-\tilde{v}\|_5^3\big)\|g\|_3\big].\end{aligned}\tag{C.5}$$

Then, there exist $\varepsilon > 0$ *and a continuous map,*

$$\begin{aligned}\Pi : V'_\beta &\to E_3,\\ f &\mapsto u,\end{aligned}$$

where

$$V'_\beta := \big\{f \in F'_\beta;\ \|f\|'_\beta < \varepsilon\big\},$$

such that, for every $f \in V'_\beta$,

$$\Phi\big(\Pi(f)\big) = \Phi(0) + f.$$

Proof. The map Π is the composition of the following maps:

$$\begin{aligned}F'_\beta &\to F'_\beta \to E_3,\\ f &\mapsto g \quad \mapsto u,\end{aligned}\tag{C.6}$$

where $f = g + T(g)$ and u is the limit of the sequence built in the proof of Theorem 6.

First, we prove the continuity of the map:

$$\begin{aligned}F'_\beta &\to F'_\beta,\\ f &\mapsto g.\end{aligned}$$

It is sufficient to prove that $T : F'_\beta \to F'_\beta$ is a contraction. Indeed, the inequality,

$$\big\|T(g) - T(\tilde{g})\big\|'_\beta \leqslant \delta\|g - \tilde{g}\|'_\beta,$$

with $\delta \in (0, 1)$ gives:

$$\|g - \tilde{g}\|'_\beta \leqslant \frac{1}{1-\delta}\big\|f - \tilde{f}\big\|'_\beta.$$

Let $g, \tilde{g} \in F'_\beta$. Let (u_j), $(\dot{u}_j)$ and (v_j) the sequences built in the proof of Theorem 6, associated to g. Let $(\tilde{u}_j)$, $(\dot{\tilde{u}}_j)$ and $(\tilde{v}_j)$ the sequences associated to $\tilde{g}$.

Then, there exists $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 > 0$ such that, for every $j \in \mathbb{N}$,

$$\|\dot{u}_j - \dot{\tilde{u}}_j\|_a \leqslant \mathcal{C}_1 \|g - \tilde{g}\|'_\beta \theta_j^{a-\alpha-1}, \quad a \in \{1, 3, 5, 7\}, \tag{C.7}$$

$$\|v_j - \tilde{v}_j\|_a \leqslant \mathcal{C}_2 \|g - \tilde{g}\|'_\beta \theta_j^{a-\alpha}, \quad a \in \{5, 7, 9\}, \tag{C.8}$$

$$\big\|(u_j - v_j) - (\tilde{u}_j - \tilde{v}_j)\big\|_a \leqslant \mathcal{C}_3 \|g - \tilde{g}\|'_\beta \theta_j^{a-\alpha}, \quad a \in \{1, 3, 5, 7\}. \tag{C.9}$$

The proof is exactly the same as the one of (3.25)–(3.27).

Remark. At this step, we have the continuity of the second map in (C.6),

$$F'_\beta \to E_3,$$
$$g \mapsto u.$$

Indeed, (C.7) gives:

$$\|u - \tilde{u}\|_3 \leqslant \mathcal{C}_1 \left(\sum_{j=0}^{\infty} \Delta_j \theta_j^{2-\alpha} \right) \|g - \tilde{g}\|'_\beta.$$

We have:

$$T(g) - T(\tilde{g}) = \sum_{j=0}^{\infty} \Delta_j \big[(e'_j - \tilde{e}'_j) + (e''_j - \tilde{e}''_j) \big].$$

Let us prove that there exists $\mathcal{C}_4, \mathcal{C}_5 > 0$ such that, for every $j \in \mathbb{N}^*$,

$$\|e'_j - \tilde{e}'_j\|_7 \leqslant \mathcal{C}_4 \max\big\{ \|g\|'_\beta, \|\tilde{g}\|'_\beta \big\} \|g - \tilde{g}\|_7, \tag{C.10}$$

$$\|e''_j - \tilde{e}''_j\|_7 \leqslant \mathcal{C}_5 \max\big\{ \|g\|'_\beta, \|\tilde{g}\|'_\beta \big\} \|g - \tilde{g}\|_7. \tag{C.11}$$

These bounds give:

$$\big\|T(g) - T(\tilde{g})\big\|'_\beta \leqslant \mathcal{C}_6 \max\big\{ \|g\|'_\beta, \|\tilde{g}\|'_\beta \big\} \|g - \tilde{g}\|_7,$$

which proves that T is a contraction of a small neighbourhood of 0 in F'_β.

We have:

$$e'_j - \tilde{e}'_j = \Delta_j \int_0^1 (1-t) \big[\Phi''(u_j + t\Delta_j \dot{u}_j; \dot{u}_j, \dot{u}_j) - \Phi''(\tilde{u}_j + t\Delta_j \dot{\tilde{u}}_j; \dot{u}_j, \dot{u}_j) \big] \,\mathrm{d}t$$
$$+ \Delta_j \int_0^1 (1-t) \Phi''(\tilde{u}_j + t\Delta_j \dot{\tilde{u}}_j; \dot{\tilde{u}}_j - \dot{u}_j, \dot{u}_j) \,\mathrm{d}t$$

$$+\Delta_j\int_0^1(1-t)\Phi''(\tilde u_j+t\Delta_j\dot{\tilde u}_j;\dot{\tilde u}_j,\dot u_j-\dot{\tilde u}_j)\,\mathrm{d}t.$$

Using (C.1) for the first line, (3.17) for the second and the third lines of the right-hand side, and proceeding as in the previous proof, we get (C.10). The inequality (C.11) can be proved in the same way. □

C.2. Application of Theorem 52

The aim of this subsection is to apply Theorem 52 to the map:

$$\Phi_\gamma:(\psi_0,v)\mapsto(\psi_0,\psi_T),$$

in order to get the following controllability result.

Theorem 14. *There exists $\gamma_0>0$ such that, for every $\gamma\in(0,\gamma_0)$, there exist $\delta>0$, $C>0$ and a continuous map,*

$$\Gamma_\gamma:\mathcal{V}_\gamma(0)\times\mathcal{V}_\gamma(T)\to H_0^1\big((0,T),\mathbb{R}\big),$$
$$(\psi_0,\psi_f)\mapsto v,$$

where

$$\mathcal{V}_\gamma(0):=\big\{\psi_0\in\mathcal{S}\cap H^7_{(\gamma)}(I,\mathbb{C});\ \big\|\psi_0-\psi_{1,\gamma}(0)\big\|_{H^7(I,\mathbb{C})}<\delta\big\},$$
$$\mathcal{V}_\gamma(T):=\big\{\psi_f\in\mathcal{S}\cap H^7_{(\gamma)}(I,\mathbb{C});\ \big\|\psi_f-\psi_{1,\gamma}(T)\big\|_{H^7(I,\mathbb{C})}<\delta\big\},$$

such that, for every $\psi_0\in\mathcal{V}_\gamma(0)$, $\psi_f\in\mathcal{V}_\gamma(T)$, the unique solution of (Σ) with control $u:=\gamma+\Gamma_\gamma(\psi_0,\psi_f)$ with $\psi(0)=\psi_0$ satisfies $\psi(T)=\psi_f$ and

$$\big\|\Gamma_\gamma(\psi_0,\psi_f)\big\|_{H_0^1((0,T),\mathbb{R})}\leqslant C\big[\big\|\psi_0-\psi_{1,\gamma}(0)\big\|_{H^7(I,\mathbb{C})}+\big|\psi_f-\psi_{1,\gamma}(T)\big\|_{H^7(I,\mathbb{C})}\big].$$

The bound (C.1) can be proved exactly in the same way as the bound (3.17) in Proposition 10.

Let us recall that we built $\mathrm{d}\Phi_\gamma(\psi_0,v)^{-1}.(\Psi_0,\Psi_f)$ in the following way:

$$\mathrm{d}\Phi_\gamma(\psi_0,v)^{-1}.(\Psi_0,\Psi_f)=M^{-1}_{(\psi_0,u)}\big(d_{(\psi_0,u)}(\Psi_0,\Psi_f)\big).$$

We will use the following decomposition:

$$\begin{aligned}&\big[\mathrm{d}\Phi_\gamma(\psi_0,u)^{-1}-\mathrm{d}\Phi_\gamma(\tilde\psi_0,\tilde u)^{-1}\big].(\Psi_0,\Psi_f)\\&\quad=M^{-1}_{(\psi_0,u)}\big[d_{(\psi_0,u)}(\Psi_0,\Psi_f)-d_{(\tilde\psi_0,\tilde u)}(\Psi_0,\Psi_f)\big]\\&\qquad+\big[M^{-1}_{(\psi_0,u)}-M^{-1}_{(\tilde\psi_0,\tilde u)}\big]\big(d_{(\tilde\psi_0,\tilde u)}(\Psi_0,\Psi_f)\big).\end{aligned}\tag{C.12}$$

In order to prove the bounds (C.2)–(C.5), we use the following corollary of Proposition 15.

Proposition 53. *Let us consider the same assumptions as in Proposition* 15*. We assume we have another map $\widetilde{M}$ and constants $\tilde{\Delta}_3, \tilde{\Delta}_5, \tilde{\Delta}_7, \tilde{\Delta}_9$ with the same properties as the map M and the constants $\Delta_3, \Delta_5, \Delta_7, \Delta_9$. We also assume that there exists some constants $C, \eta_3, \eta_5, \eta_7, \eta_9$ such that*

$$\begin{aligned}
&\big\|(\widetilde{M}-M)(w)\big\|_{h^3} \leqslant C_3\eta_3\|w\|_{L^2},\\
&\big\|(\widetilde{M}-M)(w)\big\|_{h^5} \leqslant C_3\big[\eta_3\|w\|_{H_0^1}+\eta_5\|w\|_{L^2}\big],\\
&\big\|(\widetilde{M}-M)(w)\big\|_{h^7} \leqslant C_3\big[\eta_3\|w\|_{H_0^2}+\eta_5\|w\|_{H_0^1}+\eta_7\|w\|_{L^2}\big],\\
&\big\|(\widetilde{M}-M)(w)\big\|_{h^9} \leqslant C_3\big[\eta_3\|w\|_{H_0^3}+\eta_5\|w\|_{H_0^2}+\eta_7\|w\|_{H_0^1}+\eta_9\|w\|_{L^2}\big].
\end{aligned}\tag{C.13}$$

Then, there exists $C_4>0$ such that, when η_3 is small enough, the right inverses M^{-1} and $\widetilde{M}^{-1}$ built in Proposition 15 *satisfy*:

$$\begin{aligned}
\big\|\big(\widetilde{M}^{-1}-M^{-1}\big)(d)\big\|_{L^2} &\leqslant C_4\eta_3\|d\|_{h^3},\\
\big\|\big(\widetilde{M}^{-1}-M^{-1}\big)(d)\big\|_{H_0^1} &\leqslant C_4\big[\eta_3\|d\|_{h^5}+\eta_5\|d\|_{h^3}\big],\\
\big\|\big(\widetilde{M}^{-1}-M^{-1}\big)(d)\big\|_{H_0^2} &\leqslant C_4\big[\eta_3\|d\|_{h^7}+\eta_5\|d\|_{h^5}+\eta_7\|d\|_{h^3}\big],\\
\big\|\big(\widetilde{M}^{-1}-M^{-1}\big)(d)\big\|_{H_0^3} &\leqslant C_4\big[\eta_3\|d\|_{h^9}+\eta_5\|d\|_{h^7}+\eta_7\|d\|_{h^5}+\eta_9\|d\|_{h^3}\big].
\end{aligned}$$

Proof. In the proof of Proposition 15, the functions $w:=M^{-1}(d)$ and $\widetilde{M}^{-1}(d)$ are the sum of the following series:

$$w=\sum_{k=0}^{\infty}w_k \quad \text{with } w_0:=M_\gamma^{-1}(d) \text{ and } w_k:=M_\gamma^{-1}\big[(M_\gamma-M)(w_{k-1})\big],$$

$$\widetilde{w}=\sum_{k=0}^{\infty}\widetilde{w}_k \quad \text{with } \widetilde{w}_0:=M_\gamma^{-1}(d) \text{ and } \widetilde{w}_k:=M_\gamma^{-1}\big[\big(M_\gamma-\widetilde{M}\big)(\widetilde{w}_{k-1})\big].$$

Let $k\in\mathbb{N}^*$. We have:

$$\begin{aligned}
\|w_k-\widetilde{w}_k\|_{L^2} &= \big\|M_\gamma^{-1}\big[(M_\gamma-M)(w_{k-1}-\widetilde{w}_{k-1})-\big(M-\widetilde{M}\big)(\widetilde{w}_{k-1})\big]\big\|_{L^2}\\
&\leqslant C_0\big[C_1\Delta_3\|w_{k-1}-\widetilde{w}_{k-1}\|_{L^2}+C_3\eta_3\|\widetilde{w}_{k-1}\|_{L^2}\big]\\
&\leqslant C_0\big[C_1\Delta_3\|w_{k-1}-\widetilde{w}_{k-1}\|_{L^2}+C_3C_0\eta_3\big(C_2\tilde{\Delta}_3\big)^{k-1}\|d\|_{h^3}\big].
\end{aligned}$$

By induction, we get, for every $k\in\mathbb{N}$,

$$\|w_k-\widetilde{w}_k\|_{L^2}\leqslant k\big(C_2\max\big\{\Delta_3,\tilde{\Delta}_3\big\}\big)^{k-1}C_5\eta_3\|d\|_{h^3} \quad \text{where } C_5:=C_3C_0.$$

Since $C_2\Delta_3, C_2\tilde{\Delta}_3 < 1/2$, we have:

$$\|w-\widetilde{w}\|_{L^2} \leqslant 4C_5\eta_3\|d\|_{h^3}.$$

The other bounds can be obtained in the same way. □

First, we apply this proposition with the map M_γ (respectively, M, respectively, $\widetilde{M}$) replaced by $M_{(\varphi_{1,\gamma})}$ (respectively $M_{(\psi_0,u)}$, respectively $M_{(\tilde{\psi}_0,\tilde{u})}$) defined in Section 3.6.1 and the constants:

$$\Delta_k := \big\|(\psi_0,u)-(\varphi_{1,\gamma},\gamma)\big\|_{E_k^0}, \qquad \tilde{\Delta}_k := \big\|\big(\tilde{\psi}_0,\tilde{u}\big)-(\varphi_{1,\gamma},\gamma)\big\|_{E_k^0},$$
$$\eta_k := \big\|(\psi_0,u)-\big(\tilde{\psi}_0,\tilde{u}\big)\big\|_{E_k^0},$$

for $k=3,5,7,9$, in order to get a bound on the second term of (C.12). Let us check the first bound of (C.13); the other ones can be obtained in the same way.

Let us recall that

$$M_{(\psi_0,u)}(w)_1 = \int_0^T w(t)\big\langle q\psi(t),\psi(t)\big\rangle\,\mathrm{d}t,$$

$$\begin{aligned}M_{(\psi_0,u)}(w)_k = \int_0^T \Bigg[& w(t)\big\langle q\psi(t),\varphi_{k,u(t)}\big\rangle - \mathrm{i}\dot{u}(t)\Bigg\langle \Psi_2(t), \frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(t)}\Bigg\rangle \\ &+\mathrm{i}\dot{u}(t)\frac{\langle\Psi_2(t),\varphi_{1,u(t)}\rangle}{\langle\psi(t),\varphi_{1,u(t)}\rangle}\Bigg\langle\psi(t),\frac{\mathrm{d}\varphi_{k,\gamma}}{\mathrm{d}\gamma}\bigg|_{u(t)}\Bigg\rangle \\ &+\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\langle\Psi_2(t),\varphi_{1,u(t)}\rangle}{\langle\psi(t),\varphi_{1,u(t)}\rangle}\right)\big\langle\psi(t),\varphi_{k,u(t)}\big\rangle\Bigg]\mathrm{e}^{\mathrm{i}\int_0^t\lambda_{k,u(s)}\,\mathrm{d}s}\,\mathrm{d}t, \quad k\geqslant 2.\end{aligned}$$

The computations are similar to the ones in Section 3.6.

Proposition 54. *Let $T:=4/\pi$. There exist constants $\delta, C_3>0$ such that, for every $(\tilde{\psi}_0,\tilde{v}),(\psi_0,v)\in E_9^\gamma$ satisfying $\Delta_3,\tilde{\Delta}_3<\delta$, we have, for every $w\in L^2((0,T),\mathbb{R})$,*

$$\big|(M_{(\psi_0,u)}-M_{(\tilde{\psi}_0,\tilde{u})})(w)_1\big| \leqslant C_3\eta_3\|w\|_{L^2}.$$

Proof. We have:

$$\big(M-\widetilde{M}\big)(w)_1 = \int_0^T \big(w(t)\big\langle q\psi(t),\big(\psi-\tilde{\psi}\big)(t)\big\rangle + \big\langle q\tilde{\psi}(t),\big(\psi-\tilde{\psi}\big)(t)\big\rangle\,\mathrm{d}t,$$

$$\big|\big(M-\widetilde{M}\big)(w)_1\big| \leqslant \sqrt{T}\|w\|_{L^2}\big\|\psi-\tilde{\psi}\big\|_{C^0([0,T],L^2)} \leqslant C\|w\|_{L^2}\eta_3. \quad \square$$

Proposition 55. *Let $T := 4/\pi$. There exist constants $\delta, C_3 > 0$ such that, for every $(\tilde{\psi}_0, \tilde{v}), (\psi_0, v) \in E_9^\gamma$ satisfying $\Delta_3, \tilde{\Delta}_3 < \delta$, for every $w \in L^2((0,T),\mathbb{R})$, the sequence $(X_k(w))_{k\geqslant 2}$ defined by*:

$$X_k(w) := \int_0^T w(t)\big[\langle q\psi(t), \varphi_{k,u(t)}\rangle e^{i\int_0^t \lambda_{k,u(s)}\,ds} - \langle q\tilde{\psi}(t), \varphi_{k,\tilde{u}(t)}\rangle e^{i\int_0^t \lambda_{k,\tilde{u}(s)}\,ds}\big]\,dt,$$

satisfies

$$\|X\|_{h^3} \leqslant C_3\eta_3\|w\|_{L^2}.$$

Proof. We study one by one the terms of the following decomposition:

$$X_k(w) = \int_0^T w(t)\Big\{\big[\langle q(\psi - \tilde{\psi})(t), \varphi_{k,u(t)}\rangle + \langle q\tilde{\psi}(t), \varphi_{k,u(t)} - \varphi_{k,\tilde{u}(t)}\rangle\big]e^{i\int_0^t \lambda_{k,u(s)}\,ds} + \langle q\tilde{\psi}(t), \varphi_{k,\tilde{u}(t)}\rangle\big[e^{i\int_0^t \lambda_{k,u(s)}\,ds} - e^{i\int_0^t \lambda_{k,\tilde{u}(s)}\,ds}\big]\Big\}\,dt. \tag{C.14}$$

We apply Proposition 16 to the first term of the right-hand side of (C.14). On the second term of the right-hand side of (C.14), we use an integration by parts (with respect to q) and Proposition 42. In the third term of the right-hand side of (C.14), we use an integration by parts (with respect to q) and the following consequence of Proposition 42:

$$|\lambda_{k,u} - \lambda_{k,\tilde{u}}| \leqslant \frac{C^*}{k}|u - \tilde{u}|. \quad \square$$

The strategy is exactly the same with each term in $M_{(\psi_0,u)}(w)$, we omit the end of the proof.

In a similar way, we prove bounds on,

$$d_{(\psi_0,u)}(\Psi_0, \Psi_f) - d_{(\tilde{\psi}_0,\tilde{u})}(\Psi_0, \Psi_f),$$

in order to get a suitable bound on the first term in (C.12).

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