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Controllability of a quantum particle in a moving potential well

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Abstract

We consider a nonrelativistic charged particle in a 1D moving potential well. This quantum system is subject to a control, which is the acceleration of the well. It is represented by a wave function solution of a Schrödinger equation, the position of the well together with its velocity. We prove the following controllability result for this bilinear control system: given ψ_0 close enough to an eigenstate and ψ_f close enough to another eigenstate, the wave function can be moved exactly from ψ_0 to ψ_f in finite time. Moreover, we can control the position and the velocity of the well. Our proof uses moment theory, a Nash–Moser implicit function theorem, the return method and expansion to the second order.

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1. Introduction

Following Rouchon [16], we consider a quantum particle with a potential $V(z)$ in a non-Galilean frame of absolute position $D(t)$, in a one-dimensional space. This system is represented by a complex valued wave function $(t, z) \mapsto \phi(t, z)$ solution of the Schrödinger equation

$$i\hbar \frac{\partial \phi}{\partial t}(t, z) = -\frac{\hbar^2}{2m}\frac{\partial^2 \phi}{\partial z^2}(t, z) + V(z - D(t))\phi(t, z). \tag{1.1}$$

Up to a change of variables, we can assume $\hbar = 1$, $m = 1$. It was already noted in [16] that the change of space variable $z \to q$ and function $\phi \to \psi$, defined by

$$q := z - D,$$

$$\psi(t, q) := e^{i(-z\dot{D}+D\dot{D}-\frac{1}{2}\int_0^t \dot{D}^2)}\phi(t, z),$$

transforms (1.1) into

$$i\frac{\partial \psi}{\partial t}(t, q) = -\frac{1}{2}\frac{\partial^2 \psi}{\partial q^2}(t, q) + (V(q) - u(t)q)\,\psi(t, q), \tag{1.2}$$

where $u := -\ddot{D}$. This equation also describes the nonrelativistic motion of a particle with a potential V in a uniform electric field $t \mapsto u(t)$.

We study this quantum system in the case of the following potential well (a box)

$$V(q) = 0 \text{ for } q \in I := (-\tfrac{1}{2}, \tfrac{1}{2}) \text{ and } V(q) = +\infty \text{ for } q \notin I.$$

Therefore, our system is

$$(\Sigma_0) \quad \begin{cases} i\frac{\partial \psi}{\partial t}(t, q) = -\frac{1}{2}\frac{\partial^2 \psi}{\partial q^2}(t, q) - u(t)q\psi(t, q), t \in \mathbb{R}_+, q \in I, \\ \psi(t, -1/2) = \psi(t, 1/2) = 0, \\ \dot{S}(t) = u(t), \\ \dot{D}(t) = S(t). \end{cases}$$

This is a control system, where

- the state is (ψ, S, D) with $\int_I |\psi(t, q)|^2\, dq = 1$ for every t,
- the control is the function $t \mapsto u(t) \in \mathbb{R}$.

It means that we want to control at the same time the wave function ψ of the particle, the speed S and the position D of the control is the acceleration of the box. The box (with an easy change of variable, we could instead take the force applied to the box).

Definition 1. Let $T_1 < T_2$ be two real numbers and $u \in C^0([T_1, T_2], \mathbb{R})$. A function (ψ, S, D) is a solution of (Σ_0) if

- ψ belongs to $C^0([T_1, T_2], H^2 \cap H^1_0(I, \mathbb{C})) \cap C^1([T_1, T_2], L^2(I, \mathbb{C}))$ and the first equality of (Σ_0) holds in $L^2(I, \mathbb{C})$, for every $t \in [T_1, T_2]$,
- $S \in C^1([T_1, T_2], \mathbb{R})$ and satisfies the third equality of (Σ_0), for every $t \in [T_1, T_2]$,
- $D \in C^2([T_1, T_2], \mathbb{R})$ and satisfies the fourth equality of (Σ_0), for every $t \in [T_1, T_2]$.

Then, we say that (ψ, S, D, u) is a trajectory of the control system (Σ_0) (on $[T_1, T_2]$).

Note that the first equation of (Σ_0) guarantees the conservation of the $L^2(I, \mathbb{C})$-norm of the wave function. Indeed, we have

$$\frac{d}{dt}\|\psi(t)\|^2_{L^2(I,\mathbb{C})} = \left\langle \psi(t), \frac{\partial \psi}{\partial t}(t) \right\rangle + \left\langle \frac{\partial \psi}{\partial t}(t), \psi(t) \right\rangle = 0,$$

where $\langle ., . \rangle$ denotes the usual scalar product on $L^2(I, \mathbb{C})$,

$$\langle \psi, \varphi \rangle := \int_I \psi(q)\overline{\varphi(q)}\, dq$$

and $\psi(t) := \psi(t, .)$.

It has already been proved in [1] that the subsystem

$$(\Sigma) \quad \begin{cases} i\frac{\partial \psi}{\partial t}(t,q) = -\frac{1}{2}\frac{\partial^2 \psi}{\partial q^2}(t,q) - u(t)q\psi(t,q), t \in \mathbb{R}_+, q \in I, \\ \psi(t, -1/2) = \psi(t, 1/2) = 0, \end{cases}$$

where the state is ψ and the control is u, is locally controllable around any eigenstate state for $u \equiv 0$, which are the functions

$$\psi_n(t, q) := \varphi_n(q) e^{-i\lambda_n t}, n \in \mathbb{N}^*.$$

Here $\lambda_n := (n\pi)^2/2$ are the eigenvalues of the operator A defined on

$$D(A) := H^2 \cap H^1_0(I, \mathbb{C}) \quad \text{by} \quad A\varphi := -\tfrac{1}{2}\varphi''$$

and the functions φ_n are the associated eigenvectors,

$$\varphi_n(q) := \begin{cases} \sqrt{2}\sin(n\pi q), & \text{when } n \text{ is even}, \\ \sqrt{2}\cos(n\pi q), & \text{when } n \text{ is odd}. \end{cases} \tag{1.3}$$

Thus, we know that, for every eigenstate, the wave function can be moved arbitrarily in a neighborhood of this eigenstate, in finite time.

The aim of this paper is to prove that we can also change the energy level. For example, we can move the wave function from any point in a neighborhood of the ground state ψ_1 to any point in a neighborhood of the first excited state ψ_2. We also prove that we can control the position D and the speed S of the box at the same time.

Let us introduce few notations in order to state this result,

$$\mathcal{S} := \{\varphi \in L^2(I, \mathbb{C});\ \|\varphi\|_{L^2(I,\mathbb{C})} = 1\},$$

$$H^7_{(0)}(I, \mathbb{C}) := \{\varphi \in H^7(I, \mathbb{C});\ A^n\varphi \in H^1_0(I, \mathbb{C}) \text{ for } n = 0, 1, 2, 3\}.$$

Our main result is the following one.

Theorem 1. *For every $n \in \mathbb{N}^*$, there exists $\eta_n > 0$ such that, for every $n_0, n_f \in \mathbb{N}^*$,* for every $(\psi_0, S_0, D_0), (\psi_f, S_f, D_f) \in [\mathcal{S} \cap H^7_{(0)}(I, \mathbb{C})] \times \mathbb{R} \times \mathbb{R}$ *with*

$$\|\psi_0 - \varphi_{n_0}\|_{H^7} + |S_0| + |D_0| < \eta_{n_0},\ \ \|\psi_f - \varphi_{n_f}\|_{H^7} + |S_f| + |D_f| < \eta_{n_f},$$

there exists a time $\mathcal{T} > 0$ and a trajectory (ψ, S, D, u) of (Σ_0) on $[0, \mathcal{T}]$ *which satisfies $(\psi(0), S(0), D(0)) = (\psi_0, S_0, D_0)$, $(\psi(\mathcal{T}), S(\mathcal{T}), D(\mathcal{T})) = (\psi_f, S_f, D_f)$ and $u \in H^1_0((0, \mathcal{T}), \mathbb{R})$.*

Thus, we also have the following corollary.

Corollary 1. *For every $n_0, n_f \in \mathbb{N}^*$, there exists a time $\mathcal{T} > 0$ and a trajectory (ψ, S, D, u) of (Σ_0) on $[0, \mathcal{T}]$ such that $(\psi(0), S(0), D(0)) = (\varphi_{n_0}, 0, 0)$, $(\psi(\mathcal{T}), S(\mathcal{T}), D(\mathcal{T})) = (\varphi_{n_f}, 0, 0)$, and $u \in H^1_0(0, \mathcal{T})$.*

For other results about the controllability of Schrödinger equations, we refer to the survey [17]. Note also that a negative controllability result for the control system (Σ) has been obtained by Turinici [18] for other spaces for the controls and for the states.

2. Sketch of the proof

2.1. Global strategy

Thanks to the reversibility of the control system (Σ_0), in order to get Theorem 1, it is sufficient to prove it with $n_f = n_0 + 1$. We prove it with $n_0 = 1$ and $n_f = 2$ to simplify the notations.

First, we prove the local controllability of (Σ_0) around the trajectory $(Y^{\theta,0,0}, u \equiv 0)$ for every $\theta \in [0, 1]$, where

$$Y^{\theta,0,0}(t) := (\psi_\theta(t), S(t) \equiv 0, D(t) \equiv 0),$$

$$\psi_\theta(t) := \sqrt{1-\theta}\psi_1(t) + \sqrt{\theta}\psi_2(t) \text{ for } \theta \in (0, 1),$$

$$Y^{k,0,0}(t) = (\psi_{k-1}(t), S(t) \equiv 0, D(t) \equiv 0) \text{ for } k = 0, 1.$$

Thus we know that

- there exists a nonempty open ball V_0 (resp. V_1) centered at $Y^{0,0,0}(0)$ (resp. $Y^{1,0,0}(0)$) such that (Σ_0) can be moved in finite time between any two points in V_0 (resp. V_1),
- for every $\theta \in (0, 1)$, there exists a nonempty open ball V_θ centered at $Y^{\theta,0,0}(0)$ such that (Σ_0) can be moved in finite time between any two points in V_θ.

Then, we conclude thanks to a compactness argument: the curve

$$[Y^{0,0,0}(0), Y^{1,0,0}(0)] := \{\sqrt{\lambda}Y^{0,0,0}(0) + \sqrt{(1-\lambda)}Y^{1,0,0}(0); \lambda \in [0, 1]\}$$

is compact in $L^2(I, \mathbb{R}) \times \mathbb{R} \times \mathbb{R}$ and covered by $\cup_{0 \leqslant \theta \leqslant 1} V_\theta$ thus there exists a increasing finite family $(\theta_n)_{1 \leqslant n \leqslant N}$ such that $[Y^{0,0,0}(0), Y^{1,0,0}(0)]$ is covered by $\cup_{1 \leqslant n \leqslant N} V_{\theta_n}$. We can assume $V_n \cap V_{n+1} \neq \emptyset$ for $n = 1, \ldots, N-1$. Given $Y_0 \in V_1$ and $Y_f \in V_N$, we move (Σ_0) from Y_0 to a point $Y_1 \in V_{\theta_1} \cap V_{\theta_2}$ in finite time, from Y_1 to a point $Y_2 \in V_{\theta_2} \cap V_{\theta_3}$ in finite time, etc. and we reach Y_f in finite time.

Now, let us explain the proof of the local controllability of (Σ_0) around $Y^{\theta,0,0}$ for every $\theta \in [0, 1]$. The strategy for $\theta \in (0, 1)$ is different from the one for $\theta \in \{0, 1\}$ but involves the same ideas. In the next sections, we details the two approaches. We start with the simplest case $\theta \in (0, 1)$.

2.2. *Local controllability of (Σ_0) around $Y^{\theta,0,0}$ for $\theta \in (0, 1)$*

A classical approach to prove the local controllability around a trajectory consists in proving the controllability of the linearized system around the trajectory studied and concluding with an inverse mapping theorem. This strategy does not work here because the linearized system around $(Y^{\theta,0,0}(t), u \equiv 0)$ is not controllable. In Section 3.1, we justify that the linearized system misses exactly two directions, which are $(\psi, S, D) = (\pm i\varphi_1, 0, 0)$. We call this situation "controllability up to codimension one".

First, we prove the local controllability up to codimension one of the nonlinear system (Σ_0), in Section 3.2. In Section 3.2.1, we explain that the situation is the same as in [1]: because of a loss of regularity in the controllability (up to codimension one) of the linearized system, the inverse mapping theorem cannot be applied. We deal with this difficulty by using a Nash–Moser theorem stated in Section 3.2.2. This theorem is an adaptation of Hörmander's one in [13], it is slightly different from the one used in [1]. Sections 3.2.3 and 3.2.4 are dedicated to the application of this theorem.

Then, in Section 3.3, we justify that the nonlinear term in (Σ_0) allows to move in the two directions which are missed by the linearized system. We fix the time, we perform a power series expansion and we prove that the second order term allows to move in the two directions $(\psi, S, D) = (\pm i\varphi_1, 0, 0)$. This method is classical to study the local controllability of finite dimensional systems. It has already been used for an infinite-dimensional one, the Korteweg–de Vries equation, in [7]. In this reference, an expansion to the second order was not sufficient and it was needed to compute the third-order term.

In Section 3.4, we get the local controllability of (Σ_0) around $Y^{\theta,0,0}$ by applying the intermediate values theorem.

2.3. *Local controllability of* (Σ_0) *around* $Y^{k,0,0}$ *for* $k \in \{0, 1\}$

Again, the classical approach does not work because the linearized system around $(Y^{k,0,0}, u \equiv 0)$ is not controllable for $k \in \{0, 1\}$. This result was proved by Rouchon in [16]. He proved this linearized system is steady-state controllable, but this result does not imply the same property for the nonlinear system. As noticed in Section 4.1, the situation is even worse than the previous one because the linearized system misses an infinite number of directions (half of the projections).

The proof of the local controllability of (Σ_0) around $Y^{k,0,0}$ for $k \in \{0, 1\}$ relies on the return method, a method introduced in [2,3] to solve a stabilisation problem and a controllability problem, together with quasi-static transformations as in [6]. The return method has already been used for controllability problems of partial differential equations by Beauchard in [1], by Coron [4–6], by Coron and Fursikov [8], Fursikov and Imanuvilov [9], Glass [10,11] and Horsin [14].

This strategy is divided in two steps. We explain it with $Y^{0,0,0}$ but everything works similarly with $Y^{1,0,0}$ instead of $Y^{0,0,0}$. First, in Section 4.2, we propose an other trajectory $(Y^{\gamma,\alpha,\beta}, u \equiv \gamma)$ such that (Σ_0) is locally controllable around $Y^{\gamma,\alpha,\beta}$ in time T^*. Then, we deduce the local controllability around $Y^{0,0,0}$ in Section 4.3, by using quasi-static transformations, in the same way as in [6,1]. We fix Y_0 close to $Y^{0,0,0}(t_0)$ and Y_f close to $Y^{0,0,0}(t_f)$ for some real constants t_0 and t_f. We use quasi-static transformations in order to move the system

- from Y_0 to a point Y_1, which is close to $Y^{\gamma,\alpha,\beta}(0)$, for some real constants α, β, γ,
- from a point Y_2, which is close to $Y^{\gamma,\alpha,\beta}(T^*)$, to Y_f.

Thanks to the local controllability around $Y^{\gamma,\alpha,\beta}$, we can move the system from Y_1 to Y_2 in finite time, it gives the conclusion. By "quasi-static transformations", we mean that we use controls $u(t)$ which change slowly.

Finally, in Section 5, we prove the local controllability of (Σ_0) around $Y^{\gamma,\alpha,\beta}$. Again, this local controllability result cannot be proved by using the classical approach because the linearized system around $Y^{\gamma,\alpha,\beta}$ is not controllable. In Section 5.1, we explain that this linearized system misses the two directions $(\psi, S, D) = (0, \pm 1, 0)$. We conclude with the same strategy as in Section 2.2.

In Section 5.2 we prove that the same strategy as in [1] leads to the local controllability, in time T, of (Σ_0), when the state is (ψ, D) and the control is u, around $Y^{\gamma,\alpha,\beta}$.

A loss of regularity in the controllability (up to codimension one) of the linearized system around $(Y^{\gamma,\alpha,\beta}, u \equiv \gamma)$ prevents us from applying the inverse mapping theorem. We use the Nash–Moser theorem stated in Section 3.2.2, in the context given in Section 5.2.1. Sections 5.2.2 and 5.2.3 are dedicated to the application of this theorem.

In Section 5.3, we prove that the second-order term allows to move in the two directions ($\psi = 0, S = \pm 1, D = 0$) which are missed by the linearized system.

In Section 5.4, we get the local controllability around $Y^{\gamma,\alpha,\beta}$ by applying the intermediate values theorem.

3. Local controllability of (Σ_0) around $Y^{\theta,0,0}$

In all Section 3, $\theta \in (0,1)$ is fixed. The aim of this section is the proof of the following result:

Theorem 2. *Let $T := 4/\pi$. There exists $\eta > 0$ such that, for every (ψ_0, S_0, D_0), $(\psi_f, S_f, D_f) \in [\mathcal{S} \cap H^7_{(0)}(I,\mathbb{C})] \times \mathbb{R} \times \mathbb{R}$ with*

$$\|\psi_0 - \psi_\theta(0)\|_{H^7} + |S_0| + |D_0| < \eta,$$

$$\|\psi_f - \psi_\theta(T)\|_{H^7} + |S_f| + |D_f| < \eta,$$

there exists a trajectory (ψ, S, D) of (Σ_0) on $[0, 2T]$ such that

$$(\psi(0), S(0), D(0)) = (\psi_0, S_0, D_0),$$

$$(\psi(2T), S(2T), D(2T)) = (\psi_f, S_f, D_f),$$

and $u \in H^1_0((0,2T),\mathbb{R})$.

3.1. Controllability up to codimension one of the linearized system around $(Y^{\theta,0,0}, u \equiv 0)$

Let us introduce, for $\psi \in \mathcal{S}$, the tangent space $T_{\mathcal{S}}(\psi)$ to the $L^2(I,\mathbb{C})$-sphere at the point ψ,

$$T_{\mathcal{S}}\psi := \{\varphi \in L^2(I,\mathbb{C}); \Re\langle \varphi, \psi\rangle = 0\}$$

and for $k = 2, \dots, 9$, the following subspace of $H^k(I,\mathbb{C})$,

$$H^k_{(0)}(I,\mathbb{C}) := \{\varphi \in H^k(I,\mathbb{C}); A^n\varphi \in H^1_0(I,\mathbb{C}) \text{ for } n \in \mathbb{N}, n \leqslant (k-1)/2\}.$$

The linearized control system around $(Y_\theta, u \equiv 0)$ is

$$(\Sigma_\theta^l) \quad \begin{cases} i\frac{\partial \Psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \Psi}{\partial q^2} - wq\psi_\theta, \\ \Psi(t, \pm 1/2) = 0, \\ \dot{s} = w, \\ \dot{d} = s. \end{cases}$$

It is a control system where

- the state is (Ψ, s, d) with $\Psi(t) \in T_{\mathcal{S}}(\psi_\theta(t))$,
- the control is the real valued function $t \mapsto w(t)$.

Proposition 1. *Let $T > 0$ and (Ψ, s, d) be a trajectory of (Σ_θ^l) on $[0, T]$. Then, the function*

$$t \mapsto \Im(\langle \Psi(t), \sqrt{1-\theta}\psi_1(t) - \sqrt{\theta}\psi_2(t)\rangle)$$

is constant on $[0, T]$. Thus, the control system (Σ_θ^l) is not controllable.

Proof. Let us consider the function $\xi_\theta(t) := \sqrt{1-\theta}\psi_1(t) - \sqrt{\theta}\psi_2(t)$. We have

$$i\frac{\partial \xi_\theta}{\partial t} = -\frac{1}{2}\frac{\partial^2 \xi_\theta}{\partial q^2},$$

$$\frac{d}{dt}(\Im\langle \Psi(t), \xi_\theta(t)\rangle) = \Im(iw\langle q\psi_\theta(t), \xi_\theta(t)\rangle).$$

The explicit expressions of ψ_θ and ξ_θ provide, for every t,

$$\langle q\psi_\theta(t), \xi_\theta(t)\rangle \in i\mathbb{R},$$

which gives the conclusion.

Let $T > 0$, and $\Psi_0 \in T_{\mathcal{S}}(\psi_\theta(0))$, $\Psi_f \in T_{\mathcal{S}}(\psi_\theta(T))$. A necessary condition for the existence of a trajectory of (Σ_θ^l) satisfying $\Psi(0) = \Psi_0$ and $\Psi(T) = \Psi_f$ is

$$\Im(\langle \Psi_f, \sqrt{1-\theta}\psi_1(T) - \sqrt{\theta}\psi_2(T) >) = \Im(\langle \Psi_0, \sqrt{1-\theta}\varphi_1 - \sqrt{\theta}\varphi_2\rangle).$$

This equality does not happen for an arbitrary choice of Ψ_0 and Ψ_f. Thus (Σ_θ^l) is not controllable. □

Proposition 2. *Let* $T > 0$, $(\Psi_0, s_0, d_0), (\Psi_f, s_f, d_f) \in H^3_{(0)}(I, \mathbb{R}) \times \mathbb{R} \times \mathbb{R}$ *be such that*

$$\Re\langle \Psi_0, \psi_\theta(0)\rangle = \Re\langle \Psi_f, \psi_\theta(T)\rangle = 0, \tag{3.1}$$

$$\Im\langle \Psi_f, \sqrt{1-\theta}\varphi_1 e^{-i\lambda_1 T} - \sqrt{\theta}\varphi_2 e^{-i\lambda_2 T}\rangle = \Im\langle \Psi_0, \sqrt{1-\theta}\varphi_1 - \sqrt{\theta}\varphi_2\rangle. \tag{3.2}$$

There exists $w \in L^2((0,T), \mathbb{R})$ *such that the solution of* (Σ^l_θ) *with control* w *and such that* $(\Psi(0), s(0), d(0)) = (\Psi_0, s_0, d_0)$ *satisfies* $(\Psi(T), s(T), d(T)) = (\Psi_f, s_f, d_f)$.

Remark 1. Condition (3.2) means that we miss exactly two directions, which are $(\Psi, s, d) = (\pm i\xi_\theta, 0, 0)$. Thus, if we want to control the components $\langle \Psi, \varphi_k\rangle$ for $k \geqslant 2$ and $\Re\langle \Psi, \varphi_1\rangle$ then, we cannot control $\Im\langle \Psi, \varphi_1\rangle$. This is why we say that we miss the two directions $(\Psi, s, d) = (\pm i\varphi_1, 0, 0)$.

Proof. Let $(\Psi_0, s_0, d_0) \in L^2(I, \mathbb{R}) \times \mathbb{R} \times \mathbb{R}$ with $\Psi_0 \in T_{\mathcal{S}}(\psi_\theta(0))$ and $T > 0$. Let (Ψ, s, d) be a solution of (Σ^l_θ) with $(\Psi(0), s(0), d(0)) = (\Psi_0, s_0, d_0)$ and a control $w \in L^2((0,T), \mathbb{R})$. We have the following equality in $L^2(I, \mathbb{C})$:

$$\Psi(t) = \sum_{k=1}^{\infty} x_k(t)\varphi_k \text{ where } x_k(t) := \langle \Psi(t), \varphi_k\rangle \forall k \in \mathbb{N}.$$

Using the equation satisfied by Ψ, we get

$$x_{2k}(t) = \left(\langle \Psi_0, \varphi_{2k}\rangle + i\sqrt{1-\theta}\, b_{2k} \int_0^t w(\tau) e^{i(\lambda_{2k}-\lambda_1)\tau}\, d\tau\right) e^{-i\lambda_{2k} t}, \tag{3.3}$$

$$x_{2k-1}(t) = \left(\langle \Psi_0, \varphi_{2k-1}\rangle + i\sqrt{\theta}\, c_{2k-1} \int_0^t w(\tau) e^{i(\lambda_{2k-1}-\lambda_2)\tau}\, d\tau\right) e^{-i\lambda_{2k-1} t}, \tag{3.4}$$

where, for every $k \in \mathbb{N}^*$, $b_k := \langle q\varphi_k, \varphi_1\rangle$ and $c_k := \langle q\varphi_k, \varphi_2\rangle$. Thanks to the explicit expression of the functions φ_k (see (1.3)), we get

$$b_k = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \frac{-8(-1)^{k/2} k}{\pi^2 (1+k)^2 (1-k)^2} & \text{if } k \text{ is even,} \end{cases}$$

$$c_k = \begin{cases} \frac{16(-1)^{(k-1)/2} k}{\pi^2 (k+2)^2 (k-2)^2} & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases} \tag{3.5}$$

Let $(\Psi_f, s_f, d_f) \in L^2(I,\mathbb{R}) \times \mathbb{R} \times \mathbb{R}$ with $\Psi_f \in T_{\mathcal{S}}(\psi_\theta(T))$. The equality $(\Psi(T), s(T), d(T)) = (\Psi_f, s_f, d_f)$ is equivalent to the following moment problem on w:

$$\begin{cases} \int_0^T w(t) e^{i(\lambda_{2k}-\lambda_1)t}\,dt = \frac{-i}{\sqrt{1-\theta} b_{2k}} \Big(\langle \Psi_f, \varphi_{2k}\rangle e^{i\lambda_{2k}T} - \langle \Psi_0, \varphi_{2k}\rangle \Big), \\ \quad \forall k \in \mathbb{N}^*, \\ \int_0^T w(t) e^{i(\lambda_{2k-1}-\lambda_2)t}\,dt = \frac{-i}{\sqrt{\theta} c_{2k-1}} \Big(\langle \Psi_f, \varphi_{2k-1}\rangle e^{i\lambda_{2k-1}T} - \langle \Psi_0, \varphi_{2k-1}\rangle \Big), \\ \quad \forall k \in \mathbb{N}^*, \\ \int_0^T w(t)\,dt = s_f - s_0, \\ \int_0^T (T-t) w(t)\,dt = d_f - d_0 - s_0 T. \end{cases} \tag{3.6}$$

In the two first equalities of (3.6) with $k = 1$, the left-hand sides are complex conjugate numbers because w is real valued. Thus a necessary condition on Ψ_0 and Ψ_f for the existence of $w \in L^2((0,T),\mathbb{R})$ solution of (3.6) is

$$\frac{1}{\sqrt{1-\theta}} \Big(\overline{\langle \Psi_f, \varphi_2\rangle} e^{-i\lambda_2 T} - \overline{\langle \Psi_0, \varphi_2\rangle} \Big) = \frac{-1}{\sqrt{\theta}} \Big(\langle \Psi_f, \varphi_1\rangle e^{i\lambda_1 T} - \langle \Psi_0, \varphi_1\rangle \Big). \tag{3.7}$$

The equality of the real parts of the two sides in (3.7) is guaranteed by (3.1). The equality of the imaginary parts of the two sides in (3.7) is equivalent to (3.2). Under the assumption $\Psi_0, \Psi_f \in H^3_{(0)}(I,\mathbb{C})$, the right-hand side of (3.6) defines a sequence in l^2. Then, the existence, for every $T > 0$, of $w \in L^2((0,T),\mathbb{R})$ solution of (3.6) is a classical result on trigonometric moment problems. □

3.2. Local controllability up to codimension one of (Σ_0) around $(Y^{\theta,0,0}, u \equiv 0)$

Let us introduce the following closed subspace of $L^2(I,\mathbb{C})$:

$$V := \overline{\mathrm{Span}\{\varphi_k; k \geqslant 2\}}$$

and the orthogonal projection $\mathcal{P} : L^2(I,\mathbb{C}) \to V$. The aim of this section is the proof of the following result.

Theorem 3. *Let $T := 4/\pi$. There exists $C > 0$, $\delta > 0$ and a continuous map*

$$\begin{array}{lcccl} \Gamma: & \mathcal{V}(0) & \times & \mathcal{V}(T) & \to H^1_0((0,T),\mathbb{R}) \\ & ((\psi_0, S_0, D_0) & , & (\widetilde{\psi}_f, S_f, D_f)) & \mapsto \quad u \end{array}$$

where

$$\mathcal{V}(0) := \{(\psi_0, S_0, D_0) \in [\mathcal{S} \cap H^7_{(0)}(I,\mathbb{C})] \times \mathbb{R} \times \mathbb{R}; \|\psi_0 - \psi_\theta(0)\|_{H^7} + |S_0| + |D_0| < \delta\},$$

$$\mathcal{V}(T) := \{(\widetilde{\psi_f}, S_f, D_f) \in [H^7_{(0)}(I,\mathbb{C}) \cap V \cap B_{L^2}(0,1)] \times \mathbb{R} \times \mathbb{R}; \|\widetilde{\psi_f} -\mathcal{P}\psi_\theta(T)\|_{H^7} + |S_f| + |D_f| < \delta\},$$

such that, for every $((\psi_0, S_0, D_0), (\widetilde{\psi_f}, S_f, D_f)) \in \mathcal{V}(0) \times \mathcal{V}(T)$, *the trajectory of* (Σ_0) *with control* $\Gamma(\psi_0, S_0, D_0, \widetilde{\psi_f}, S_f, D_f)$ *such that* $(\psi(0), S(0), D(0)) = (\psi_0, S_0, D_0)$ *satisfies*

$$(\mathcal{P}\psi(T), S(T), D(T)) = (\widetilde{\psi_f}, S_f, D_f)$$

and

$$\|\Gamma(\psi_0, S_0, D_0, \widetilde{\psi_f}, S_f, D_f)\|_{H^1_0((0,T),\mathbb{R})} \leqslant C[\|\mathcal{P}(\psi_0 - \psi_\theta(0))\|_{H^7(I,\mathbb{C})} + |S_0| + |D_0| + \|\widetilde{\psi_f} - \mathcal{P}\psi_\theta(T)\|_{H^7(I,\mathbb{C})} + |S_f| + |D_f|].$$

3.2.1. The inverse mapping theorem cannot be applied

In our situation, in order to prove the Theorem 3 with the classical approach, we would like to apply the inverse mapping theorem to the map

$$\Phi : (\psi_0, S_0, D_0, u) \mapsto (\psi_0, S_0, D_0, \mathcal{P}\psi(T), S(T), D(T)),$$

where ψ solves

$$\begin{cases} i\dot{\psi} = -\frac{1}{2}\psi'' - uq\psi, \\ \dot{S} = w, \\ \dot{D} = S, \end{cases}$$

with $(\psi(0), S(0), D(0)) = (\psi_0, S_0, D_0)$.

The map Φ is C^1 between the following spaces:

$$\Phi : [\mathcal{S} \cap H^2_{(0)}(I,\mathbb{C})] \times \mathbb{R} \times \mathbb{R} \times L^2((0,T),\mathbb{R}) \to [\mathcal{S} \cap H^2_{(0)}(I,\mathbb{C})] \times \mathbb{R} \times \mathbb{R} \times [V \cap B_{L^2}(0,1) \cap H^2_{(0)}(I,\mathbb{C})] \times \mathbb{R} \times \mathbb{R},$$

$$\Phi : [\mathcal{S} \cap H^3_{(0)}(I,\mathbb{C})] \times \mathbb{R} \times \mathbb{R} \times H^1_0((0,T),\mathbb{R}) \to [\mathcal{S} \cap H^3_{(0)}(I,\mathbb{C})] \times \mathbb{R} \times \mathbb{R} \times [V \cap B_{L^2}(0,1) \cap H^3_{(0)}(I,\mathbb{C})] \times \mathbb{R} \times \mathbb{R}.$$

Thus, in order to apply the inverse mapping theorem, we would need to construct a right inverse to the map $d\Phi(\psi_\theta(0), 0, 0, 0)$ which maps the following spaces:

$$[T_{\mathcal{S}}(\psi_\theta(0)) \cap H^2_{(0)}] \times \mathbb{R} \times \mathbb{R} \times [V \cap H^2_{(0)}] \times \mathbb{R} \times \mathbb{R} \to [T_{\mathcal{S}}(\psi_\theta(0)) \cap H^2_{(0)}] \times \mathbb{R} \times \mathbb{R} \times L^2((0,T),\mathbb{R})$$

or

$$[T_{\mathcal{S}}(\psi_\theta(0)) \cap H^3_{(0)}] \times \mathbb{R} \times \mathbb{R} \times [V \cap H^3_{(0)}] \times \mathbb{R} \times \mathbb{R}$$
$$\to [T_{\mathcal{S}}(\psi_\theta(0)) \cap H^3_{(0)}] \times \mathbb{R} \times \mathbb{R} \times H^1_0((0,T),\mathbb{R}).$$

The controllability up to codimension one proved for the linearized system around $(Y_\theta, u \equiv 0)$ only provides a right inverse for $d\Phi(\psi_\theta(0),0,0,0)$ which maps the following spaces:

$$[T_{\mathcal{S}}(\psi_\theta(0)) \cap H^3_{(0)}] \times \mathbb{R} \times \mathbb{R} \times [V \cap H^3_{(0)}] \times \mathbb{R} \times \mathbb{R}$$
$$\to [T_{\mathcal{S}}(\psi_\theta(0)) \cap H^3_{(0)}] \times \mathbb{R} \times \mathbb{R} \times L^2((0,T),\mathbb{R}).$$

In order to deal with this loss of regularity in the controllability of the linearized system around $(Y^{\theta,0,0}, u \equiv 0)$, we use a Nash–Moser implicit function theorem stated in the following section. It is an adaptation of Hörmander's one in [13], it is slightly different from the one proved in [1, Section 3.2]. The use of the projection $\mathcal{P}$ introduce changes in the statement and the proof so we write them completely.

3.2.2. The Nash–Moser theorem used

As in [1], we consider a decreasing family of Hilbert spaces $(E_a)_{a\in\{1,\ldots,9\}}$ with continuous injections $E_b \to E_a$ of norm $\leqslant 1$ when $b \geqslant a$. Suppose we have given linear operators $S_\lambda : E_1 \to E_9$ for $\lambda \geqslant 1$. We assume there exists a constant $K > 0$ such that for every $a \in \{1,\ldots,9\}$, for every $\lambda \geqslant 1$ and for every $u \in E_a$ we have

$$\|S_\lambda u\|_b \leqslant K\|u\|_a, \forall b \in \{1,\ldots,a\}, \tag{3.8}$$

$$\|S_\lambda u\|_b \leqslant K\lambda^{b-a}\|u\|_a, \forall b \in \{a+1,\ldots,9\}, \tag{3.9}$$

$$\|u - S_\lambda u\|_b \leqslant K\lambda^{b-a}\|u\|_a, \forall b \in \{1,\ldots,a-1\}, \tag{3.10}$$

$$\left\|\frac{d}{d\lambda}S_\lambda u\right\|_b \leqslant K\lambda^{b-a-1}\|u\|_a, \forall b \in \{1,\ldots,9\}. \tag{3.11}$$

Then, we have the convexity of the norms (see [13] for the proof): there exists a constant $c \geqslant 1$ such that, for every $\lambda \in [0,1]$, for every $a, b \in \{1,\ldots,9\}$ such that $a \leqslant b$, $\lambda a + (1-\lambda)b \in \mathbb{N}$ and for every $u \in E_b$,

$$\|u\|_{\lambda a+(1-\lambda)b} \leqslant c\|u\|_a^\lambda \|u\|_b^{1-\lambda}.$$

We fix a sequence $1 = \theta_0 < \theta_1 < \cdots \to \infty$ of the form $\theta_j = (j+1)^\delta$ where $\delta > 0$. We set $\Delta_j := \theta_{j+1} - \theta_j$ and we introduce

$$R_j u := \frac{1}{\Delta_j}(S_{\theta_{j+1}} - S_{\theta_j})u \quad \text{if } j > 0 \text{ and } R_0 u := \frac{1}{\Delta_0} S_{\theta_1} u.$$

Thanks to (3.10), we have

$$u = \sum_{j=0}^{\infty} \Delta_j R_j u$$

with convergence in E_b when $u \in E_a$ and $a > b$. As noted in [1], it follows from (3.11) that there exists $K' > 0$ such that, for every $a \in \{1, \dots, 9\}$, for every $u \in E_a$, for every $b \in \{1, \dots, 9\}$, for every $\delta \in (0, 2]$ and for every $j \in \mathbb{N}^*$,

$$\|R_j u\|_b \leqslant K' \theta_j^{b-a-1} \|u\|_a.$$

Let $a_1, a_2 \in \mathbb{N}$ and $a \in \mathbb{R}$ be such that $1 \leqslant a_1 < a < a_2 \leqslant 9$. We define the space

$$E'_a := \left\{ \sum_{j=0}^{\infty} \Delta_j u_j ; u_j \in E_{a_2}, \exists M > 0 / \forall j, \|u_j\|_b \leqslant M \theta_j^{b-a-1} \text{ for } b = a_1, a_2 \right\},$$

with the norm $\|u\|'_a$ given by the infimum of M over all such decomposition of u. This space does not depend on the choice of a_1 and a_2 (see [13] for the proof). The norm $\|.\|'_a$ is stronger than the norm $\|.\|_b$ when $b < a$,

$$\|u\|_b \leqslant M_{b,a} \|u\|'_a \tag{3.12}$$

and $\|.\|'_a$ is weaker than $\|.\|_a$,

$$\|u\|'_a \leqslant K' \|u\|_a.$$

As noticed in [1,13], there exists a constant K'' such that, for every $a \in \{1, \dots, 9\}$, for every $\theta \geqslant 1$, for every $b < a$ and for every $u \in E'_a$ we have

$$\|u - S_\theta u\|_b \leqslant K'' \theta^{b-a} \|u\|'_a. \tag{3.13}$$

We have another family $(F_a)_{a \in \{1, \dots, 9\}}$ with the same properties as above, we use the same notations for the smoothing operators S_λ. Moreover, we assume the injection $F_b \to F_a$ is compact when $b > a$.

Theorem 4. *Let α and β be fixed positive real numbers such that*

$$4 < \alpha < \beta < 7 \text{ and } \beta - \alpha \geqslant 2. \tag{3.14}$$

Let $\mathcal{P}$ be a continuous linear operator from F_b to F_b of norm $\leqslant 1$, for $b = 1, \ldots, 9$, such that $\mathcal{P}S_\theta = S_\theta\mathcal{P}$. Let V be a convex E'_α-neighborhood of 0 and Φ a map from $V \cap E_7$ to F_β which is twice differentiable and satisfies

$$\|\Phi''(u; v, w)\|_7 \leqslant C \sum (1 + \|u\|_{m'_j})\|v\|_{m''_j}\|w\|_{m'''_j}, \tag{3.15}$$

where the sum is finite, all the subscripts belong to $\{1, 3, 5, 7\}$ and satisfy

$$\max(m'_j - \alpha, 0) + \max(m''_j, 2) + m'''_j < 2\alpha, \ \forall j. \tag{3.16}$$

We assume that $\Phi : E_3 \to F_3$ is continuous. We also assume that $\Phi'(v)$, for $v \in V \cap E_9$, has a right inverse $\psi(v)$ mapping F_9 into E_7, that $(v, g) \mapsto \psi(v)g$ is continuous from $(V \cap E_9) \times F_9$ to E_7 and that there exists a constant C such that for every $(v, g) \in (V \cap E_9) \times F_9$,

$$\|\psi(v)g\|_1 \leqslant C[\|\mathcal{P}g\|_3 + \|v\|_3\|g\|_3], \tag{3.17}$$

$$\|\psi(v)g\|_3 \leqslant C[\|\mathcal{P}g\|_5 + \|v\|_3\|g\|_5 + \|v\|_5\|g\|_3], \tag{3.18}$$

$$\|\psi(v)g\|_5 \leqslant C[\|\mathcal{P}g\|_7 + \|v\|_3\|g\|_7 + \|v\|_5\|g\|_5 + (\|v\|_7 + \|v\|_5^2)\|g\|_3], \tag{3.19}$$

$$\begin{aligned}\|\psi(v)g\|_7 \leqslant\ & C[\|\mathcal{P}g\|_9 + \|v\|_3\|g\|_9 + \|v\|_5\|g\|_7 + (\|v\|_7 + \|v\|_5^2)\|g\|_5 \\ & +(\|v\|_9 + \|v\|_7\|v\|_5 + \|v\|_5^3)\|g\|_3].\end{aligned} \tag{3.20}$$

For every $f \in F'_\beta$ with sufficiently small norm, there exists $u \in E_3$ such that $\Phi(u) = \Phi(0) + f$.

Remark 2. The Nash–Moser theorem used in [1] corresponds to the case $\mathcal{P} = \mathrm{Id}$. In what follows, we only emphasize where the projection $\mathcal{P}$ appears in the proof of [1, Section 3.2].

Proof. Let $g \in F'_\beta$. There exist decompositions (see [1, Proof of Theorem 6])

$$g = \sum \Delta_j g_j \text{ with } \|g_j\|_b \leqslant K'\theta_j^{b-\beta-1}\|g\|_\beta \text{ for every } b \in \{1, \ldots, 9\}, \tag{3.21}$$

$$\mathcal{P}g = \sum \Delta_j \mathcal{P}g_j \text{ with } \|\mathcal{P}g_j\|_b \leqslant K'\theta_j^{b-\beta-1}\|\mathcal{P}g\|'_\beta \text{ for every } b \in \{1, \ldots, 9\}. \tag{3.22}$$

To get (3.22), we have used $\mathcal{P}S_\theta = S_\theta\mathcal{P}$. We claim that if $\|g\|'_\beta$ is small enough, we can define a sequence $u_j \in E_7 \cap V$ with $u_0 = 0$ by the recursive formula

$$u_{j+1} := u_j + \Delta_j \dot{u}_j, \ \dot{u}_j := \psi(v_j)g_j, \ v_j := S_{\theta_j}u_j. \tag{3.23}$$

We also claim that there exist constants C_1, C_2, C_3 such that for every $j \in \mathbb{N}$,

$$\|\dot{u}_j\|_a \leqslant C_1 \|\mathcal{P}g\|'_\beta \theta_j^{a-\alpha-1}, a \in \{1, 3, 5, 7\}, \tag{3.24}$$

$$\begin{aligned} &\|v_j\|_a \leqslant C_2 \|\mathcal{P}g\|'_\beta \theta_j^{a-\alpha}, a \in \{5, 7, 9\}, \\ &\|v_j\|_3 \leqslant C_2 \|\mathcal{P}g\|'_\beta, \end{aligned} \tag{3.25}$$

$$\|u_j - v_j\|_a \leqslant C_3 \|\mathcal{P}g\|'_\beta \theta_j^{a-\alpha}, a \in \{1, 3, 5, 7\}. \tag{3.26}$$

More precisely, we prove by induction on k the following property

$$\begin{aligned} (\mathrm{P}_k) : \ & u_j \text{ is well defined for } j = 0, \ldots, k+1, \\ & (3.24) \text{ is satisfied for } j = 0, \ldots, k, \\ & (3.25), (3.26) \text{ are satisfied for } j = 0, \ldots, k+1. \end{aligned}$$

Property (P_0) is easy to be checked. Let $k \in \mathbb{N}^*$. We suppose property (P_{k-1}) is true and we prove (P_k).

Let us introduce a real number $\rho > 0$ such that, for every $u \in E'_\alpha$, $\|u\|'_\alpha \leqslant \rho$ implies $u \in V$. With the same kind of calculus as in [1], we get (3.24)–(3.26) with

$$C_1 := 8CK',$$

$$C_2 := KC_1 \max\left\{\frac{1}{7-\alpha}, \frac{2^{\delta(\alpha-4)}}{5-\alpha}, \frac{2^{\delta(\alpha-2)}}{\alpha-1}\right\},$$

$$C_3 := C_1 \max\left\{\frac{1+K}{7-\alpha}, K''\right\},$$

for every $g \in F'_\beta$ with

$$\|g\|'_\beta \leqslant \min\left\{\frac{\rho}{KC_1}, \frac{1}{C_2}\right\}.$$

Inequality (3.24) proves that (u_k) converges in E_3 to the vector $u := \sum_{j=0}^{\infty} \Delta_j \dot{u}_j$ and

$$\|u\|_3 \leqslant \widetilde{C_1} \|\mathcal{P}g\|'_\beta, \text{ where } \widetilde{C_1} := C_1 \left(\sum_{j=0}^{\infty} \Delta_j \theta_j^{2-\alpha}\right). \tag{3.27}$$

Now, let us consider the limit of $(\Phi(u_k))_{k\in\mathbb{N}}$. We have

$$\Phi(u_{j+1}) - \Phi(u_j) = \Phi(u_j + \Delta_j \dot{u}_j) - \Phi(u_j) = \Delta_j(e'_j + e''_j + g_j),$$

where

$$e'_j := \frac{1}{\Delta_j}\left(\Phi(u_j + \Delta_j \dot{u}_j) - \Phi(u_j) - \Phi'(u_j)\Delta_j \dot{u}_j\right),$$

$$e''_j := \left(\Phi'(u_j) - \Phi'(v_j)\right)\dot{u}_j.$$

Thanks to (3.15), (3.16) and (3.24)–(3.26) and the same calculus as in [1] we get the existence of $\varepsilon, C_4, C_5 > 0$ such that, for every $j \in \mathbb{N}$,

$$\|e'_j\|_7 \leqslant C_4 \|\mathcal{P}g\|'^2_\beta \theta_j^{-1-\varepsilon}, \; \|e''_j\|_7 \leqslant C_5 \|\mathcal{P}g\|'^2_\beta \theta_j^{-1-\varepsilon}. \tag{3.28}$$

Thus $\sum \Delta_j(e'_j + e''_j)$ converges in F_7. Let us denote $T(g)$ its sum,

$$T(g) := \sum_{j=0}^{\infty} \Delta_j(e'_j + e''_j).$$

Thanks to (3.28), we get the existence of $C_6 > 0$ such that

$$\|Tg\|_7 \leqslant C_6 \|\mathcal{P}g\|'^2_\beta.$$

The continuity of Φ gives $\Phi(u_k) \to \Phi(u)$ in F_3, thus we have the following equality in F_3:

$$\Phi(u) = \Phi(0) + T(g) + g.$$

Let us fix $f \in F'_\beta$. We search u such that $\Phi(u) = \Phi(0) + f$. It is sufficient to find $g \in F'_\beta$ such that $g + Tg = f$. This is equivalent to prove the existence of a fixed point for the map

$$\begin{array}{rcl} F : F'_\beta & \to & F'_\beta, \\ g & \mapsto & f - T(g). \end{array}$$

We conclude by applying the Leray–Shauder fixed point theorem. $\square$

In our situation, we need the continuity of the map $f \mapsto u$ in order to apply the intermediate values theorem in Section 3.4. This property can be proved by applying the Banach fixed point theorem instead of the Leray–Shauder fixed point theorem in the previous proof. In order to do this, we need more assumptions, which are given in the next theorem.

Theorem 5. *Let us consider the same assumptions as in Theorem* 4. *We assume moreover that*, *for every* $u, \tilde{u} \in V \cap E_7$,

$$\|\Phi''(u; v, w) - \Phi''(\tilde{u}; v, w)\|_7 \leqslant C \sum (1 + \|u - \tilde{u}\|_{n'_j}) \|v\|_{n''_j} \|w\|_{n'''_j}, \tag{3.29}$$

where the sum is finite, *all the subscripts belong to* $\{1, 3, 5, 7\}$ *and satisfy* (3.16) *with* $m_j \leftarrow n_j$. *We also assume that*, *for every* $v, \tilde{v} \in V \cap E_9$,

$$\|(\psi(v) - \psi(\tilde{v}))g\|_1 \leqslant C\|v - \tilde{v}\|_3 \|g\|_3, \tag{3.30}$$

$$\|(\psi(v) - \psi(\tilde{v}))g\|_3 \leqslant C[\|v - \tilde{v}\|_3 \|g\|_5 + \|v - \tilde{v}\|_5 \|g\|_3], \tag{3.31}$$

$$\begin{aligned}\|(\psi(v) - \psi(\tilde{v}))g\|_5 &\leqslant C[\|v - \tilde{v}\|_3 \|g\|_7 + \|v - \tilde{v}\|_5 \|g\|_5 \\ &+ (\|v - \tilde{v}\|_7 + \|v - \tilde{v}\|_5^2)\|g\|_3],\end{aligned} \tag{3.32}$$

$$\begin{aligned}\|(\psi(v) - \psi(\tilde{v}))g\|_7 \leqslant\ & C[\|v - \tilde{v}\|_3 \|g\|_9 + \|v - \tilde{v}\|_5 \|g\|_7 \\ & + (\|v - \tilde{v}\|_7 + \|v - \tilde{v}\|_5^2)\|g\|_5 \\ & + (\|v - \tilde{v}\|_9 + \|v - \tilde{v}\|_7 \|v - \tilde{v}\|_5 + \|v - \tilde{v}\|_5^3)\|g\|_3].\end{aligned} \tag{3.33}$$

Then, *there exists* $C' > 0$, $\varepsilon > 0$ *and a continuous map*

$$\begin{aligned}\Pi : V'_\beta &\to E_3, \\ f &\mapsto u,\end{aligned}$$

where

$$V'_\beta := \{f \in F'_\beta; \|f\|'_\beta < \varepsilon\},$$

such that, for every $f \in V'_\beta$,

$$\Phi(\Pi(f)) = \Phi(0) + f,$$

$$\|\Pi(f)\|_3 \leqslant C' \|\mathcal{P} f\|'_\beta. \tag{3.34}$$

Proof. The first part of Theorem 5 has already been proved in [1, Appendix C]. Here, we justify bound (3.34). Let us recall that under assumptions (3.29)–(3.33), the map T

is a contraction on a small enough neighborhood of zero in F'_β: there exists $\delta \in (0,1)$ such that

$$\|T(g) - T(\tilde{g})\|'_\beta \leqslant \delta \|g - \tilde{g}\|'_\beta.$$

Thus, when $f = g + T(g)$ and $\tilde{f} = \tilde{g} + T(\tilde{g})$, we also have

$$\|g - \tilde{g}\|'_\beta \leqslant \frac{1}{1-\delta} \|f - \tilde{f}\|'_\beta.$$

Let $f \in F'_\beta$ small enough. Let $g \in F'_\beta$ be the solution of $f = g + T(g)$ given by the Banach fixed point theorem. Using $\tilde{f} = 0$, we have

$$\|g\|'_\beta \leqslant \frac{1}{1-\delta} \|f\|'_\beta.$$

Let $u \in E_3$ be the vector built in the proof of Theorem 4. Using (3.27) and

$$\mathcal{P}g = \mathcal{P}f - \mathcal{P}T(g), \ \|\mathcal{P}Tg\|'_\beta \leqslant \|Tg\|'_\beta \leqslant C_6 \|\mathcal{P}g\|'^2_\beta,$$

we get

$$\|\mathcal{P}g\|'_\beta \leqslant 2\|\mathcal{P}f\|'_\beta \text{ when } \|f\|'_\beta \leqslant \frac{1-\delta}{2C_6},$$

thus

$$\|u\|_3 \leqslant 2\widetilde{C_1} \|\mathcal{P}f\|'_\beta. \qquad \square$$

We apply Theorems 4 and 5 to the map Φ defined in Section 3.2.1, in a neighborhood of $(\psi_\theta(0), 0, 0, 0)$. Our spaces are defined, for $k = 1, 3, 5, 7, 9$, by

$$E_k := [\mathcal{S} \cap H^k_{(0)}(I, \mathbb{C})] \times \mathbb{R} \times \mathbb{R} \times H_0^{(k-1)/2}((0,T), \mathbb{R}),$$

$$F_k := [\mathcal{S} \cap H^k_{(0)}(I, \mathbb{C})] \times \mathbb{R} \times \mathbb{R} \times [V \cap H^k_{(0)}(I, \mathbb{C})] \times \mathbb{R} \times \mathbb{R}.$$

We work on the manifold $\mathcal{S}$ instead of a whole space. It does not matter because we can move the problem to an hyperplane of $L^2(I, \mathbb{C})$ by studying a new map

$$\widetilde{\Phi}(\widetilde{\psi_0}, S_0, D_0, u) := \Phi(p^{-1}(\widetilde{\psi_0}), S_0, D_0, u),$$

where p is a suitable local diffeomorphism from a neighborhood of the trajectory ψ_θ in the sphere $\mathcal{S}$ to an hyperplane of $L^2(I,\mathbb{C})$. For example, we can use the following one.

Proposition 3. *Let* $\mathcal{U} := \{\psi \in \mathcal{S}; \exists t \in [0, 4/\pi], \|\psi - \psi_\theta(t)\|_{L^2(I,\mathbb{C})} < \varepsilon\}$ *where* $\varepsilon > 0$ *is small enough,* $\mathcal{H} := \{\psi \in L^2(I,\mathbb{C}); \Re\langle\psi, \varphi_3\rangle = 0\}$ *and* $p : L^2(I,\mathbb{C}) \to \mathcal{H}$ *defined by*

$$p(\psi) := \psi - \Re(\langle\psi, \varphi_3\rangle)\varphi_3 + \Re(\langle\psi, \varphi_3\rangle)\langle\psi, \varphi_1\rangle\varphi_1.$$

The map p is a C^1 *diffeomorphism from* $\mathcal{U}$ *to an open subset of* $\mathcal{H}$. *Moreover, the norm of* $dp(\psi)$ *as linear operator from* $(T_{\mathcal{S}}(\psi), \|.\|_{H^s})$ *to* $(\mathcal{H}, \|.\|_{H^s})$ *is uniformly bounded on* $\mathcal{U}$ *for every integer* $s \in [1, 7]$.

The proof is similar to the one of [1, Proposition 2, Section 3.2].

Now, we build smoothing operators. First, we smooth the wave function. Note that we need a smoothing operator preserving the space $\mathcal{H}$ defined in Proposition 3. Let $s \in C^\infty(\mathbb{R},\mathbb{R})$ be such that

$$s = 1 \text{ on } [0,1], 0 \leqslant s \leqslant 1, s = 0 \text{ on } [2, +\infty).$$

We define

$$\widetilde{S}_\lambda \varphi := \sum_{k=1}^{\infty} s\left(\frac{k}{\lambda}\right)\langle\varphi, \varphi_k\rangle\varphi_k.$$

The proof of the following proposition is easy.

Proposition 4. *There exists a constant K such that, for every* $a \in \{1, \dots, 9\}$, *for every* $\varphi \in H^a_{(0)}(I,\mathbb{C})$ *and for every* $\lambda \geqslant 1$, *we have*

$$\|\widetilde{S}_\lambda\varphi\|_{H^b} \leqslant K\|\varphi\|_{H^a}, \quad b \in \{1, \dots, a\},$$

$$\|\widetilde{S}_\lambda\varphi\|_{H^b} \leqslant K\lambda^{b-a}\|\varphi\|_{H^a}, \quad b \in \{a+1, \dots, 9\},$$

$$\|\varphi - \widetilde{S}_\alpha\varphi\|_{H^b} \leqslant K\lambda^{b-a}\|\varphi\|_{H^a}, \quad b \in \{1, \dots, a-1\},$$

$$\left\|\frac{d}{d\lambda}\widetilde{S}_\lambda\varphi\right\|_{H^b} \leqslant K\lambda^{b-a-1}\|\varphi\|_{H^a}, \quad b \in \{1, \dots, 9\}.$$

The suitable smoothing operators for the control, $\widehat{S}_\lambda u$, can be built with convolution products and truncations with a C^∞-function with compact support as in [1, Section 3.3.2]. This construction is inspired from [12].

Finally, we take on the spaces E_k

$$S_\lambda(\psi_0, S_0, D_0, u) := (\widetilde{S}_\lambda\psi_0, S_0, D_0, \widehat{S}_\lambda(u)),$$

and on the spaces F_k

$$S_\lambda(\psi_0, S_0, D_0, \psi_f, S_f, D_f) := (\widetilde{S}_\lambda\psi_0, S_0, D_0, \widetilde{S}_\lambda(\psi_f), S_f, D_f).$$

Bounds (3.15), (3.16), and (3.29)–(3.33) can be checked in the same way as in [1]. In the following two sections, we focus on the most difficult part in the application of the Nash–Moser theorem, which is the proof of the existence of a right inverse for the differential, with the bounds (3.17)–(3.20).

3.2.3. Controllability up to codimension one of the linearized system around $(Y^{\theta,0,0}, u \equiv 0)$ *and bounds (3.17)–(3.20)*

The aim of this section is the proof of the following proposition.

Proposition 5. *Let* $T := 4/\pi$. *There exists* $C > 0$ *such that, for every*

$$(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f) \in [T_{\mathcal{S}}(\psi_\theta(0)) \cap H^9_{(0)}(I, \mathbb{C})] \times \mathbb{R} \times \mathbb{R} \times [V \cap H^9_{(0)}(I, \mathbb{C})] \times \mathbb{R} \times \mathbb{R},$$

there exists $w \in H^3_0((0, T), \mathbb{R})$ *such that the solution of* (Σ^l_θ) *with control* w *such that* $(\Psi(0), s(0), d(0)) = (\Psi_0, s_0, d_0)$ *satisfies* $(\mathcal{P}\Psi(T), s(T), d(T)) = (\widetilde{\Psi_f}, s_f, d_f)$ *and*

$$\|w\|_{L^2((0,T),\mathbb{R})} \leqslant C\|(\mathcal{P}\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_{F_3},$$

$$\|w\|_{H^1_0((0,T),\mathbb{R})} \leqslant C\|(\mathcal{P}\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_{F_5},$$

$$\|w\|_{H^2_0((0,T),\mathbb{R})} \leqslant C\|(\mathcal{P}\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_{F_7},$$

$$\|w\|_{H^3_0((0,T),\mathbb{R})} \leqslant C\|(\mathcal{P}\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_{F_9}.$$

Moreover, the map

$$\begin{array}{ccccccccc} [T_{\mathcal{S}}(\psi_\theta(0)) \cap H^9_{(0)}] & \times \mathbb{R} & \times \mathbb{R} & \times [V \cap H^9_{(0)}] & \times \mathbb{R} & \times \mathbb{R} & \to & H^3_0((0,T),\mathbb{R}), \\ (\Psi_0 & , s_0 & , d_0 & , \widetilde{\Psi_f} & , s_f & , d_f) & \mapsto & w \end{array}$$

is continuous.

Remark 3. The function $\Psi(T)$ is the unique function $\Psi_f \in T_{\mathcal{S}}(\psi_\theta(T))$ which satisfies (3.2) and $\mathcal{P}\Psi_f = \widetilde{\Psi_f}$.

Let us introduce the notations, for $s \in \{0, \dots, 6\}$

$$h^s(\mathbb{N}, \mathbb{C}) := \left\{ d = (d_k)_{k\in\mathbb{N}};\ \|d\|_{h^s(\mathbb{N},\mathbb{C})} := \left(|d_0| + \sum_{k=1}^{\infty} |k^s d_k|^2 \right)^{1/2} < +\infty \right\},$$

$$h^s_r(\mathbb{N}, \mathbb{C}) := \{d = (d_k)_{k\in\mathbb{N}} \in h^s(\mathbb{N}, \mathbb{C}); d_0, d_1 \in \mathbb{R}\},$$

we write l^2_r instead of h^0_r.

Proof. Let

$$(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f) \in [T_{\mathcal{S}}(\psi_\theta(0)) \cap H^9_{(0)}(I, \mathbb{C})] \times \mathbb{R} \times \mathbb{R} \times [V \cap H^9_{(0)}(I, \mathbb{C})] \times \mathbb{R} \times \mathbb{R},$$

and (Ψ, s, d) be a solution of (Σ^l_θ) with $(\Psi(0), s(0), d(0)) = (\Psi_0, s_0, d_0)$ and a control $w \in H^3_0((0, T), \mathbb{R})$. As noticed in Section 3.1, the equality $(\mathcal{P}\Psi(T), s(T), d(T)) = (\widetilde{\Psi_f}, s_f, d_f)$ is equivalent to

$$Z(w) = D(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f),$$

where

$$Z(w) := (Z(w)_k)_{k\in\mathbb{N}} \quad \text{and} \quad D(\Psi_0, s_0, d_0, \Psi_f, s_f, d_f) := (D_k)_{k\in\mathbb{N}}$$

are defined by

$$\begin{aligned} &Z(w)_0 := \int_0^T (T-t) w(t)\, dt, && Z(w)_1 := \int_0^T w(t)\, dt, \\ &Z(w)_{2k} := \int_0^T w(t) e^{i(\lambda_{2k}-\lambda_1)t}\, dt, && Z(w)_{2k+1} := \int_0^T w(t) e^{i(\lambda_{2k+1}-\lambda_2)t}\, dt, k \in \mathbb{N}^*, \end{aligned}$$

$$\begin{aligned} &D_0 := d_f - d_0 - s_0 T, && D_1 := s_f - s_0, \\ &D_{2k} := \tfrac{-i}{\sqrt{1-\theta b_{2k}}} \langle \widetilde{\Psi_f} - \Psi_0, \varphi_{2k} \rangle, && D_{2k+1} := \tfrac{-i}{\sqrt{\theta} c_{2k+1}} \langle \widetilde{\Psi_f} - \Psi_0, \varphi_{2k+1} \rangle, k \in \mathbb{N}^*. \end{aligned}$$

Using the behavior of the coefficients c_k and b_k given by (3.5) and standard results about Fourier series, we get a constant $C > 0$ such that, for every $(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)$, for $s = 0, 2, 4, 6$,

$$\|D(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_{h^s(\mathbb{N},\mathbb{C})} \leqslant C \|(\mathcal{P}\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_{F_{s+3}}.$$

Thus, it is sufficient to prove the following proposition to end this proof. □

Proposition 6. *The linear map Z is continuous from E to F for every*

$$(E, F) \in \{(L^2, l_r^2), (H_0^1, h_r^2), (H_0^2, h_r^4), (H_0^3, h_r^6)\}.$$

There exist $C > 0$ and a continuous right inverse

$$Z^{-1} : h_r^6(\mathbb{N}, \mathbb{C}) \to H_0^3((0, T), \mathbb{R})$$

such that, for every $d \in h_r^6(\mathbb{N}, \mathbb{C})$,

$$\|Z^{-1}(d)\|_{L^2} \leqslant C\|d\|_{l^2}, \quad \|Z^{-1}(d)\|_{H_0^1} \leqslant C\|d\|_{h^2},$$
$$\|Z^{-1}(d)\|_{H_0^2} \leqslant C\|d\|_{h^4}, \quad \|Z^{-1}(d)\|_{H_0^3} \leqslant C\|d\|_{h^6}.$$

Proof. The first statement comes from integrations by parts and standard results about Fourier series. Let us introduce the notations

$$\omega_1 := 0, \ \omega_{2k} := \lambda_{2k} - \lambda_1, \ \omega_{2k+1} := \lambda_{2k+1} - \lambda_2 \quad \text{for } k \in \mathbb{N}^*.$$

Let $d \in h_r^6(\mathbb{N}, \mathbb{C})$. A suitable candidate for $Z^{-1}(d)$ is the function

$$w(t) := \tfrac{1}{T}\Big[\tfrac{d_1}{6} + a_2 e^{-i\omega_2 t} + \overline{a_2} e^{i\omega_2 t} + a_3 e^{-i\omega_3 t} + \overline{a_3} e^{i\omega_3 t} + \sum_{k=4}^{\infty}\left(\tfrac{d_k}{6} e^{-i\omega_k t} + \tfrac{\overline{d_k}}{6} e^{i\omega_k t}\right) + \alpha e^{-i\omega t} + \overline{\alpha} e^{i\omega t}\Big]\left(e^{i\frac{1}{2}\pi^2 t} - 1\right)^2\left(e^{-i\frac{1}{2}\pi^2 t} - 1\right)^2,$$

where

$$a_2 := \tfrac{6d_2 - d_3}{35}, \ a_3 := \tfrac{6d_3 - d_2}{35}.$$

$$\omega = \frac{1}{2}m\pi^2 \text{ with } m \in \mathbb{N} \text{ and } \{m, m \pm 1, m \pm 2\} \cap \left\{\frac{2}{\pi^2}\omega_k, \frac{2}{\pi^2}\omega_k \pm 1, \frac{2}{\pi^2}\omega_k \pm 2; k \in \mathbb{N}^*\right\} = \emptyset$$

and $\alpha \in \mathbb{C}$ is such that $\int_0^T (T - t)w(t)\,dt = d_0$. □

3.2.4. Controllability up to codimension one of the linearized system around (Y, u) and bounds (3.17)–(3.20)

Let $(\psi_0, S_0, D_0, u) \in E_9$ The aim of this section is the proof of the existence of a right inverse to $d\Phi(\psi_0, S_0, D_0, u)$ with estimates (3.17)–(3.20).

Let (ψ, S, D) be the solution of (Σ_0) with control u such that $(\psi(0), S(0), D(0)) = (\psi_0, S_0, D_0)$. The linearized system around (ψ, S, D, u) is

$$\begin{cases} i\dot{\Psi} = -\frac{1}{2}\Psi'' - uq\Psi - wq\psi, \\ \Psi(t, \pm 1/2) = 0, \\ \dot{s} = w, \\ \dot{d} = s. \end{cases} \tag{3.35}$$

It is a control system where

- the state is (Ψ, s, d) with $\Psi(t) \in T_{\mathcal{S}}(\psi(t))$, for every t,
- the control is the real valued function w.

Let $T := 4/\pi$ and

$$(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f) \in [T_{\mathcal{S}}(\psi_0) \cap H^9_{(0)}(I, \mathbb{C})] \times \mathbb{R} \times \mathbb{R} \times [V \cap H^9_{(0)}(I, \mathbb{C})] \times \mathbb{R} \times \mathbb{R}.$$

We look for $w \in H^3_0((0, T), \mathbb{R})$ such that the solution of (3.35) with $(\Psi(0), s(0), d(0)) = (\Psi_0, s_0, d_0)$ satisfies

$$(\Psi(T), s(T), d(T)) = (\mathcal{P}\widetilde{\Psi_f}, s_f, d_f) \tag{3.36}$$

and

$$\begin{aligned}
\|w\|_{L^2} &\leqslant C[\|(\mathcal{P}\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_3 + \Delta_3\|(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_3], \\
\|w\|_{H^1_0} &\leqslant C[\|(\mathcal{P}\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_5 + \Delta_3\|(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_5 \\
&\quad + \Delta_5\|(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_3], \\
\|w\|_{H^2_0} &\leqslant C[\|(\mathcal{P}\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_7 + \Delta_3\|(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_7 \\
&\quad + \Delta_5\|(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_5 + (\Delta_7 + \Delta_5^2)\|(\Psi_0, s_0, d_0, \\
&\quad \widetilde{\Psi_f}, s_f, d_f)\|_3], \\
\|w\|_{H^3_0} &\leqslant C[\|(\mathcal{P}\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_9 + \Delta_3\|(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_9 \\
&\quad + \Delta_5\|(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_7 + (\Delta_7 + \Delta_5^2)\|(\Psi_0, s_0, d_0, \\
&\quad \widetilde{\Psi_f}, s_f, d_f)\|_5 \\
&\quad + (\Delta_9 + \Delta_7\Delta_5 + \Delta_5^3)\|(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_3],
\end{aligned} \tag{3.37}$$

where

$$\Delta_k := \|(\psi_0 - \psi_\theta(0), S_0, D_0, u)\|_{E_k} \text{ for } k = 3, 5, 7, 9.$$

Let us consider the decomposition $(\Psi, s, d) = (\Psi_1, s_1, d_1) + (\Psi_2, s_2, d_2)$ where

$$\begin{cases} i\dot{\Psi}_1 = -\frac{1}{2}\Psi_1'' - uq\Psi_1, \\ \Psi_1(t, \pm 1/2) = 0, \\ \Psi_1(0) = \Psi_0, \\ \dot{s}_1 = 0, s_1(0) = s_0, \\ \dot{d}_1 = s_1, d_1 = d_0. \end{cases} \quad \begin{cases} i\dot{\Psi}_2 = -\frac{1}{2}\Psi_2'' - uq\Psi_2 - wq\psi, \\ \Psi_2(t, \pm 1/2) = 0, \\ \Psi_2(0) = 0, \\ \dot{s}_2 = w, s_2(0) = 0, \\ \dot{d}_2 = s_2, d_2(0) = 0. \end{cases}$$

Equality (3.36) is equivalent to

$$(\mathcal{P}\Psi_2(T), s_2(T), d_2(T)) = (\widetilde{\Psi_f} - \mathcal{P}\Psi_1(T), s_f - s_0, d_f - d_0 - s_0 T). \tag{3.38}$$

Let us introduce, for $\gamma \in \mathbb{R}$ the operator A_γ defined on

$$D(A_\gamma) := H^2 \cap H_0^1(I, \mathbb{C}) \text{ by } A_\gamma \varphi := -\tfrac{1}{2}\varphi'' - \gamma q \varphi$$

and $(\lambda_{k,\gamma})_{k \in \mathbb{N}^*}$ the increasing sequence of eigenvalues for A_γ. We know from [15, Chapter 7, Example 2.14] that $\lambda_{k,\gamma}$ are analytic functions of the parameter γ.

Equality (3.38) is equivalent to

$$M_{(\psi_0,u)}(w) = D_{(\psi_0,u)}(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f),$$

where

$$M_{(\psi_0,u)}(w) := \left(d_2(T), s_2(T), \langle \Psi_2(T), \varphi_2 \rangle e^{i\int_0^T \lambda_{2,u(s)}\,ds}, \langle \Psi_2(T), \varphi_3 \rangle e^{i\int_0^T \lambda_{3,u(s)}\,ds}, \dots\right),$$

$$\begin{aligned} D_{(\psi_0,u)}(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f) := \big(&d_f - d_0 - s_0 T, s_f - s_0, \langle \widetilde{\Psi_f} - \\ &\Psi_1(T), \varphi_2 \rangle e^{i\int_0^T \lambda_{2,u(s)}\,ds}, \\ &\langle \widetilde{\Psi_f} - \Psi_1(T), \varphi_3 \rangle e^{i\int_0^T \lambda_{3,u(s)}\,ds}, \dots\big). \end{aligned}$$

Proposition 7. *The linear map $M_{(\psi_0,u)}$ is continuous from E to F for every*

$$(E, F) \in \{(L^2, h_r^3), (H_0^1, h_r^5), (H_0^2, h_r^7), (H_0^3, h_r^9)\}.$$

There exist $C > 0$ and $\eta > 0$ such that, for every $(\psi_0, u) \in H^9_{(0)}(I, \mathbb{C}) \times \mathbb{R} \times \mathbb{R}$ with

$$\|\psi_0 - \psi_\theta(0)\|_{H^3(I,\mathbb{C})} + \|u\|_{H_0^1((0,T),\mathbb{R})} < \eta,$$

there exists a continuous right inverse

$$M^{-1}_{(\psi_0,u)} : h^9_r(\mathbb{N},\mathbb{C}) \to H^3_0((0,T),\mathbb{R})$$

such that, for every $d \in h^9_r(\mathbb{N},\mathbb{C})$,

$$\begin{aligned}
&\|M^{-1}_{(\psi_0,u)}(d)\|_{L^2} \leqslant C\|d\|_{h^3},\\
&\|M^{-1}_{(\psi_0,u)}(d)\|_{H^1_0} \leqslant C[\|d\|_{h^5} + \Delta_5\|d\|_{h^3}],\\
&\|M^{-1}_{(\psi_0,u)}(d)\|_{H^2_0} \leqslant C[\|d\|_{h^7} + \Delta_5\|d\|_{h^5} + (\Delta_7 + \Delta_5^2)\|d\|_{h^3}],\\
&\|M^{-1}_{(\psi_0,u)}(d)\|_{H^3_0} \leqslant C[\|d\|_{h^9} + \Delta_5\|d\|_{h^7} + (\Delta_7 + \Delta_5^2)\|d\|_{h^5} + (\Delta_9 + \Delta_7\Delta_5 + \Delta_5^3)\|d\|_{h^3}],
\end{aligned}$$

where

$$\Delta_k := \|(\psi_0 - \psi_\theta(0), S_0, D_0, u)\|_{E_k} \text{ for } k = 3, 5, 7, 9.$$

In order to get this result, we prove that when (ψ_0, u) is close to $(\psi_\theta(0), 0)$ in $H^3_{(0)}(I,\mathbb{C}) \times H^1_0((0,T),\mathbb{R})$, the map $M_{(\psi_0,u)}$ is close enough to the map $M_{(\psi_\theta(0),0)}$, in a sense presented in the following proposition, so that

- the existence of a right inverse $M^{-1}_{(\psi_\theta(0),0)}$ guarantees the existence of a right inverse $M^{-1}_{(\psi_0,u)}$,
- the bounds proved on $M^{-1}_{(\psi_\theta(0),0)}$ give the same kind of bounds on $M^{-1}_{(\psi_0,u)}$.

More precisely, we apply the following proposition already proved in [1, Proposition 15, Section 3.6.1].

Proposition 8. *Let* $T := 4/\pi$, M and M_θ *be bounded linear operators from* $L^2((0,T),\mathbb{R})$ to $h^3(\mathbb{N},\mathbb{C})$, *from* $H^1_0((0,T),\mathbb{R})$ to $h^5(\mathbb{N},\mathbb{C})$, *from* $H^2_0((0,T),\mathbb{R})$ to $h^7(\mathbb{N},\mathbb{C})$ *and from* $H^3_0((0,T),\mathbb{R})$ to $h^9(\mathbb{N},\mathbb{C})$. *We assume there exist a continuous linear operator* $M^{-1}_\theta : h^9(\mathbb{N},\mathbb{C}) \to H^3_0((0,T),\mathbb{R})$ *and a positive constant* C_0 *such that for every* $d \in h^9(\mathbb{N},\mathbb{C})$, $M_\theta \circ M^{-1}_\theta(d) = d$ *and* $\|M^{-1}_\theta(d)\|_E \leqslant C_0\|d\|_F$ *for every* $(E,F) \in \{(L^2,h^3),(H^1_0,h^5),(H^2_0,h^7),(H^3_0,h^9)\}$. *We also assume there exist positive constants* $C_1, \Delta_3, \Delta_5, \Delta_7, \Delta_9$ *with* $C_0C_1\Delta_3 \leqslant 1/2$, *satisfying, for every* $w \in H^3_0((0,T),\mathbb{R})$

$$\begin{aligned}
&\|(M - M_\theta)(w)\|_{h^3} \leqslant C_1\Delta_3\|w\|_{L^2},\\
&\|(M - M_\theta)(w)\|_{h^5} \leqslant C_1[\Delta_3\|w\|_{H^1_0} + \Delta_5\|w\|_{L^2}],\\
&\|(M - M_\theta)(w)\|_{h^7} \leqslant C_1[\Delta_3\|w\|_{H^2_0} + \Delta_5\|w\|_{H^1_0} + \Delta_7\|w\|_{L^2}],\\
&\|(M - M_\theta)(w)\|_{h^9} \leqslant C_1[\Delta_3\|w\|_{H^3_0} + \Delta_5\|w\|_{H^2_0} + \Delta_7\|w\|_{H^1_0} + \Delta_9\|w\|_{L^2}].
\end{aligned} \tag{3.39}$$

Then, there exists a continuous linear operator $M^{-1} : h^9(\mathbb{N}, \mathbb{C}) \to H_0^3((0,T), \mathbb{R})$ *such that for every* $d \in h^9(\mathbb{N}, \mathbb{C})$, $M \circ M^{-1}(d) = d$ *and the function* $w := M^{-1}(d)$ *satisfies*

$$\begin{aligned}
\|w\|_{L^2} &\leqslant 2C_0\|d\|_{h^3},\\
\|w\|_{H_0^1} &\leqslant 2C_0[\|d\|_{h^5} + 2C_2\Delta_5\|d\|_{h^3}],\\
\|w\|_{H_0^2} &\leqslant 2C_0[\|d\|_{h^7} + 2C_2\Delta_5\|d\|_{h^5} + (2C_2\Delta_7 + 8C_2^2\Delta_5^2)\|d\|_{h^3}],\\
\|w\|_{H_0^3} &\leqslant 2C_0[\|d\|_{h^9} + 2C_2\Delta_5\|d\|_{h^7} + (2C_2\Delta_7 + 8C_2^2\Delta_5^2)\|d\|_{h^5}\\
&\quad +(2C_2\Delta_9 + 16C_2^2\Delta_7\Delta_5 + 48C_2^3\Delta_5^3)\|d\|_{h^3}],
\end{aligned}$$

where $C_2 := C_0C_1$.

Let us recall that, for $\gamma \in \mathbb{R}$, the space $L^2(I, \mathbb{C})$ has a complete orthonormal system $(\varphi_{k,\gamma})_{k\in\mathbb{N}^*}$ of eigenvectors for A_γ:

$$A_\gamma \varphi_{k,\gamma} = \lambda_{k,\gamma}\varphi_{k,\gamma}.$$

We know from [15, Chapter 7, Example 2.14] that $\varphi_{k,\gamma}$ are analytic functions of the parameter γ. This result gives sense to the notation

$$\frac{d\varphi_{k,\gamma}}{d\gamma}]_{\gamma_0}$$

which means the derivative of the map $\gamma \mapsto \varphi_{k,\gamma}$ with respect to γ evaluated at the point $\gamma = \gamma_0$.

Proof of Proposition 7. Let us consider the decomposition

$$\Psi_2(t) = \sum_{k=1}^{\infty} x_k(t)\varphi_{k,u(t)} \text{ where } x_k(t) := \langle \Psi_2(t), \varphi_{k,u(t)}\rangle.$$

Using $u(T) = 0$, we get

$$\begin{aligned}
&M_{(\psi_0,u)}(w)\\
&= \left(\int_0^T (T-t)w(t)\,dt, \int_0^T w(t)\,dt, x_2(T)e^{i\int_0^T \lambda_{2,u(s)}\,ds}, x_3(T)e^{i\int_0^T \lambda_{3,u(s)}\,ds}, \ldots\right).
\end{aligned}$$

The partial differential equation satisfied by Ψ_2 provides, for every $k \in \mathbb{N}^*$, an ordinary differential equation satisfied by the component x_k,

$$\dot{x}_k(t) = \left\langle \frac{\partial \Psi_2}{\partial t}(t), \varphi_{k,u(t)}\right\rangle + \dot{u}(t)\left\langle \Psi_2(t), \frac{d\varphi_{k,\gamma}}{d\gamma}]_{u(t)}\right\rangle,$$

$$\left\langle \frac{\partial \Psi_2}{\partial t}(t), \varphi_{k,u(t)} \right\rangle = \langle -iA_{u(t)}\Psi_2(t) + iw(t)q\psi(t), \varphi_{k,u(t)} \rangle$$

$$= -i\lambda_{k,u(t)} x_k(t) + iw(t)\langle q\psi(t), \varphi_{k,u(t)} \rangle,$$

$$\dot{x}_k(t) = -i\lambda_{k,u(t)} x_k(t) + iw(t)\langle q\psi(t), \varphi_{k,u(t)} \rangle + \dot{u}(t)\left\langle \Psi_2(t), \frac{d\varphi_{k,\gamma}}{d\gamma}]_{u(t)} \right\rangle.$$

Solving this equation, we get

$$M_{(\psi_0,u)}(w)_k = \int_0^T \left(iw(t)\langle q\psi(t), \varphi_{k,u(t)} \rangle + \dot{u}(t)\left\langle \Psi_2(t), \frac{d\varphi_{k,\gamma}}{d\gamma}]_{u(t)} \right\rangle \right) \times e^{i\int_0^t \lambda_{k,u(s)}\,ds}\,dt, k \geqslant 2.$$

We introduce the following decomposition:

$$(M_{(\psi_0,u)} - M_{(\psi_\theta(0),0)})(w) = \delta M(w)^1 + \delta M(w)^2,$$

where

$$\delta M(w)_k^j = 0 \text{ for } j = 1, 2 \text{ and } k = 0, 1,$$

$$\delta M(w)_k^1 = i\int_0^T w(t)[\langle q\psi(t), \varphi_{k,u(t)} \rangle e^{i\int_0^t \lambda_{k,u(s)}\,ds} - \langle q\psi_\theta(t), \varphi_k \rangle e^{i\lambda_k t}]\,dt, k \geqslant 2,$$

$$\delta M(w)_k^2 = \int_0^T \dot{u}(t)\left\langle \Psi_2(t), \frac{d\varphi_{k,\gamma}}{d\gamma}]_{u(t)} \right\rangle e^{i\int_0^t \lambda_{k,u(s)}\,ds}\,dt, k \geqslant 2.$$

Let us justify bounds (3.39) on the terms $\delta M(w)^j$ for $j = 1, 2$. The study of $\delta M(w)^1$ can be done in the same way as in the proof of [1, Section 3.6.2, Proposition 27] (with $\gamma = 0$). The study of $\delta M(w)^2$ can be done by applying [1, Propositions 18, 20, 23, 25, Section 3.6.2]. □

Proposition 9. *We assume $\Delta_3 \leqslant 1$. There exists $C > 0$ such that, for every*

$$(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f) \in [T_{\mathcal{S}}(\psi_0) \cap H^9_{(0)}(I, \mathbb{C})] \times \mathbb{R} \times \mathbb{R} \times [V \cap H^9_{(0)}(I, \mathbb{C})] \times \mathbb{R} \times \mathbb{R},$$

we have

$$\|D(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_{h^3} \leqslant C[\|(\mathcal{P}\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_3 + \Delta_3\|(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_3],$$

$$\|D(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_{h^5} \leqslant C[\|(\mathcal{P}\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_5 + \Delta_3\|(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_5 + \Delta_5\|(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_3],$$

$$\|D(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_{h^7} \leqslant C[\|(\mathcal{P}\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_7 + \Delta_3\|(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_7 + \Delta_5\|(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_5 + (\Delta_7 + \Delta_5^2)\|(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_3],$$

$$\|D(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_{h^9} \leqslant C[\|(\mathcal{P}\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_9 + \Delta_3\|(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_9 + \Delta_5\|(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_7 + (\Delta_7 + \Delta_5^2)\|(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_5 \times(\Delta_9 + \Delta_7\Delta_5 + \Delta_5^3)\|(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_3].$$

Proof. Standard results about Fourier series provide the existence of $C > 0$ such that, for every

$$(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f) \in [T_{\mathcal{S}}(\psi_0) \cap H^9_{(0)}(I, \mathbb{C})] \times \mathbb{R} \times \mathbb{R} \times [V \cap H^9_{(0)}(I, \mathbb{C})] \times \mathbb{R} \times \mathbb{R},$$

for every $s \in \{3, 5, 7, 9\}$,

$$\|D_{(\psi_0,u)}(\Psi_0, s_0, d_0, \widetilde{\Psi_f}, s_f, d_f)\|_{h^s(\mathbb{N},\mathbb{C})} \leqslant C[\|\mathcal{P}\Psi_1(T)\|_{H^s} + \|\widetilde{\Psi_f}\|_{H^s} + |s_0| + |d_0| + |s_f| + |d_f|].$$

Thus, it is sufficient to prove the existence of $C > 0$ such that

$$\begin{aligned}
\|\mathcal{P}\Psi_1(T)\|_{H^3} &\leqslant C[\|\mathcal{P}\Psi_0\|_{H^3} + \Delta_3\|(\Psi_0, u)\|_{H^3\times H^1_0}],\\
\|\mathcal{P}\Psi_1(T)\|_{H^5} &\leqslant C[\|\mathcal{P}\Psi_0\|_{H^5} + \Delta_3\|(\Psi_0, u)\|_{H^5\times H^2_0} + \Delta_5\|(\Psi_0, u)\|_{H^3\times H^1_0}],\\
\|\mathcal{P}\Psi_1(T)\|_{H^7} &\leqslant C[\|\mathcal{P}\Psi_0\|_{H^7} + \Delta_3\|(\Psi_0, u)\|_{H^7\times H^3_0} + \Delta_5\|(\Psi_0, u)\|_{H^5\times H^2_0}\\
&\quad +\Delta_7\|(\Psi_0, u)\|_{H^3\times H^1_0}],\\
\|\mathcal{P}\Psi_1(T)\|_{H^9} &\leqslant C[\|\mathcal{P}\Psi_0\|_{H^9} + \Delta_3\|(\Psi_0, u)\|_{H^9\times H^4_0} + \Delta_5\|(\Psi_0, u)\|_{H^7\times H^3_0}\\
&\quad +\Delta_7\|(\Psi_0, u)\|_{H^5\times H^2_0} + \Delta_9\|(\Psi_0, u)\|_{H^3\times H^1_0}].
\end{aligned} \tag{3.40}$$

For every $s \in \{3, 5, 7, 9\}$, we have

$$\|\mathcal{P}\Psi_1(T)\|_{H^s} \leqslant C \left(\sum_{k=2}^{\infty} |k^s x_k(T)|^2 \right)^{1/2} \quad \text{where } x_k(t) := \langle \Psi_1(t), \varphi_{k,u(t)} \rangle,$$

because $u(T) = 0$. Thanks to the equation satisfied by Ψ_1, we get

$$x_k(T) = \left(\langle \Psi_0, x_k \rangle + \int_0^T \dot{u}(t) \left\langle \Psi_1(t), \frac{d\varphi_{k,\gamma}}{d\gamma}]_{u(t)} \right\rangle e^{i \int_0^t \lambda_{k,u(s)}\, ds}\, dt \right) e^{-i \int_0^T \lambda_{k,u(s)}\, ds},$$

$$\left(\sum_{k=2}^{\infty} |k^s x_k(T)|^2 \right)^{1/2}$$

$$\leqslant \|\mathcal{P}\Psi_0\|_{H^s} + \left(\sum_{k=2}^{\infty} \left| k^s \int_0^T \dot{u}(t) \langle \Psi_1(t), \frac{d\varphi_{k,\gamma}}{d\gamma}]_{u(t)} \rangle e^{i \int_0^t \lambda_{k,u(s)}\, ds}\, dt \right|^2 \right)^{1/2}.$$

Using [1, Propositions 17, 20, 23, 29], we get (3.40). □

In conclusion, using Propositions 7 and 9, we get bounds (3.37).

3.3. *Motion in the directions* $(\psi, S, D) = (\pm i\varphi_1, 0, 0)$

The aim of this section is the proof of the following theorem.

Theorem 6. *Let $T := 4/\pi$. There exists $w_\pm \in H^4 \cap H_0^3((0,T),\mathbb{R})$, $v_\pm \in H_0^3((0,T),\mathbb{R})$ such that the solutions of*

$$\begin{cases} i\dot{\Psi}_\pm = -\frac{1}{2}\Psi_\pm'' - w_\pm q\psi_\theta, \\ \Psi_\pm(0) = 0, \\ \Psi_\pm(t,-1/2) = \Psi_\pm(t,1/2) = 0, \\ \dot{s}_\pm = w_\pm,\ s_\pm(0) = 0, \\ \dot{d}_\pm = s_\pm,\ d_\pm(0) = 0, \end{cases} \tag{3.41}$$

$$\begin{cases} i\dot{\xi}_\pm = -\frac{1}{2}\xi_\pm'' - w_\pm q\Psi_\pm - v_\pm q\psi_\theta, \\ \xi_\pm(0) = 0, \\ \xi_\pm(t,-1/2) = \xi_\pm(t,1/2) = 0, \\ \dot{\sigma}_\pm = v_\pm,\ \sigma_\pm(0) = 0, \\ \dot{\delta}_\pm = \sigma_\pm,\ \delta_\pm(0) = 0, \end{cases} \tag{3.42}$$

satisfy $\Psi_\pm(T) = 0$, $s_\pm(T) = 0$, $d_\pm(T) = 0$, $\xi_\pm(T) = \pm i\varphi_1$, $\sigma_\pm = 0$, $\delta_\pm = 0$.

We introduce the following subspace of $L^2((0,T),\mathbb{C})$:

$$\mathcal{X} := \mathrm{Span}(1, t, e^{\pm i(\lambda_{2k}-\lambda_1)t}, e^{\pm i(\lambda_{2k+1}-\lambda_2)t}; k \in \mathbb{N}^*).$$

The symbol $\mathcal{X}^\perp$ denotes the orthogonal subspace to $\mathcal{X}$ in $L^2((0,T),\mathbb{C})$.

Proposition 10. *There exists $w \in H^4 \cap H_0^3((0,T),\mathbb{R}) \cap \mathcal{X}^\perp$ such that*

$$\int_0^T w(t)\langle q\Psi(t), \varphi_1\rangle e^{i\lambda_1 t}\,dt - \frac{\sqrt{\theta}}{\sqrt{1-\theta}}\int_0^T w(t)\langle q\varphi_2, \Psi(t)\rangle e^{-i\lambda_2 t}\,dt \in (0,+\infty)\ (resp. (-\infty,0)), \tag{3.43}$$

where Ψ is the solution of

$$\begin{cases} i\dot{\Psi} = -\frac{1}{2}\Psi'' - wq\psi_\theta, \\ \Psi(0) = 0, \\ \Psi(t,\pm 1/2) = 0. \end{cases} \tag{3.44}$$

Remark 4. If $w \in \mathcal{X}^\perp$ and Ψ solves the previous system, then

$$\int_0^T w(t)\langle q\Psi(t), \varphi_1\rangle e^{i\lambda_1 t}\,dt - \frac{\sqrt{\theta}}{\sqrt{1-\theta}}\int_0^T w(t)\langle q\varphi_2, \Psi(t)\rangle e^{-i\lambda_2 t}\,dt \in \mathbb{R}.$$

Indeed, we have (see (3.3) and (3.4))

$$\Psi(t) = \sum_{k=1}^\infty x_k(t)\varphi_k, \quad \text{where}$$

$$\begin{cases} x_{2k}(t) = i\sqrt{1-\theta}\, b_{2k} e^{-i\lambda_{2k}t}\int_0^t w(\tau)e^{i(\lambda_{2k}-\lambda_1)\tau}\,d\tau, \\ x_{2k-1}(t) = i\sqrt{\theta}\, c_{2k-1} e^{-i\lambda_{2k-1}t}\int_0^t w(\tau)e^{i(\lambda_{2k-1}-\lambda_2)\tau}\,d\tau, \end{cases} \tag{3.45}$$

where b_k and c_k are given by (3.5). Thus, we get

$$\int_0^T w(t)\langle q\Psi(t), \varphi_1\rangle e^{i\lambda_1 t}\,dt = i\sqrt{1-\theta}\sum_{k=1}^\infty b_{2k}^2 f_{2k}, \tag{3.46}$$

$$\int_0^T w(t)\langle q\varphi_2, \Psi(t)\rangle e^{-i\lambda_2 t}\,dt = -i\sqrt{\theta}\sum_{k=0}^\infty c_{2k+1}^2 f_{2k+1}, \tag{3.47}$$

where

$$f_{2k} := \int_0^T w(t)e^{i(\lambda_1-\lambda_{2k})t}\int_0^t w(\tau)e^{i(\lambda_{2k}-\lambda_1)\tau}\,d\tau\,dt, \forall k \in \mathbb{N}^*,$$

$$f_{2k+1} := \int_0^T w(t)e^{i(\lambda_{2k+1}-\lambda_2)t}\int_0^t w(\tau)e^{i(\lambda_2-\lambda_{2k+1})\tau}\,d\tau\,dt, \forall k \in \mathbb{N}.$$

Thanks to integrations by parts and the property $w \in \mathcal{X}^\perp$, we get, for every $k \in \mathbb{N}^*$, $f_k \in i\mathbb{R}$.

Proof of Proposition 10. Let us consider functions of the form

$$w(t) = a_1 \sin(\tfrac{1}{2}n_1\pi^2 t) + a_2 \sin(\tfrac{1}{2}n_2\pi^2 t) + \sin(\tfrac{1}{2}n_3\pi^2 t),$$

where n_1, n_2, n_3 are three distinct positive integers such that

$$n_1, n_2, n_3 \notin \{0, \pm[(2k)^2-1], \pm[(2k-1)^2-4]; k \in \mathbb{N}^*\},$$

and a_1, a_2 are defined by

$$a_1 := \tfrac{n_1(n_3^2-n_2^2)}{n_3(n_2^2-n_1^2)},\ a_2 := \tfrac{n_2(n_1^2-n_3^2)}{n_3(n_2^2-n_1^2)}.$$

Then,

$$w \in H^4 \cap H_0^3((0,T),\mathbb{R}) \cap \mathcal{X}^\perp.$$

Let Ψ be the solution of (3.44). Condition (3.43) is equivalent to

$$i\sum_{k=1}^\infty b_{2k}^2 f_{2k} + i\frac{\theta}{1-\theta}\sum_{k=0}^\infty c_{2k+1}^2 f_{2k+1} \in (0,+\infty) \quad (\text{resp.}(-\infty,0)). \tag{3.48}$$

Using (3.5), the two previous infinite sums can be computed explicitly. We find

$$i\sum_{k=1}^\infty b_{2k}^2 f_{2k} = \frac{32T}{\pi^6}(a_1^2 A_{n_1} + a_2^2 A_{n_2} + A_{n_3}),$$

$$i\sum_{k=0}^\infty c_{2k+1}^2 f_{2k+1} = \frac{32T}{\pi^6}(a_1^2 B_{n_1} + a_2^2 B_{n_2} + B_{n_3}),$$

where

$$A_n := \sum_{k=1}^{\infty} \frac{(2k)^2}{(1+2k)^4(1-2k)^4}\left(\frac{1}{n+4k^2-1}+\frac{1}{-n+4k^2-1}\right),$$

$$B_n := \sum_{k=0}^{\infty} \frac{4(2k+1)^2}{(3+2k)^4(1-2k)^4}\left(\frac{1}{-n+4-(2k+1)^2}+\frac{1}{n+4-(2k+1)^2}\right).$$

Let us choose $n_1 = 1$, $n_2 = 2$, $n_3 = 4$ (resp. $n_1 = 1$, $n_2 = 4$, $n_3 = 6$) then

$$a_1^2 A_{n_1} + a_2^2 A_{n_2} + A_{n_3} > 0 \text{ (resp. } < 0),$$

$$a_1^2 B_{n_1} + a_2^2 B_{n_2} + B_{n_3} > 0 \text{ (resp. } < 0),$$

thus, for every $\theta \in (0, 1)$, we have (3.48). □

Proof of Theorem 6. Let $w \in H^4 \cap H_0^3((0,T),\mathbb{R}) \cap \mathcal{X}^{\perp}$ be such that

$$\int_0^T w(t)\langle q\Psi(t),\varphi_1\rangle e^{i\lambda_1 t}\,dt - \frac{\sqrt{\theta}}{\sqrt{1-\theta}}\int_0^T w(t)\langle q\varphi_2,\Psi(t)\rangle e^{-i\lambda_2 t}\,dt$$
$$= +1 \text{ (resp. } -1). \tag{3.49}$$

Using (3.45) and the assumption $w \in \mathcal{X}^{\perp}$, we get $\Psi(T) = 0$, $s(T) = 0$, $d(T) = 0$. Let us prove that there exists $v \in H_0^3((0,T),\mathbb{R})$ such that the solution ξ of (3.42) satisfies $\xi(T) = i\varphi_1$ (resp. $-i\varphi_1$), $\sigma(T) = 0$, $\delta(T) = 0$. We have

$$\xi(t) = \sum_{k=1}^{\infty} y_k(t)\varphi_k,$$

$$y_{2k}(t) = i\left(\int_0^t [w(\tau)\langle q\Psi(\tau),\varphi_{2k}\rangle + v(\tau)\sqrt{1-\theta}\,b_{2k}e^{-i\lambda_1\tau}]e^{i\lambda_{2k}\tau}d\tau\right)e^{-i\lambda_{2k}t},$$

$$y_{2k+1}(t) = i\left(\int_0^t [w(\tau)\langle q\Psi(\tau),\varphi_{2k+1}\rangle + v(\tau)\sqrt{\theta}\,c_{2k+1}e^{-i\lambda_2\tau}]e^{i\lambda_{2k+1}\tau}d\tau\right)e^{-i\lambda_{2k+1}t}.$$

Thus the equality $(\xi(T),\sigma(T),\delta(T)) = (\pm i\varphi_1, 0, 0)$ is equivalent to

$$\int_0^T v(t)e^{i(\lambda_1-\lambda_2)t}\,dt = \frac{1}{\sqrt{\theta}c_1}\left(\pm 1 - \int_0^T w(t)\langle q\Psi(t),\varphi_1\rangle e^{i\lambda_1 t}\,dt\right),$$

$$\int_0^T v(t)e^{i(\lambda_{2k}-\lambda_1)t}\,dt = \frac{-1}{\sqrt{1-\theta}b_{2k}}\int_0^T w(t)\langle q\Psi(t),\varphi_{2k}\rangle e^{i\lambda_{2k}t}\,dt, \forall k\in\mathbb{N}^*,$$

$$\int_0^T v(t)e^{i(\lambda_{2k+1}-\lambda_2)t}\,dt = \frac{-1}{\sqrt{\theta}c_{2k+1}}\int_0^T w(t)\langle q\Psi(t),\varphi_{2k+1}\rangle e^{i\lambda_{2k+1}t}\,dt, \forall k\in\mathbb{N}^*,$$

$$\int_0^T v(t)\,dt = 0,$$

$$\int_0^T (T-t)v(t)\,dt = 0.$$

The left-hand sides of the two first equalities with $k = 1$ are complex conjugate numbers when v is real valued. Thus, a necessary condition for the existence of real-valued solution v to this problem is

$$\frac{1}{\sqrt{\theta}}\left(+1-\int_0^T w(t)\langle q\Psi(t),\varphi_1\rangle e^{i\lambda_1 t}\,dt\right) = \frac{-1}{\sqrt{1-\theta}}\int_0^T w(t)\langle q\varphi_2,\Psi(t)\rangle e^{-i\lambda_2 t}\,dt,$$

$$\left(\text{resp. } \frac{1}{\sqrt{\theta}}\left(-1-\int_0^T w(t)\langle q\Psi(t),\varphi_1\rangle e^{i\lambda_1 t}\,dt\right)\right.$$
$$\left. = \frac{-1}{\sqrt{1-\theta}}\int_0^T w(t)\langle q\varphi_2,\Psi(t)\rangle e^{-i\lambda_2 t}\,dt\right).$$

This property is satisfied thanks to (3.49).

Let $d = (d_k)_{k\in\mathbb{N}}$ be the sequence defined by

$$d_0 := 0,\ \ d_1 = 0,\ \ d_{2k} := -\frac{1}{b_{2k}\sqrt{1-\theta}}\int_0^T w(t)\langle q\Psi(t),\varphi_k\rangle e^{i\lambda_{2k}t}\,dt, \forall k\geqslant 1,$$

$$d_{2k+1} := -\frac{1}{c_{2k+1}\sqrt{\theta}}\int_0^T w(t)\langle q\Psi(t),\varphi_{2k+1}\rangle e^{i\lambda_{2k+1}t}\,dt, \forall k\geqslant 1$$

The previous moment problem can be written $Z(v) = d$, where the map Z has been defined in Section 3.2.3. Thanks to (3.5) and Proposition 6, a sufficient condition for the existence of $v \in H_0^3((0,T),\mathbb{R})$ solution of this equation is $d \in h^6(\mathbb{N},\mathbb{C})$. We can get this result by applying [1, Proposition 24, Section 3.6.2]. □

3.4. Proof of Theorem 2

In all this section $T := 4/\pi$. Let $\rho \in \mathbb{R}$, $\psi_0, \psi_f \in H^7_{(0)}(I,\mathbb{C})$, $S_0, D_0, S_f, D_f \in \mathbb{R}$. Let us consider, for $t \in [0,T]$

$$u(t) := \sqrt{|\rho|}w + |\rho|v,$$

where $w := w_+, v := v_+$ if $\rho \geqslant 0$ and $w := w_-, v := v_-$ if $\rho \leqslant 0$ and $w_\pm, v_\pm$ are defined in Theorem 6. Let (ψ, S, D) be the solution of (Σ_0) on $[0, T]$ with control u and such that

$$(\psi(0), S(0), D(0)) = (\psi_0, S_0, D_0).$$

Then, we have

$$S(T) = S_0,\ D(T) = D_0.$$

We have $u \in W^{3,1}((0,T), \mathbb{R})$ and $u(0) = u(T) = \dot{u}(0) = \dot{u}(T) = 0$ thus (see [1, Appendix B, Proposition 51]) the function $\psi(T)$ belongs to $H^7_{(0)}(I, \mathbb{C})$.

Proposition 11. *There exists* $\mathcal{C} > 0$ *such that, for every* $\rho \in (-1, 1)$, *we have*

$$\|\psi(T) - (\psi_\theta(T) + i\rho\varphi_1)\|_{H^7(I,\mathbb{C})} \leqslant \mathcal{C}[\|\psi_0 - \psi_\theta(0)\|_{H^7(I,\mathbb{C})} + |\rho|^{3/2}].$$

Proof. We have $\psi(T) - (\psi_\theta(T) + i\rho\varphi_1) = (\psi - Z)(T)$ where $Z := \psi_\theta + \Psi + \xi$ and Ψ, ξ are the solutions of

$$\begin{cases} i\dot{\Psi} = -\frac{1}{2}\Psi'' - \sqrt{|\rho|}wq\psi_\theta, \\ \Psi(t, \pm 1/2) = 0, \\ \Psi(0) = 0, \end{cases}$$

$$\begin{cases} i\dot{\xi} = -\frac{1}{2}\xi'' - \sqrt{|\rho|}wq\Psi - |\rho|vq\psi_\theta, \\ \xi(t, \pm 1/2) = 0, \\ \xi(0) = 0. \end{cases}$$

The function $\Delta := \psi - Z$ solves

$$\begin{cases} i\dot{\Delta} = -\frac{1}{2}\Delta'' - uq\Delta - |\rho|vq(\Psi + \xi) - \sqrt{|\rho|}wq\xi, \\ \Delta(t, \pm 1/2) = 0, \\ \Delta(0) = \psi_0 - \psi_\theta(0). \end{cases}$$

We know from [1, Proposition 51, Appendix B], that the following quantities

$$\|\Psi\|_{C^0([0,T],H^7)}, \|\Psi\|_{C^1([0,T],H^5)}, \|\Psi\|_{C^2([0,T],H^3)}, \|\Psi\|_{C^3([0,T],H^1)},$$

are bounded by

$$A_7(\Psi) := C[\|f\|_{C^0([0,T],H^5)} + \|f\|_{C^1([0,T],H^3)} + \|f\|_{W^{2,1}((0,T),H^2)} + \|f\|_{W^{3,1}((0,T),H^1)}],$$

where C is a positive constant and $f := \sqrt{|\rho|} w q \psi_\theta$. Thus, there exists a constant C_1 such that

$$A_7(\Psi) \leqslant C_1 \sqrt{|\rho|}.$$

In the same way, we prove that there exists a constant C_2 such that

$$\|\xi\|_{C^0([0,T],H^7)}, \|\xi\|_{C^1([0,T],H^5)}, \|\xi\|_{C^2([0,T],H^3)}, \|\xi\|_{C^3([0,T],H^1)},$$

are bounded by

$$A_7(\xi) \leqslant C_2 |\rho|.$$

Using [1, Appendix B, Proposition 51] we get the existence of a constant $C_3 > 0$ such that

$$\|\Delta(T)\|_{H^7} \leqslant C_3 [\|\psi_0 - \psi_\theta(0)\|_{H^7} + \sqrt{|\rho|} A_7(\xi) + |\rho| A_7(\Psi)]. \qquad \square$$

Now, we use the local controllability up to codimension one around Y_θ. Let $\delta > 0$ be as in Theorem 3. We assume

$$\|\psi_0 - \psi_\theta(0)\|_{H^7(I,\mathbb{C})} < \frac{\delta}{4\mathcal{C}},$$

$$|S_0| + |D_0| < \frac{\delta}{2},$$

$$\|\mathcal{P}[\psi_f - \psi_\theta(2T)]\|_{H^7} + |S_f| + |D_f| < \delta.$$

When ρ satisfies

$$|\rho| < \eta := \min\left\{1; \frac{\delta}{4(\|\varphi_1\|_{H^7} + \mathcal{C})}\right\},$$

the previous proposition proves that

$$\|\psi(T) - \psi_\theta(0)\|_{H^7} \leqslant (\|\varphi_1\|_{H^7} + \mathcal{C})|\rho|^{3/2} + \frac{\delta}{4} < \frac{\delta}{2}.$$

Thus $(\psi(T), S_0, D_0) \in \mathcal{V}(0)$ and $(\mathcal{P}\psi_f, S_f, D_f) \in \mathcal{V}(T)$. Thanks to Theorem 3, there exists

$$\tilde{u} := \Gamma(\psi(T), S_0, D_0, \mathcal{P}\psi_f, S_f, D_f) \in H_0^1((T, 2T), \mathbb{R})$$

such that

$$(\mathcal{P}\psi(2T), S(2T), D(2T)) = (\mathcal{P}\psi_f, S_f, D_f),$$

where (ψ, S, D) is the solution of (Σ_0) with control u on $[0, 2T]$, with u extended to $[0, 2T]$ by $u := \tilde{u}$ on $[T, 2T]$. Theorem 3 and the previous proposition give the existence of a constant C such that

$$\|u\|_{H^1((T,2T),\mathbb{R})} \leqslant C[|\rho|^{3/2} + \|\psi_0 - \psi_\theta(0)\|_{H^7} + |S_0| + |D_0| + \|\mathcal{P}(\psi_f - \psi_\theta(2T))\|_{H^7} + |S_f| + |D_f|]. \tag{3.50}$$

We define the map

$$\begin{aligned} F : (-\eta, \eta) &\rightarrow \mathbb{R}, \\ \rho &\mapsto \Im(\langle \psi(2T), \varphi_1 \rangle). \end{aligned}$$

Thanks to Theorem 3, F is continuous on $(-\eta, \eta)$. We can assume δ is small enough so that

$$\Re(\langle \psi(2T), \varphi_1 \rangle) > 0,$$

because ψ is close enough to ψ_θ. Since $\psi \in \mathcal{S}$ and $\Re(\langle \psi(2T), \varphi_1 \rangle)$ is positive, we have

$$\psi(2T) = \psi_f \text{ if and only if } F(\rho) = \Im(\langle \psi_f, \varphi_1 \rangle).$$

Therefore, in order to get Theorem 2, it is sufficient to prove that F is surjective on a neighborhood of 0.

Let $x(t) := \langle \psi(t), \varphi_1 \rangle$ on $[T, 2T]$. We have

$$x(2T) = x(T) + i \int_T^{2T} u(t) \langle q\psi(t), \varphi_1 \rangle e^{i\lambda_1 t} \, dt.$$

Thus

$$F(\rho) = \rho + [\Im(x(T)) - \rho] + \Im \left(i \int_T^{2T} u(t) \langle q\psi(t), \varphi_1 \rangle e^{i\lambda_1 t} \right),$$

where

$$|\Im(x(T)) - \rho| \leqslant \|\psi(T) - (\psi_\theta(T) + i\rho)\|_{L^2} \leqslant C[|\rho|^{3/2} + \|\psi_0 - \psi_\theta(0)\|_{H^7}],$$

$$\left|\int_T^{2T} u(t)\langle q\psi(t), \varphi_1\rangle e^{i\lambda_1 t}\, dt\right| \leqslant T\|u\|_{L^\infty((T,2T),\mathbb{R})}.$$

Using (3.50), we get the existence of a constant K such that

$$|F(\rho)-\rho| \leqslant K[|\rho|^{3/2} + \|\psi_0 - \psi_\theta(0)\|_{H^7} + \|\mathcal{P}[\psi_f - \psi_\theta(2T)]\|_{H^7} + |S_f| \\ + |D_f| + |S_0| + |D_0|].$$

There exists $\tau \in (0,\eta)$ such that

$$K|\tau|^{3/2} < \frac{\tau}{3}.$$

Let us assume that

$$K[\|\psi_0 - \psi_\theta(0)\|_{H^7} + \|\mathcal{P}[\psi_f - \psi_\theta(2T)]\|_{H^7} + |S_f| + |D_f| + |S_0| + |D_0|] < \frac{\tau}{3}.$$

Then

$$F(\tau) > \frac{\tau}{3} \text{ and } F(-\tau) < -\frac{\tau}{3},$$

thus the intermediate values theorem guarantees that F is surjective on a neighborhood of zero, this ends the proof of Theorem 2.

4. Local controllability of (Σ_0) around $Y^{0,0,0}$

The aim of this section is the proof of the following theorem.

Theorem 7. *Let $\phi_0, \phi_1 \in \mathbb{R}$. There exist $\mathcal{T} > 0$ and $\eta > 0$ such that, for every* (ψ_0, S_0, D_0), $(\psi_f, S_f, D_f) \in [\mathcal{S} \cap H^7_{(0)}(I,\mathbb{C})] \times \mathbb{R} \times \mathbb{R}$ with

$$\|\psi_0 - \varphi_1 e^{i\phi_0}\|_{H^7(I,\mathbb{C})} + |S_0| + |D_0| < \eta,$$

$$\|\psi_f - \varphi_1 e^{i\phi_1}\|_{H^7(I,\mathbb{C})} + |S_f| + |D_f| < \eta,$$

there exists a trajectory (ψ, S, D, u) of (Σ_0) on $[0,\mathcal{T}]$ such that

$$(\psi(0), S(0), D(0)) = (\psi_0, S_0, D_0),$$

$$(\psi(\mathcal{T}), S(\mathcal{T}), D(\mathcal{T})) = (\psi_f, S_f, D_f)$$

and $u \in H^1_0((0,\mathcal{T}),\mathbb{R})$.

4.1. Noncontrollability of the linearized system around $(Y^{0,0,0}, u \equiv 0)$

The linearized system around $(Y^{0,0,0}, u \equiv 0)$ is

$$(\Sigma_0^l) \quad \begin{cases} i\dot{\Psi} = -\frac{1}{2}\Psi'' - wq\psi_1, \\ \Psi(t, \pm 1/2) = 0, \\ \dot{s} = w, \\ \dot{d} = s. \end{cases}$$

It is a control system where

- the state is (Ψ, s, d) with $\Psi(t) \in T_{\mathcal{S}}(\psi_1(t))$ for every t,
- the control is the real valued function w.

Let $(\Psi_0, s_0, d_0) \in T_{\mathcal{S}}(\psi_1(0)) \times \mathbb{R} \times \mathbb{R}$ and (Ψ, s, d) be the solution of (Σ_0^l) such that $(\Psi(0), s(0), d(0)) = (\Psi_0, s_0, d_0)$, with some control $w \in L^2((0,T), \mathbb{R})$. We have the following equality in $L^2(I, \mathbb{C})$

$$\Psi(t) = \sum_{k=1}^{\infty} x_k(t)\varphi_k \quad \text{where} \quad x_k(t) := \langle \Psi(t), \varphi_k \rangle \forall k \in \mathbb{N}^*.$$

Using the parity of the functions φ_k and the equation solved by Ψ, we get

$$i\dot{x}_{2k+1} = \lambda_{2k+1} x_{2k+1}, \forall k \in \mathbb{N}.$$

Half of the components have a dynamic independent of the control w. Thus the control system (Σ_0^l) is not controllable.

4.2. Local controllability of (Σ_0) around $Y^{\gamma,0,0}$ for $\gamma \neq 0$

Let $\gamma \in \mathbb{R}^*$. The ground state for $u \equiv \gamma$ is the function

$$\psi_{1,\gamma}(t,q) := \varphi_{1,\gamma}(q) e^{-i\lambda_{1,\gamma} t},$$

where $\lambda_{1,\gamma}$ is the first eigenvalue and $\varphi_{1,\gamma}$ the associated normalized eigenvector of the operator A_γ defined on

$$D(A_\gamma) := H^2 \cap H_0^1(I, \mathbb{C}) \text{ by } A_\gamma \varphi := -\frac{1}{2}\varphi'' - \gamma q \varphi.$$

When $\alpha, \beta \in \mathbb{R}$, the function

$$Y^{\gamma,\alpha,\beta}(t) := (\psi_{1,\gamma}(t), \alpha + \gamma t, \beta + \alpha t + \gamma t^2/2)$$

solves (Σ_0) with $u \equiv \gamma$. We define $T := 4/\pi$, $T^* := 2T$ and, for $s = 1, 3, 5, 7, 9$ the space

$$H^s_{(\gamma)}(I, \mathbb{C}) := \{\varphi \in H^s(I, \mathbb{C});\, A^n_\gamma \varphi \in H^1_0(I, \mathbb{C}) \text{ for } n = 0, ..., (s-1)/2\}.$$

We admit the following result which will be proved in section 5.

Theorem 8. *There exists $\gamma_0 > 0$ such that, for every $\gamma \in (0, \gamma_0)$, there exists $\delta = \delta(\gamma) > 0$, such that, for every (ψ_0, S_0, D_0), $(\psi_f, S_f, D_f) \in [\mathcal{S} \cap H^7_{(\gamma)}(I, \mathbb{C})] \times \mathbb{R} \times \mathbb{R}$ with*

$$\|\psi_0 - \psi_{1,\gamma}(0)\|_{H^7} + |S_0 - \alpha| + |D_0 - \beta| < \delta,$$

$$\|\psi_f - \psi_{1,\gamma}(T^*)\|_{H^7(I,\mathbb{C})} + |S_f - \alpha - \gamma T^*| + |D_f - \beta - \alpha T^* - \gamma T^{*2}/2| < \delta,$$

for some real constants α, β, there exists $v \in H^1_0((0, T^*), \mathbb{R})$ such that, the unique solution of (Σ_0) on $[0, T^*]$, with control $u := \gamma + v$, such that $(\psi(0), S(0), D(0)) = (\psi_0, S_0, D_0)$ satisfies

$$(\psi(T^*), S(T^*), D(T^*)) = (\psi_f, S_f, D_f).$$

4.3. Quasi-static transformations

Let $\gamma \in (0, \gamma_0)$ with γ_0 as in Theorem 8. Let $f \in C^4([0, 3], \mathbb{R})$ be such that

$$f \equiv 0 \text{ on } [0, 1/2] \cup [5/2, 3], \tag{4.1}$$

$$f(t) = t \text{ for } t \in [1, 3/2], \tag{4.2}$$

$$\int_0^3 f(t)\, dt = 0. \tag{4.3}$$

For $\varepsilon > 0$, we define

$$\begin{array}{rcl} u_\varepsilon : [0, 3/\varepsilon] & \to & \mathbb{R}, \\ t & \mapsto & \gamma f'(\varepsilon t). \end{array}$$

Let $\phi_0, \phi_1 \in \mathbb{R}$. Let $\psi_\varepsilon, S_\varepsilon, D_\varepsilon)$ be the solution on $[0, 1/\varepsilon]$ of

$$\begin{cases} i\frac{\partial \psi_\varepsilon}{\partial t}(t,q) = -\frac{1}{2}\frac{\partial^2 \psi_\varepsilon}{\partial q^2}(t,q) - u_\varepsilon(t) q \psi_\varepsilon, \\ \psi_\varepsilon(0,q) = \varphi_1(q) e^{i\phi_0}, \\ \psi_\varepsilon(t,-1/2) = \psi_\varepsilon(t,1/2) = 0, \\ \dot{S}_\varepsilon(t) = u_\varepsilon(t), S_\varepsilon(0) = 0, \\ \dot{D}_\varepsilon(t) = S_\varepsilon(t), D_\varepsilon(0) = 0. \end{cases}$$

The following result has been proved in [1, Section 4].

Proposition 12. *There exist* $\varepsilon_0 > 0$, $C_0 > 0$ *such that, for every* $\varepsilon \in (0, \varepsilon_0]$,

$$\|\psi_\varepsilon(1/\varepsilon) - \varphi_{1,\gamma} e^{i(\phi_0 - \int_0^{1/\varepsilon} \lambda_{1,\gamma f'(\varepsilon s)}\, ds)}\|_{H^7(I,\mathbb{C})} \leqslant C_0 \gamma^{1/8} \varepsilon^{1/32}.$$

The continuity with respect to initial conditions gives the following proposition.

Proposition 13. *Let* $\varepsilon \in (0, \varepsilon_0)$. *There exists* $\eta_0 = \eta_0(\varepsilon) > 0$ *such that, for every* $(\psi_0, S_0, D_0) \in H^7_{(0)}(I, \mathbb{C}) \times \mathbb{R} \times \mathbb{R}$, with

$$\|\psi_0 - \varphi_1 e^{i\phi_0}\|_{H^7(I,\mathbb{C})} \leqslant \eta_0,$$

the solution (ψ, S, D) of (Σ_0) on $[0, 1/\varepsilon]$ *with initial condition* $(\psi(0), S(0), D(0)) = (\psi_0, S_0, D_0)$ *and control* u_ε *satisfies*

$$\|\psi(1/\varepsilon) - \varphi_{1,\gamma} e^{i(\phi_0 - \int_0^{1/\varepsilon} \lambda_{1,\gamma f'(\varepsilon s)}\, ds)}\|_{H^7(I,\mathbb{C})} \leqslant 2C_0 \gamma^{1/8} \varepsilon^{1/32},$$

$$S(1/\varepsilon) = S_0 + \frac{\gamma}{\varepsilon}, \quad D(1/\varepsilon) = D_0 + \frac{S_0}{\varepsilon} + \frac{\gamma}{\varepsilon^2} \int_0^1 f.$$

Let $(\xi_\varepsilon, s_\varepsilon, d_\varepsilon)$ be the solution on $[(1/\varepsilon) + T^*, (3/\varepsilon)]$ of

$$\begin{cases} i\frac{\partial \xi_\varepsilon}{\partial t}(t,q) = -\frac{1}{2}\frac{\partial^2 \xi_\varepsilon}{\partial q^2}(t,q) - u_\varepsilon(t) q \xi_\varepsilon, \\ \xi_\varepsilon(3/\varepsilon, q) = \varphi_1(q) e^{i\phi_\varepsilon}, \\ \xi_\varepsilon(t,-1/2) = \xi_\varepsilon(t,1/2) = 0, \\ \dot{s}_\varepsilon(t) = u_\varepsilon(t), s_\varepsilon(3/\varepsilon) = 0, \\ \dot{d}_\varepsilon(t) = s_\varepsilon(t), d_\varepsilon(3/\varepsilon) = 0. \end{cases}$$

where ϕ_ε is the unique solution in $[\phi_1, \phi_1 + 2\pi)$ of

$$\phi_\varepsilon + \int_{1/\varepsilon + T^*}^{3/\varepsilon} \lambda_{1,\gamma f'(\varepsilon t)} dt = \phi_0 - \int_0^{1/\varepsilon} \lambda_{1,\gamma f'(\varepsilon t)}\, dt - \lambda_{1,\gamma} T^*, \ (\text{mod } 2\pi). \tag{4.4}$$

In the same way as in [1, Section 4] and thanks to (4.4), we get the following proposition.

Proposition 14. *There exist* $\varepsilon_f > 0$, $C_f > 0$ *such that, for every* $\varepsilon \in (0, \varepsilon_f]$,

$$\|\xi_\varepsilon((1/\varepsilon) + T^*) - \varphi_{1,\gamma} e^{i(\phi_0 - \int_0^{1/\varepsilon} \lambda_{1,\gamma f'(\varepsilon s)}\, ds - \lambda_{1,\gamma} T^*)}\|_{H^7(I,\mathbb{C})} \leqslant C_f \gamma^{1/8} \varepsilon^{1/32}.$$

Let us extend ξ_ε to $[(1/\varepsilon) + T^*, (3/\varepsilon) + \tau_\varepsilon]$ in such way that $\xi_\varepsilon((3/\varepsilon) + \tau_\varepsilon) = \varphi_1 e^{i\phi_1}$. Let τ_ε be the unique solution in $[0, 2\pi/\lambda_1)$ of

$$\phi_\varepsilon - \lambda_1 \tau_\varepsilon = \phi_1 \pmod{2\pi}.$$

We extend u_ε to $[(1/\varepsilon) + T^*, (3/\varepsilon) + \tau_\varepsilon]$ by zero:

$$u_\varepsilon(t) := 0, \text{ for every } t \in [3/\varepsilon, (3/\varepsilon) + \tau_\varepsilon].$$

We still denote by $(\xi_\varepsilon, s_\varepsilon, d_\varepsilon)$ the solution of the last system on $[(1/\varepsilon)+T^*, (3/\varepsilon)+\tau_\varepsilon]$. Then,

$$\xi_\varepsilon((3/\varepsilon) + \tau_\varepsilon) = \varphi_1 e^{i\phi_1}, s_\varepsilon((3/\varepsilon) + \tau_\varepsilon) = 0, d_\varepsilon((3/\varepsilon) + \tau_\varepsilon) = 0.$$

Again, the continuity with respect to initial conditions gives the following proposition.

Proposition 15. *Let* $\varepsilon \in (0, \varepsilon_f)$ *such that* $\varepsilon < 1/(2T^*)$. *There exists* $\eta_f = \eta_f(\varepsilon) > 0$ *such that, for every* $(\psi_f, S_f, D_f) \in H^7_{(0)}(I, \mathbb{C}) \times \mathbb{R} \times \mathbb{R}$, *with*

$$\|\psi_f - \varphi_1 e^{i\phi_1}\|_{H^7(I,\mathbb{C})} \leqslant \eta_f,$$

the solution (ψ, S, D) *of* (Σ_0) *on* $[(1/\varepsilon)+T^*, (3/\varepsilon)+\tau_\varepsilon]$ *with initial condition* $((\psi(3/\varepsilon)+\tau_\varepsilon), S((3/\varepsilon) + \tau_\varepsilon), D((3/\varepsilon) + \tau_\varepsilon)) = (\psi_f, S_f, D_f)$ *and control* u_ε *satisfies*

$$\|\psi((1/\varepsilon) + T^*) - \varphi_{1,\gamma} e^{i(\phi_0 - \int_0^{1/\varepsilon} \lambda_{1,\gamma f'(\varepsilon s)}\, ds - \lambda_{1,\gamma} T^*)}\|_{H^7(I,\mathbb{C})} \leqslant 2C_f \gamma^{1/8} \varepsilon^{1/32},$$

$$S((1/\varepsilon) + T^*) = S_f + \frac{\gamma}{\varepsilon} + \gamma T^*,$$

$$D((1/\varepsilon) + T^*) = D_f + S_f \left(T^* - \frac{2}{\varepsilon} - \tau_\varepsilon\right) + \frac{\gamma}{\varepsilon^2} \int_0^1 f + \frac{\gamma}{\varepsilon} T^* + \frac{1}{2}\gamma T^{*2}.$$

Proof of Theorem 7. We fix $\varepsilon \in (0, \varepsilon_0)$ such that

$$\varepsilon < 1/(2T^*) \text{ and } 2\max(C_0, C_f)\gamma^{1/8}\varepsilon^{1/32} < \frac{\delta}{3},$$

where δ is given by Theorem 8. Let $(\psi_0, S_0, D_0), (\psi_f, S_f, D_f) \in [\mathcal{S} \cap H^7_{(0)}(I, \mathbb{C})] \times \mathbb{R} \times \mathbb{R}$ be such that

$$\|\psi_0 - \varphi_{1,\gamma} e^{i\phi_0}\|_{H^7(I,\mathbb{C})} < \eta_0(\varepsilon), \tag{4.5}$$

$$\|\psi_f - \varphi_{1,\gamma} e^{i\phi_1}\|_{H^7(I,\mathbb{C})} < \min(\eta_f(\varepsilon), \delta/3), \tag{4.6}$$

$$|S_0| < \delta/6 \ , \ |S_f| < \delta/6, \tag{4.7}$$

$$|D_0| + |D_f| + |S_0|\left(\frac{1}{\varepsilon} + T^*\right) + |S_f|\left(\frac{2}{\varepsilon} + T^* + 2\pi\right) < \delta/3. \tag{4.8}$$

Then, the solution (ψ, S, D) of (Σ_0) on $[0, 1/\varepsilon]$ with control u_ε such that $(\psi(0), S(0), D(0)) = (\psi_0, S_0, D_0)$ satisfies

$$\|\psi(1/\varepsilon) - \psi_{1,\gamma}(\theta_\varepsilon)\|_{H^7(I,\mathbb{C})} \leqslant 2C_0\gamma^{1/8}\varepsilon^{1/32} < \delta,$$

$$S(1/\varepsilon) = S_0 + \frac{\gamma}{\varepsilon}, D(1/\varepsilon) = D_0 + \frac{S_0}{\varepsilon} + \frac{\gamma}{\varepsilon^2}\int_0^1 f,$$

where θ_ε is such that

$$-\lambda_{1,\gamma}\theta_\varepsilon = \phi_0 - \int_0^{1/\varepsilon} \lambda_{1,\gamma f'(\varepsilon s)}\, ds.$$

The solution (ψ, S, D) of (Σ_0) on $[(1/\varepsilon) + T^*, (3/\varepsilon) + \tau_\varepsilon]$ with control u_ε such that

$$(\psi((3/\varepsilon) + \tau_\varepsilon), S((3/\varepsilon) + \tau_\varepsilon), D((3/\varepsilon) + \tau_\varepsilon)) = (\psi_f, S_f, D_f),$$

satisfies

$$\|\psi((1/\varepsilon) + T^*) - \psi_{1,\gamma}(\theta_\varepsilon + T^*)\|_{H^7(I,\mathbb{C})} \leqslant 2C_f\gamma^{1/8}\varepsilon^{1/32} < \delta/3,$$

$$S((1/\varepsilon) + T^*) = S_f + \frac{\gamma}{\varepsilon} + \gamma T^*,$$

$$D((1/\varepsilon) + T^*) = D_f + S_f(T^* - (2/\varepsilon) - \tau_\varepsilon) + \frac{\gamma}{\varepsilon^2}\int_0^1 f + \frac{\gamma}{\varepsilon}T^* + \frac{1}{2}\gamma T^{*2}.$$

We apply Theorem 8 with

$$\alpha := S(1/\varepsilon), \beta := D(1/\varepsilon).$$

Assumptions (4.7) and (4.8) give

$$|S((1/\varepsilon) + T^*) - \alpha - \gamma T^*| < \delta/3,$$

$$|D((1/\varepsilon) + T^*) - \beta - \alpha T^* - \gamma T^{*2}/2 < \delta/3.$$

Thus, there exists $v \in H_0^1((0, T^*), \mathbb{R})$ such that the solution $(\tilde{\psi}, \tilde{S}, \tilde{D})$ of (Σ_0) on $[0, T^*]$ with control $u := \gamma + v$ such that $(\tilde{\psi}(0), \tilde{S}(0), \tilde{D}(0)) = (\psi(1/\varepsilon), S(1/\varepsilon), D(1/\varepsilon))$ satisfies

$$(\tilde{\psi}(T^*), \tilde{S}(T^*), \tilde{D}(T^*)) = (\psi((1/\varepsilon) + T^*), S((1/\varepsilon) + T^*), D((1/\varepsilon) + T^*)).$$

Thus, the control $u : [0, (3/\varepsilon) + \tau_\varepsilon] \to \mathbb{R}$ defined by

$$u = u_\varepsilon \text{ on } [0, 1/\varepsilon] \cup [(1/\varepsilon) + T^*, (3/\varepsilon) + \tau_\varepsilon],$$

$$u(t) = \gamma + v(t - 1/\varepsilon) \text{ for every } t \in [1/\varepsilon, (1/\varepsilon) + T^*]$$

gives the result.

5. Local controllability of (Σ_0) around $Y^{\gamma,\alpha,\beta}$

The aim of this section is the proof of Theorem 8. In [1] a similar local controllability result has been proved for the subsystem (Σ) defined in the Introduction. It is the following one.

Theorem 9. *There exists $\gamma_0 > 0$ such that, for every $\gamma \in (0, \gamma_0)$, there exist $\delta > 0$, $C > 0$ and a continuous map*

$$\begin{array}{rcl} \Gamma_\gamma : \mathcal{V}_\gamma(0) \times \mathcal{V}_\gamma(T) & \to & H_0^1((0, T), \mathbb{R}), \\ (\psi_0 \quad , \quad \psi_f) & \mapsto & v, \end{array}$$

where

$$\mathcal{V}_\gamma(0) := \{\psi_0 \in \mathcal{S} \cap H^7_{(\gamma)}(I, \mathbb{C}); \|\psi_0 - \psi_{1,\gamma}(0)\|_{H^7(I,\mathbb{C})} < \delta\},$$

$$\mathcal{V}_\gamma(T) := \{\psi_f \in \mathcal{S} \cap H^7_{(\gamma)}(I, \mathbb{C}); \|\psi_f - \psi_{1,\gamma}(T)\|_{H^7(I,\mathbb{C})} < \delta\},$$

such that, for every $\psi_0 \in \mathcal{V}_\gamma(0)$, $\psi_f \in \mathcal{V}_\gamma(T)$, the unique solution of (Σ) with control $u := \gamma + v$ such that $\psi(0) = \psi_0$ satisfies $\psi(T) = \psi_f$ and

$$\|\Gamma_\gamma(\psi_0, \psi_f)\|_{H_0^1((0,T),\mathbb{R})} \leqslant C[\|\psi_0 - \psi_{1,\gamma}(0)\|_{H^7(I,\mathbb{C})} + \|\psi_f - \psi_{1,\gamma}(T)\|_{H^7(I,\mathbb{C})}].$$

Let us recall the main ideas of the proof of this theorem in order to emphasize the difficulty of Theorem 8. We proved that the linearized system of (Σ) around $(\psi_{1,\gamma}, u \equiv \gamma)$ is controllable and we concluded by applying an implicit function theorem of Nash–Moser type.

This strategy does not work with (Σ_0) because the linearized system of (Σ_0) around $Y^{\gamma,\alpha,\beta}$ is not controllable.

5.1. Controllability up to codimension one of the linearized system around $(Y^{\gamma,\alpha,\beta}, u \equiv \gamma)$

In this section, we fix $\alpha, \beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}^*$. The linearized control system around $(Y^{\gamma,\alpha,\beta}, u \equiv \gamma)$ is

$$(\Sigma_\gamma^l) \quad \begin{cases} i\frac{\partial \Psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \Psi}{\partial q^2} - \gamma q \Psi - wq\psi_{1,\gamma}, \\ \Psi(t,-1/2) = \Psi(t,1/2) = 0, \\ \dot{s} = w, \\ \dot{d} = s. \end{cases}$$

It is a control system where

- the state is (Ψ, s, d) with $\Psi(t) \in T_{\mathcal{S}}(\psi_{1,\gamma}(t))$,
- the control is the real-valued function w.

Let us recall that the space $L^2(I, \mathbb{C})$ has a complete orthonormal system $(\varphi_{k,\gamma})_{k \in \mathbb{N}^*}$ of eigenfunctions for the operator A_γ defined on

$$D(A_\gamma) := H^2 \cap H_0^1(I, \mathbb{C}) \quad \text{by} \quad A_\gamma \varphi := -\tfrac{1}{2}\varphi'' - \gamma q \varphi,$$

$$A_\gamma \varphi_{k,\gamma} = \lambda_{k,\gamma} \varphi_{k,\gamma},$$

where $(\lambda_{k,\gamma})_{k \in \mathbb{N}^*}$ is an increasing sequence of positive real numbers. For technical reasons, we introduce the notation

$$b_{k,\gamma} := \langle \varphi_{k,\gamma}, q\varphi_{1,\gamma} \rangle.$$

It has already been proved in [1, Proposition 1, Section 3.1] that, for γ small enough and different from zero, $b_{k,\gamma}$ is different from zero for every $k \in \mathbb{N}^*$ and, roughly speaking, behaves like $1/k^3$ when $k \to +\infty$. In all this section, we assume we are in this situation.

Proposition 16. *Let $T > 0$ and (Ψ, s, d) be a trajectory of (Σ_γ^l) on $[0, T]$. Then, for every $t \in [0, T]$, we have*

$$s(t) = s(0) + \frac{1}{ib_{1,\gamma}} \left(\langle \Psi(t), \varphi_{1,\gamma} \rangle e^{i\lambda_{1,\gamma} t} - \langle \Psi(0), \varphi_{1,\gamma} \rangle \right). \tag{5.1}$$

Thus, the control system (Σ_γ^l) is not controllable.

Proof. Let $x_1(t) := \langle \Psi(t), \varphi_{1,\gamma} \rangle$. We have

$$\dot{x}_1(t) = \left\langle \frac{\partial \Psi}{\partial t}(t), \varphi_{1,\gamma} \right\rangle = \langle -iA_\gamma \Psi(t) + iw(t) q \psi_{1,\gamma}(t), \varphi_{1,\gamma} \rangle,$$

$$\dot{x}_1(t) = -i\lambda_{1,\gamma} x_1(t) + ib_{1,\gamma} w(t) e^{-i\lambda_{1\gamma} t},$$

$$x_1(t) = \left(x_1(0) + ib_{1,\gamma} \int_0^t w(\tau) d\tau \right) e^{-i\lambda_{1,\gamma} t}.$$

We get (5.1) by using

$$s(t) = s(0) + \int_0^t w(\tau) d\tau.$$

Let $T > 0$, $\Psi_0 \in T_{\mathcal{S}}(\psi_{1,\gamma}(0))$, $\Psi_f \in T_{\mathcal{S}}(\psi_{1,\gamma}(T))$, $s_0, s_f \in \mathbb{R}$. A necessary condition for the existence of a trajectory of (Σ_γ^l) such that $\Psi(0) = \Psi_0$, $s(0) = s_0$, $\Psi(T) = \Psi_f$, $s(T) = s_f$ is

$$s_f - s_0 = \frac{1}{ib_{1,\gamma}} \left(\langle \Psi_f, \varphi_{1,\gamma} \rangle e^{i\lambda_{1,\gamma} T} - \langle \Psi_0, \varphi_{1,\gamma} \rangle \right).$$

This equality does not happen for an arbitrary choice of Ψ_0, Ψ_f, s_0, s_f. Thus (Σ_γ^l) is not controllable. □

Proposition 17. *Let $T > 0$, $(\Psi_0, s_0, d_0), (\Psi_f, s_f, d_f) \in H^3_{(0)}(I, \mathbb{C}) \times \mathbb{R} \times \mathbb{R}$ be such that*

$$\Re \langle \Psi_0, \psi_{1,\gamma}(0) \rangle = \Re \langle \Psi_f, \psi_{1,\gamma}(T) \rangle = 0, \tag{5.2}$$

$$s_f - s_0 = \frac{i}{b_{1,\gamma}} \left(\langle \Psi_0, \varphi_{1,\gamma} \rangle - \langle \Psi_f, \varphi_{1,\gamma} \rangle e^{i\lambda_{1,\gamma} T} \right). \tag{5.3}$$

Then there exists $w \in L^2((0,T),\mathbb{R})$ *such that the solution of* (Σ_γ^l) *with control* w *and such that* $(\Psi(0), s(0), d(0)) = (\Psi_0, s_0, d_0)$ *satisfies* $(\Psi(T), s(T), d(T)) = (\Psi_f, s_f, d_f)$.

Remark 5. We can control Ψ and d but we cannot control s. We miss only two directions which are $(\Psi, s, d) = (0, \pm 1, 0)$.

Proof. Let $(\Psi_0, s_0, d_0) \in T_{\mathcal{S}}(\psi_{1,\gamma}(0)) \times \mathbb{R} \times \mathbb{R}$ and $T > 0$. Let (Ψ, s, d) be a solution of (Σ_γ^l) with $(\Psi(0), s(0), d(0)) = (\Psi_0, s_0, d_0)$ and a control $w \in L^2((0,T),\mathbb{R})$. Let $(\Psi_f, s_f, d_f) \in T_{\mathcal{S}}(\psi_{1,\gamma}(T)) \times \mathbb{R} \times \mathbb{R}$. The equality $(\Psi(T), s(T), d(T)) = (\Psi_f, s_f, d_f)$ is equivalent to the following moment problem on w,

$$\begin{aligned}
&\int_0^T w(t) e^{i(\lambda_{k,\gamma} - \lambda_{1,\gamma})t}\, dt = \frac{i}{b_{k,\gamma}} \left(\langle \Psi_0, \varphi_{k,\gamma} \rangle - \langle \Psi_f, \varphi_{k,\gamma} \rangle e^{i\lambda_{k,\gamma} T} \right), \forall k \in \mathbb{N}^*, \\
&\int_0^T w(t)\, dt = s_f - s_0, \\
&\int_0^T (T-t) w(t)\, dt = d_f - d_0 - s_0 T.
\end{aligned} \tag{5.4}$$

The left-hand sides of the two first equalities with $k = 1$ are equal, the equality of the right-hand sides is guaranteed by (5.3). Under the assumption $\Psi_0, \Psi_f \in H^3_{(0)}(I,\mathbb{C})$, the right hand side of (5.4) defines a sequence in l^2. Thus, under assumptions (5.3), and $\Psi_0, \Psi_f \in H^3_{(0)}(I,\mathbb{C})$, the existence of a solution $w \in L^2((0,T),\mathbb{R})$ of (5.4) can be proved in the same way as in [1, Theorem 5]. □

5.2. *Local controllability up to codimension one of* (Σ_0) *around* $Y^{\gamma,\alpha,\beta}$

In this section, we fix $\alpha, \beta \in \mathbb{R}$. The aim of this section is the proof of the following result.

Theorem 10. *There exists* $\gamma_0 > 0$ *such that, for every* $\gamma \in (0, \gamma_0)$, *for every* $S_0 \in \mathbb{R}$, *there exist* $\delta > 0$, $C > 0$ *and a continuous map*

$$\begin{array}{rccccl}
\Gamma_{\gamma,S_0}: & \mathcal{V}_\gamma(0) & \times & \mathcal{V}_{\gamma,S_0}(T) & \to & H_0^1((0,T),\mathbb{R}), \\
& ((\psi_0, D_0) & , & (\psi_f, D_f)) & \mapsto & v,
\end{array}$$

where

$$\mathcal{V}_\gamma(0) := \{(\psi_0, D_0) \in [\mathcal{S} \cap H^7_{(\gamma)}(I,\mathbb{C})] \times \mathbb{R};\ \|\psi_0 - \psi_{1,\gamma}(0)\|_{H^7(I,\mathbb{C})} + |D_0 - \beta| < \delta\},$$

$$\begin{aligned}
\mathcal{V}_{\gamma,S_0}(T) := \{&(\psi_f, D_f) \in [\mathcal{S} \cap H^7_{(\gamma)}(I,\mathbb{C})] \times \mathbb{R}; \\
&\|\psi_f - \psi_{1,\gamma}(T)\|_{H^7(I,\mathbb{C})} + |D_f - \beta - S_0 T - \gamma T^2/2| < \delta\},
\end{aligned}$$

such that, for every $(\psi_0, D_0) \in \mathcal{V}_\gamma(0)$, $(\psi_f, D_f) \in \mathcal{V}_{\gamma,S_0}(T)$, *the unique solution of* (Σ_0) *with control* $u := \gamma + v$, *such that*

$$(\psi(0), S(0), D(0)) = (\psi_0, S_0, D_0)$$

satisfies $\psi(T) = \psi_f$, $D(T) = D_f$ *and*

$$\|\Gamma_{\gamma,S_0}(\psi_0, \psi_f)\|_{H_0^1((0,T),\mathbb{R})} \leqslant C[\|\psi_0 - \psi_{1,\gamma}(0)\|_{H^7(I,\mathbb{C})} + |D_0 - \beta| + \|\psi_f - \psi_{1,\gamma}(T)\|_{H^7(I,\mathbb{C})} + |D_f - \beta - S_0 T - \gamma T^2/2|].$$

Remark 6. The same result is true if one replaces, $\psi_{1,\gamma}(0)$ by $\psi_{1,\gamma}(\theta)$ and $\psi_{1,\gamma}(T)$ by $\psi_{1,\gamma}(\theta + T)$ for some $\theta \in \mathbb{R}$. Indeed, if (ψ, S, D) solves (Σ_0) on $[0, T]$ with initial condition $(\psi(0), S(0), D(0)) = (\psi_0, S_0, D_0)$ and control u, then, $(\tilde{\psi} := \psi e^{-i\lambda_{1,\gamma}\theta}, S, D)$ solves (Σ_0) on $[0, T]$ with initial condition $(\tilde{\psi}(0), S(0), D(0)) = (\psi_0 e^{-i\lambda_{1,\gamma}\theta}, S_0, D_0)$ and control u.

The same loss of regularity as in Section 3.2.1 prevents us from using he inverse mapping theorem. We use exactly the same strategy as in [1]. We expose in the next sections the few differences in the proof.

In [1], the local controllability of (Σ) in a neighborhood of $(\psi_{1,\gamma}, u \equiv \gamma)$ was got by proving a local surjectivity result on the map $\Phi_\gamma : (\psi_0, v) \mapsto (\psi_0, \psi(T))$, where ψ is the solution of (Σ) with $u := \gamma + v$ such that $\psi(0) = \psi_0$. Thus, in order to prove the local controllability of (Σ_0) in a neighborhood of $(\psi_{1,\gamma}, \beta + \alpha t + \gamma t^2/2)$ we consider the map

$$\widetilde{\Phi}_{\gamma,S_0} : (\psi_0, D_0, v) \mapsto (\psi_0, D_0, \psi(T), D(T)),$$

where (ψ, S, D) is the solution of (Σ_0) with control $u := \gamma + v$ such that $\psi(0) = \psi_0$, $S(0) = S_0$, $D(0) = D_0$. As in [1] we get a local surjectivity result on this map by applying a Nash–Moser theorem, stated in Section 3.2.2.

5.2.1. Context for the Nash–Moser theorem

We apply the Theorem 4 to the map $\widetilde{\Phi}_\gamma$ with $\mathcal{P} = Id$ and the spaces defined, for $k = 1, 3, 5, 7, 9$, by

$$\widetilde{E}_k^\gamma := [\mathcal{S} \cap H_{(\gamma)}^k(I, \mathbb{C})] \times \mathbb{R} \times H_0^{(k-1)/2}((0, T), \mathbb{R}),$$

$$\widetilde{F}_k^\gamma := [\mathcal{S} \cap H_{(\gamma)}^k(I, \mathbb{C})] \times \mathbb{R} \times [\mathcal{S} \cap H_{(\gamma)}^k(I, \mathbb{C})] \times \mathbb{R}.$$

The smoothing operators defined on the spaces

$$E_k^\gamma := [\mathcal{S} \cap H_{(\gamma)}^k(I, \mathbb{C})] \times H_0^{(k-1)/2}((0, T), \mathbb{R})$$

and

$$F_k^\gamma := [\mathcal{S} \cap H_{(\gamma)}^k(I, \mathbb{C}) \times [\mathcal{S} \cap H_{(\gamma)}^k(I, \mathbb{C})] \times \mathbb{R}$$

in [1, Section 3.3], give easily suitable smoothing operators on the spaces $\widetilde{E}_k^\gamma$ and $\widetilde{F}_k^\gamma$: we will not do anything on the constants in $\mathbb{R}$.

As in [1, Section 3.4], the map $\widetilde{\Phi}_{\gamma,S_0} : \widetilde{E}_7^\gamma \to \widetilde{F}_\beta^\gamma$ is twice differentiable. The maps $\widetilde{\Phi}'_{\gamma,S_0}$ and $\widetilde{\Phi}''_{\gamma,S_0}$ do not depend on S_0, thus, we just write $\widetilde{\Phi}'_\gamma$ and $\widetilde{\Phi}''_\gamma$. The map $\widetilde{\Phi}''_\gamma$ satisfies inequality (3.15). Indeed, if we write

$$\Phi''_\gamma(\psi_0, v).((\phi_0, v), (\xi_0, \mu)) = (0, h(T)),$$

then, we have

$$\widetilde{\Phi}''_\gamma(\psi_0, D_0, v).((\phi_0, d_0, v), (\xi_0, \delta_0, \mu)) = (0, 0, h(T), 0),$$

and inequality (3.15) was already proved for Φ_γ in [1, Section 3.4].

The assumptions of Theorem 5 can be checked in the same way as in [1, Appendix C]. In the following two sections, we focus on the most difficult part in the application of the Nash–Moser theorem, which is the existence of a right inverse to the differential with bounds (3.17)–(3.20).

5.2.2. Controllability up to codimension one of the linearized system around $(\psi_{1,\gamma}(t), D(t) = \beta + S_0 t + \gamma t^2/2, u \equiv \gamma)$ and bounds (3.17)–(3.20)

In [1, Section 3.5], in order to study the controllability of Ψ, we introduced the map

$$Z_\gamma : w \mapsto \left(\int_0^T w(t) e^{i(\lambda_{k,\gamma} - \lambda_{1,\gamma})t} dt\right)_{k \in \mathbb{N}^*}.$$

Thus, in order to study the controllability of (Ψ, d), it is natural to introduce the map $\widetilde{Z}_\gamma$, defined by

$$\widetilde{Z}_\gamma(w)_0 := \int_0^T (T - t) w(t)\, dt,$$

$$\widetilde{Z}_\gamma(w)_k := Z_\gamma(w)_k, \quad \forall k \in \mathbb{N}^*.$$

Let Ψ_0, $\Psi_f \in L^2(I, \mathbb{C})$, s_0, d_0, $d_f \in \mathbb{R}$, $T > 0$ and (Ψ, s, d) the solution of (Σ_γ^l) such that $(\Psi(0), s(0), d(0)) = (\Psi_0, s_0, d_0)$ with some control $w \in L^2((0, T), \mathbb{R})$. As noticed in Section 5.1, the equality $(\Psi(T), s(T), d(T)) = (\Psi_f, s_f, d_f)$ is equivalent to $Z_\gamma(w) = D$ where $D = (D_k)_{k \in \mathbb{N}}$ is defined by

$$D_0 := d_f - d_0 - s_0 T,$$

$$D_k := \frac{i}{b_{k,\gamma}} \left(\langle \Psi_0, \varphi_{k,\gamma} \rangle - \langle \Psi_f, \varphi_{k,\gamma} \rangle e^{i\lambda_{k,\gamma} T}\right), \forall k \in \mathbb{N}^*.$$

Proposition 18. *Let $T = 4/\pi$. There exists $\gamma_0 > 0$, $C_1 > 0$ such that, for every $\gamma \in (-\gamma_0, \gamma_0)$,*

1. *the linear map $\widetilde{Z}_\gamma$ is continuous from $L^2((0,T),\mathbb{R})$ to $l^2_r(\mathbb{N},\mathbb{C})$, from $H^1_0((0,T),\mathbb{R})$ to $h^2_r(\mathbb{N},\mathbb{C})$, from $H^2_0((0,T),\mathbb{R})$ to $h^4_r(\mathbb{N},\mathbb{C})$, from $H^3_0((0,T),\mathbb{R})$ to $h^6_r(\mathbb{N},\mathbb{C})$;*
2. *for every $w \in H^3_0((0,T),\mathbb{R})$,*

$$\|(\widetilde{Z}_\gamma - \widetilde{Z}_0)(w)\|_F \leqslant C_1 \gamma^2 \|w\|_E$$

for $(E,F) = (L^2, l^2_r), (H^1_0, h^2_r), (H^2_0, h^4_r), (H^3_0, h^6_r)$.

The same results have already been proved for the maps Z_γ in [1, Propositions 11,13]. The new term in $\widetilde{Z}_\gamma$ has no influence.

Proposition 19. *Let $T = 4/\pi$. There exists a continuous linear map*

$$\widetilde{Z}_0^{-1} : h^6_r(\mathbb{N},\mathbb{C}) \to H^3_0((0,T),\mathbb{R}),$$

such that, for every $d \in h^6_r(\mathbb{N},\mathbb{C})$, $\widetilde{Z}_0 \circ \widetilde{Z}_0^{-1}(d) = d$. Moreover, there exists a constant C_0 such that, for every $d \in h^6_r(\mathbb{N},\mathbb{C})$, the function $w := \widetilde{Z}_0^{-1}(d)$ satisfies

$$\|w\|_{L^2} \leqslant C_0 \|d\|_{l^2},\ \|w\|_{H^1_0} \leqslant C_0 \|d\|_{h^2},\ \|w\|_{H^2_0} \leqslant C_0 \|d\|_{h^4},\ \|w\|_{H^3_0} \leqslant C_0 \|d\|_{h^6}.$$

Proof. As in [1, Proof of Proposition 12], we introduce the notations, for $k \in \mathbb{N}^*$,

$$\omega_k := \lambda_{k+1} - \lambda_1,\ \omega_{-k} := -\omega_k.$$

Let $d \in h^6_r(\mathbb{N},\mathbb{C})$. We define $\widetilde{d} \in h^6(\mathbb{Z},\mathbb{C})$ by

$$\widetilde{d}_k := d_{k+1},\ \widetilde{d}_{-k} := \overline{\widetilde{d}_k}, \quad \text{for every } k \in \mathbb{N}$$

A candidate for $\widetilde{Z}^{-1}(d)$ is

$$w(t) := \left(\frac{1}{T}\sum_{k\in\mathbb{Z}} \tilde{d}_k e^{i\omega_k t} + \alpha\left(e^{i\frac{1}{2}n\pi^2 t} + e^{-i\frac{1}{2}n\pi^2 t}\right)\right)(1 - e^{i\frac{1}{2}\pi^2 t})^2 (1 - e^{-i\frac{1}{2}\pi^2 t})^2,$$

where $n \in \mathbb{N}$ with $\{n, n\pm 1, n\pm 2\} \cap \{\pm(k^2-1); k \in \mathbb{N}^*\} = \emptyset$ and $\alpha \in \mathbb{R}$ is such that $\int_0^T (T-t) w(t)\, dt = d_0$. There exists a constant $C = C(n)$ such that $|\alpha| \leqslant C\|d\|_{l^2(\mathbb{N},\mathbb{C})}$. $\square$

Finally, we get the following proposition, the proof of which is the same as the one of [1, Proposition 14].

Proposition 20. *Let $T = 4/\pi$. There exists $\gamma_0 > 0$, $C_2 > 0$ such that, for every $\gamma \in (-\gamma_0, \gamma_0)$, there exists a linear map*

$$\widetilde{Z}_\gamma^{-1} : h_r^6(\mathbb{N}, \mathbb{C}) \to H_0^3((0, T), \mathbb{R}),$$

such that, for every $d \in h_r^6(\mathbb{N}, \mathbb{C})$, $\widetilde{Z}_\gamma \circ \widetilde{Z}_\gamma^{-1}(d) = d$. Moreover, for every $d \in h_r^6(\mathbb{N}, \mathbb{C})$, the function $w := \widetilde{Z}_\gamma^{-1}(d)$ satisfies

$$\|w\|_{L^2} \leqslant C_2 \|d\|_{l^2}, \ \|w\|_{H_0^1} \leqslant C_2 \|d\|_{h^2}, \ \|w\|_{H_0^2} \leqslant C_2 \|d\|_{h^4}, \ \|w\|_{H_0^3} \leqslant C_2 \|d\|_{h^6}.$$

Thanks to the behavior of the coefficients $b_{k,\gamma}$, we get the following controllability result for the linearized system around $(\psi_{1,\gamma}, \beta + \alpha t + \gamma t^2/2, u \equiv \gamma)$.

Theorem 11. *There exists $\gamma_0 > 0$ such that, for every $\gamma \in (-\gamma_0, \gamma_0)$ different from zero, there exist $C > 0$ and a continuous map*

$$\begin{array}{rcccl} \Pi_\gamma : [\mathcal{T}_{\gamma,0} \cap H^9_{(\gamma)}] \times \mathbb{R} \times [\mathcal{T}_{\gamma,T} \cap H^9_{(\gamma)}] \times \mathbb{R} & \to & \widetilde{E}_7^\gamma, \\ (\Psi_0 \quad , d_0 \ , \quad \Psi_f \quad , d_f) & \mapsto & (\Psi_0, d_0, w), \end{array}$$

where, for every $t \in \mathbb{R}$,

$$\mathcal{T}_{\gamma,t} := \{\varphi \in L^2(I, \mathbb{C}); \Re(\langle \varphi, \psi_{1,\gamma}(t) \rangle) = 0\},$$

such that, for every $(\Psi_0, d_0, \Psi_f, d_f) \in \widetilde{F}_9^\gamma$ with $\Psi_0 \in \mathcal{T}_{\gamma,0}$ and $\Psi_f \in \mathcal{T}_{\gamma,T}$, we have

$$\widetilde{\Phi}'_\gamma(\varphi_{1,\gamma}, 0, 0).\Pi_\gamma(\Psi_0, d_0, \Psi_f, d_f) = (\Psi_0, d_0, \Psi_f, d_f),$$

$$\|w\|_E \leqslant C \|(\Psi_0, d_0, \Psi_f, d_f)\|_F,$$

for any $(E, F) \in \{(L^2, \widetilde{F}_3^\gamma), (H_0^1, \widetilde{F}_5^\gamma), (H_0^2, \widetilde{F}_7^\gamma), (H_0^3, \widetilde{F}_9^\gamma)\}$.

5.2.3. Controllability up to codimension one of the linearized system around $(Y(t), u(t))$

Let $\gamma \in (-\gamma_0, \gamma_0)$ different from zero, where γ_0 is as in Theorem 11. Let $T := 4/\pi$, $(\psi_0, D_0, v) \in \underline{E}_9^\gamma$ and $S_0 \in \mathbb{R}$. As in [1, Section 3.6.3] we introduce

$$\begin{aligned} &\Delta_3 := \gamma + \delta_3, && \Delta_5 := \gamma + \delta_5, \\ &\Delta_7 := \gamma + \delta_7 + \delta_5^2, && \Delta_9 := \gamma + \delta_9 + \delta_7 \delta_5 + \delta_5^3, \end{aligned}$$

where $\delta_i := \|(\psi_0, d_0, v) - (\varphi_{1,\gamma}, 0, 0)\|_{\widetilde{E}_i^0}$.

Let $Y(t) := (\psi(t), S(t), D(t))$ be the solution of (Σ_0) with control $u := \gamma + v$ such that $\psi(0) = \psi_0$, $S(0) = S_0$ and $D(0) = D_0$. The linearized system around $(\psi(t), D(t), u(t))$ is

$$\begin{cases} i\frac{\partial \Psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \Psi}{\partial q^2} - uq\Psi - wq\psi, \\ \Psi(t, \pm 1/2) = 0, \\ \dot{s} = w, \\ \dot{d} = s. \end{cases}$$

If $(\Psi(0), s(0), d(0)) = (\Psi_0, 0, d_0)$, the equality $(\Psi(T), d(T)) = (\Psi_f, d_f)$ is equivalent to $\widetilde{M}_{(\psi_0,u)}(w) = \widetilde{d}(\Psi_0, d_0, \Psi_f, d_f)$ where

$$\begin{array}{ll} \widetilde{M}_{(\psi_0,u)}(w)_0 := \int_0^T (T-t)w(t)\,dt, & \widetilde{d}(\Psi_0, d_0, \Psi_f, d_f)_0 := d_f - d_0, \\ \widetilde{M}_{(\psi_0,u)}(w)_k := M_{(\psi_0,u)}(w)_k, & \widetilde{d}(\Psi_0, d_0, \Psi_f, d_f)_k := d(\Psi_0, \Psi_f)_k, \quad \forall k \in \mathbb{N}^* \end{array}$$

and the map $M_{(\psi_0,u)}$ is defined in [1, Section 3.6.1].

As in [1], we prove a surjectivity result on $\widetilde{M}_{(\psi_0,u)}$ when Δ_3 is small enough. The argument is the following one: we know a right inverse for $\widetilde{M}_{(\varphi_{1,\gamma},\gamma)}$, built in the previous subsection, and we prove that, when Δ_3 is small, $\widetilde{M}_{(\psi_0,u)}$ and $\widetilde{M}_{(\varphi_{1,\gamma},\gamma)}$ are close enough, in order to get a right inverse of $\widetilde{M}_{(\psi_0,u)}$.

The study of $(\widetilde{M}_{(\psi_0,u)} - \widetilde{M}_{(\varphi_{1,\gamma},\gamma)})(w)$ reduces to the study of $(M_{(\psi_0,u)} - M_{(\varphi_{1,\gamma},\gamma)})(w)$ [1, Section 3.6.3], because the new terms are equal. The study of the right-hand side $\widetilde{d}(\Psi_0, d_0, \Psi_f, d_f)$ is the same as in [1, Section 3.6.4]. In this way, we get the following theorem.

Theorem 12. *Let $\gamma \in (-\gamma_0, \gamma_0)$ different from zero. Let $(\psi_0, D_0, v) \in \widetilde{E}_9^\gamma$ and (ψ, S, D) be the associated solution of (Σ_0) with $u := \gamma + v$. If $\Delta_3 := \gamma + \|(\psi_0, D_0, v) - (\varphi_{1,\gamma}, 0, 0)\|_{\widetilde{E}_3^0}$ is small enough, then there exists a constant $C > 0$ and a continuous map*

$$\begin{array}{llllllll} \Pi_{\psi,v} : [\mathcal{T}_{\psi,0} \cap H^9_{(\gamma)}] & \times \mathbb{R} & \times [\mathcal{T}_{\psi,T} \cap H^9_{(\gamma)}] & \times \mathbb{R} & \to & \widetilde{E}_7^\gamma, \\ (\Psi_0 & , d_0 , & \Psi_f & , d_f) & \mapsto & (\Psi_0, d_0, w), \end{array}$$

where

$$\mathcal{T}_{\psi,t} := \{\varphi \in L^2(I, \mathbb{C}); \Re(\langle \varphi, \psi(t) \rangle) = 0\},$$

such that, for every $(\Psi_0, d_0, \Psi_f, d_f) \in \widetilde{F}_9^\gamma$ with $\Psi_0 \in \mathcal{T}_{\psi,0}$, $\Psi_f \in \mathcal{T}_{\psi,T}$, we have

$$\widetilde{\Phi}'_{\gamma,S_0}(\psi_0, D_0, v).\Pi_{\psi,v}(\Psi_0, d_0, \Psi_f, d_f) = (\Psi_0, d_0, \Psi_f, d_f),$$

and the same bounds as in [1, Theorem 9], *with everywhere* $\|(\Psi_0, \Psi_T)\|_{F_k^\gamma}$ *replaced by* $\|(\Psi_0, d_0, \Psi_f, d_f)\|_{\widetilde{F}_k^\gamma}$.

Now we can apply the Nash–Moser implicit function theorem stated in Section 3.2.2 and we get Theorem 10.

5.3. Motion in the directions $(\psi, S, D) = (0, \pm 1, 0)$

The aim of this section is to prove of the following theorem.

Theorem 13. *There exists* $\gamma_0 > 0$ *such that, for every* $\gamma \in (0, \gamma_0)$, *there exist* $w_\pm, v_\pm \in H_0^3((0,T),\mathbb{R})$ *such that the solutions of*

$$\begin{cases} i\frac{\partial \Psi_\pm}{\partial t} = -\frac{1}{2}\frac{\partial^2 \Psi_\pm}{\partial q^2} - \gamma q \Psi_\pm - w_\pm q \psi_{1,\gamma}, \\ \Psi_\pm(0) = 0, \\ \Psi_\pm(t,-1/2) = \Psi_\pm(t,1/2) = 0, \\ \dot{s}_\pm = w_\pm, s_\pm(0) = 0, \\ \dot{d}_\pm = s_\pm, d_\pm(0) = 0, \end{cases}$$

$$\begin{cases} i\frac{\partial \xi_\pm}{\partial t} = -\frac{1}{2}\frac{\partial^2 \xi_\pm}{\partial q^2} - \gamma q \xi_\pm - w_\pm q \Psi_\pm - v_\pm q \psi_{1,\gamma}, \\ \xi_\pm(0) = 0, \\ \xi_\pm(t,-1/2) = \xi_\pm(t,1/2) = 0, \\ \dot{\sigma}_\pm = v_\pm, \sigma_\pm(0) = 0, \\ \dot{\delta}_\pm = \sigma_\pm, \delta_\pm(0) = 0, \end{cases}$$

satisfy $\Psi_\pm(T) = 0,\ s_\pm(T) = 0,\ d_\pm(T) = 0,\ \xi_\pm(T) = 0,\ \sigma_\pm(T) = \pm 1,\ \delta_\pm(T) = 0.$

Let us introduce new notations. Let $\gamma \in \mathbb{R}^*$. We define the subspace of $L^2((0,T),\mathbb{C})$

$$\mathcal{X}_\gamma := \mathrm{Span}(t, e^{i(\lambda_{k,\gamma}-\lambda_{1,\gamma})t}, e^{-i(\lambda_{k,\gamma}-\lambda_{1,\gamma})t}; k \in \mathbb{N}^*).$$

The symbol $\mathcal{X}_\gamma^\perp$ denotes the orthogonal subspace to $\mathcal{X}_\gamma$ in $L^2((0,T),\mathbb{C})$. We recall that we have

$$\lambda_k := \lambda_{k,0} = \frac{1}{2}(k\pi)^2, \quad \varphi_k := \varphi_{k,0} = \begin{cases} \sqrt{2}\sin(k\pi q), & \text{when } k \text{ in even}, \\ \sqrt{2}\cos(k\pi q), & \text{when } k \text{ in odd}. \end{cases}$$

The parity of the functions φ_k gives $b_{2k+1} := b_{2k+1,0} = 0$ for every $k \in \mathbb{N}$.

One has the following proposition.

Proposition 21. *There exists $\gamma_0 > 0$ such that, for every $\gamma \in (0, \gamma_0)$, there exists $w_\gamma \in H_0^4((0,T),\mathbb{R}) \cap \mathcal{X}_\gamma^\perp$ such that*

$$\int_0^T w_\gamma(t)\langle q\Psi_\gamma(t), \psi_{1,\gamma}(t)\rangle dt \in (0,+\infty), (resp. \in (-\infty, 0)),$$

where Ψ_γ is the solution of

$$\begin{cases} i\frac{\partial \Psi_\gamma}{\partial t} = -\frac{1}{2}\frac{\partial^2 \Psi_\gamma}{\partial q^2} - \gamma q\Psi_\gamma - w_\gamma q\psi_{1,\gamma}, \\ \Psi_\gamma(0) = 0, \\ \Psi_\gamma(t,-1/2) = \Psi_\gamma(t,1/2) = 0. \end{cases}$$

Remark 7. Let $\gamma \in \mathbb{R}$. If $w_\gamma \in \mathcal{X}_\gamma^\perp$, and Ψ_γ is the solution of the previous system, then

$$\int_0^T w_\gamma(t)\langle q\Psi_\gamma(t), \psi_{1,\gamma}(t)\rangle dt \in \mathbb{R}.$$

Indeed, we have

$$\Psi_\gamma(t) = \sum_{k=1}^{+\infty} x_k(t)\varphi_{k,\gamma}, \text{ where } x_k(t) = ib_{k,\gamma}e^{-i\lambda_{k,\gamma}t}\int_0^t w_\gamma(\tau)e^{i(\lambda_{k,\gamma}-\lambda_{1,\gamma})\tau}d\tau.$$

Thus

$$\int_0^T w_\gamma(t)\langle q\Psi_\gamma(t), \psi_{1,\gamma}(t)\rangle dt = \sum\nolimits_{k=1}^{+\infty} ib_{k,\gamma}^2 f_{k,\gamma},$$

where

$$f_{k,\gamma} := \int_0^T w_\gamma(t)e^{-i(\lambda_{k,\gamma}-\lambda_{1,\gamma})t}\int_0^t w_\gamma(\tau)e^{i(\lambda_{k,\gamma}-\lambda_{1,\gamma})\tau}d\tau dt.$$

Thanks to an integration by parts in the definition of $f_{k,\gamma}$ and the property $w_\gamma \in \mathcal{X}_\gamma^\perp$, we get: for every $k \in \mathbb{N}^*$, $f_{k,\gamma} \in i\mathbb{R}$.

Proof of Proposition 21. First, we study the case $\gamma = 0$. Let us consider functions of the form

$$w(t) := \sin(\tfrac{1}{2}n_0\pi^2 t) + a_1\sin(\tfrac{1}{2}n_1\pi^2 t) + a_2\sin(\tfrac{1}{2}n_2\pi^2 t) + a_3\sin(\tfrac{1}{2}n_3\pi^2 t), \tag{5.5}$$

where n_0, n_1, n_2, n_3 are four different positive integers such that

$$n_0, n_1, n_2, n_3 \notin \{\pm(k^2-1); k \in \mathbb{N}^*\}$$

and $a_1, a_2, a_3 \in \mathbb{R}$ solve

$$\begin{pmatrix} \frac{1}{n_1} & \frac{1}{n_2} & \frac{1}{n_3} \\ n_1 & n_2 & n_3 \\ n_1^3 & n_2^3 & n_3^3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{n_0} \\ -n_0 \\ -n_0^3 \end{pmatrix}.$$

Then,

$$w \in H_0^4((0,T),\mathbb{R}) \cap \mathcal{X}_0^{\perp}.$$

Let Ψ be the solution of

$$\begin{cases} i\frac{\partial \Psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \Psi}{\partial q^2} - wq\psi_1, \\ \Psi(0) = 0, \\ \Psi(t,-1/2) = \Psi(t,1/2) = 0. \end{cases}$$

We have

$$\Psi(t) = \sum_{k=1}^{+\infty} x_k(t)\varphi_k \text{ where } x_k(t) := \langle \Psi(t), \varphi_k \rangle, \ \forall k \in \mathbb{N}^*,$$

$$\int_0^T w(t)\langle q\Psi(t), \psi_1(t)\rangle dt = \sum_{k=1}^{+\infty} b_{2k} \int_0^T w(t)x_{2k}(t)e^{i\lambda_1 t}dt, \tag{5.6}$$

$$x_{2k}(t) = ib_{2k}\left(\int_0^t w(\tau)e^{i(\lambda_{2k}-\lambda_1)\tau}d\tau\right)e^{-i\lambda_{2k}t},$$

$$b_{2k} = -\frac{(-1)^k 16k}{\pi^2(1+2k)^2(1-2k)^2}.$$

Thus, the right-hand side of (5.6) can be explicitly computed. We find

$$\int_0^T w(t)\langle q\Psi(t), \psi_1(t)\rangle dt = \frac{32T}{\pi^6}\left(S_{n_0} + a_1^2 S_{n_1} + a_2^2 S_{n_2} + a_3^2 S_{n_3}\right),$$

where, for every $p \in \mathbb{N}$ with $p \notin \{\pm(k^2-1); k \in \mathbb{N}^*\}$, S_p is defined by

$$S_p := \sum_{k=1}^{+\infty} \frac{(2k)^2}{(1+2k)^4(1-2k)^4}\left(\frac{1}{-p+4k^2-1}+\frac{1}{p+4k^2-1}\right).$$

Let us choose $n_0=1$, $n_1=2$, $n_2=4$, $n_3=5$ (resp. $n_0=4$, $n_1=5$, $n_2=6$, $n_3=7$), we get

$$\int_0^T w(t)\langle q\Psi(t), \psi_1(t)\rangle dt \in (0,+\infty) \text{ (resp. } (-\infty,0)). \tag{5.7}$$

Now, we study the case $\gamma \neq 0$. We use the following proposition, which will be proved later on.

Proposition 22. *Let $T = 4/\pi$. There exists $\gamma_* > 0$, C_1, $C_2 > 0$ such that, for every $\gamma \in (-\gamma_*, \gamma_*)$,*

1. *the linear map $\widetilde{Z}_\gamma$ is continuous from $H_0^4((0,T),\mathbb{R})$ to $h_r^8(\mathbb{N},\mathbb{C})$,*
2. *for every $w \in H_0^4((0,T),\mathbb{R})$,*

$$\|(\widetilde{Z}_\gamma - \widetilde{Z}_0)(w)\|_{h^8} \leqslant C_1\gamma^2\|w\|_{H_0^4},$$

3. *there exists a linear map*

$$\widetilde{Z}_\gamma^{-1} : h_r^8(\mathbb{N},\mathbb{C}) \to H_0^4((0,T),\mathbb{R}),$$

such that, for every $d \in h_r^8(\mathbb{N},\mathbb{C})$, $\widetilde{Z}_\gamma \circ \widetilde{Z}_\gamma^{-1}(d) = d$ and the function $w := \widetilde{Z}_\gamma^{-1}(d)$ satisfies

$$\|w\|_{H_0^4} \leqslant C_2\|d\|_{h^8}.$$

Let $\gamma \in (-\gamma_*, \gamma_*)$ different from zero. We define

$$w_\gamma := w - \widetilde{Z}_\gamma^{-1}(\widetilde{Z}_\gamma(w)),$$

where $\widetilde{Z}_\gamma^{-1}$ is defined in Proposition 22 and w is defined in (5.5). We have $w \in H_0^4((0,T),\mathbb{R})$, so $\widetilde{Z}_\gamma(w) \in h_r^8(\mathbb{N},\mathbb{C})$ and $\widetilde{Z}_\gamma^{-1}(\widetilde{Z}_\gamma(w)) \in H_0^4((0,T),\mathbb{R})$, thus

$$w_\gamma \in H_0^4((0,T),\mathbb{R}) \cap \mathcal{X}_0^\perp.$$

We have

$$\|w - w_\gamma\|_{H_0^4} = \|\widetilde{Z}_\gamma^{-1}((\widetilde{Z}_\gamma - \widetilde{Z}_0)(w))\|_{H_0^4} \leqslant C_2 C_1 \gamma^2 \|w\|_{H_0^4}. \tag{5.8}$$

Let us consider the map

$$\begin{array}{rcl} G : (-\gamma_*, \gamma_*) & \to & \mathbb{R}, \\ \gamma & \mapsto & \int_0^T w_\gamma(t) \langle q \Psi_\gamma(t), \psi_{1,\gamma}(t) \rangle \, dt, \end{array}$$

where, for every $\gamma \in (-\gamma_*, \gamma_*)$, Ψ_γ is the solution of the system written in Proposition 21. Bound (5.8) proves that G is continuous at $\gamma = 0$. We know from (5.7), that $G(0) > 0$ (resp. < 0). Thus, there exists $\gamma_0 > 0$ such that, for every $\gamma \in (-\gamma_0, \gamma_0)$, $G(\gamma) > 0$ (resp. < 0). $\square$

Proof of Proposition 22. The strategy is the same as in Section 3.1.2. We just need to build a right inverse for $\widetilde{Z}_0$ which maps $h_r^8(\mathbb{N}, \mathbb{C})$ into $H_0^4((0,T), \mathbb{R})$. With the same notations as in the proof of Proposition 19, a suitable candidate for $\widetilde{Z}_0^{-1}(d)$ is

$$w(t) := \left(\frac{1}{T} \sum_{k \in \mathbb{Z}} \tilde{d}_k e^{i\omega_k t} + \alpha (e^{i\frac{1}{2}n\pi^2 t} + e^{-i\frac{1}{2}n\pi^2 t}) \right) (1 - e^{i\frac{1}{2}\pi^2 t})^3 (1 - e^{-i\frac{1}{2}\pi^2 t})^3,$$

where $n \in \mathbb{N}$ with $\{n, n \pm 1, n \pm 2, n \pm 3\} \cap \{\pm(k^2 - 1); k \in \mathbb{N}^*\} = \emptyset$ and $\alpha \in \mathbb{R}$ is such that $\int_0^T (T - t) w(t) dt = d_0$. $\square$

Proof of Theorem 13. Let $\gamma \in (0, \gamma_0)$, where γ_0 is given in Proposition 21. Let $w \in H_0^4((0,T), \mathbb{R}) \cap \mathcal{X}_\gamma^\perp$ be such that

$$\int_0^T w(t) \langle q \Psi(t), \psi_{1,\gamma}(t) \rangle dt = -b_{1,\gamma} \text{ (resp. } = +b_{1,\gamma}).$$

We have

$$\Psi(t) = \sum_{k=1}^{+\infty} x_k(t) \varphi_{k,\gamma} \text{ where } x_k(t) = i b_{k,\gamma} e^{-i\lambda_{k,\gamma} t} \int_0^t w(\tau) e^{i(\lambda_{k,\gamma} - \lambda_{1,\gamma})\tau} d\tau.$$

The assumption $w \in \mathcal{X}_\gamma^\perp$ gives $\Psi(T)=0$, $s(T)=0$ and $d(T)=0$. Let us prove that there exists $v \in L^2((0,T),\mathbb{R})$ such that the solution of

$$\begin{cases} i\frac{\partial \xi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \xi}{\partial q^2} - \gamma q \xi - wq\Psi - vq\psi_{1,\gamma}, \\ \xi(0)=0, \\ \xi(t,-1/2)=\xi(t,1/2)=0 \\ \dot{\sigma}=v, \sigma(0)=0, \\ \dot{\delta}=\sigma, \delta(0)=0, \end{cases}$$

satisfies $\xi(T)=0$, $\sigma(T)=1$ (resp. $=-1$), $\delta(T)=0$. We have

$$\xi(t)=\sum_{k=1}^{+\infty} y_k(t)\varphi_{k,\gamma},$$

$$y_k(t)=ie^{-i\lambda_{k,\gamma}t}\int_0^t \left(w(\tau)\langle q\Psi(\tau),\varphi_{k,\gamma}\rangle + v(\tau)b_{k,\gamma}e^{-i\lambda_{1,\gamma}\tau}\right) e^{i\lambda_{k,\gamma}\tau}d\tau.$$

The equality $(\xi(T),\sigma(T),\delta(T))=(0,1,0)$ (resp. $=(0,-1,0)$) is equivalent to

$$\int_0^T v(t)e^{i(\lambda_{k,\gamma}-\lambda_{1,\gamma})t}dt = -\frac{1}{b_{k,\gamma}}\int_0^T w(t)\langle q\Psi(t),\varphi_{k,\gamma}\rangle e^{i\lambda_{k,\gamma}t}dt, \quad \forall k\in\mathbb{N}^*,$$

$$\int_0^T v(t)dt = 1(\text{resp.} =-1),$$

$$\int_0^T (T-t)v(t)\,dt=0.$$

A necessary condition for the existence of a solution v to this moment problem is

$$-\frac{1}{b_{1,\gamma}}\int_0^T w(t)\langle q\Psi(t),\varphi_{1,\gamma}\rangle e^{i\lambda_{1,\gamma}t}dt = +1 \text{ (resp. } =-1).$$

The choice of w has been done in order to satisfy this condition.

Then, a sufficient condition for the existence of a solution $v \in H_0^3((0,T),\mathbb{R})$ is

$$\left(\int_0^T w(t)\langle q\Psi(t),\varphi_{k,\gamma}\rangle e^{i\lambda_{k,\gamma}t}dt\right)_{k\in\mathbb{N}^*} \in h_r^9(\mathbb{N}^*,\mathbb{C}) \tag{5.9}$$

(see [1, Section 3.1], for the behavior of $b_{k,\gamma}$ and Proposition 20 for the existence of $\widetilde{Z}_\gamma^{-1}$ between the suitable spaces).

The assumption $w \in H_0^4((0,T),\mathbb{R})$ implies (5.9). Indeed, integrations by parts lead to

$$\int_0^T w\langle q\Psi, \varphi_{k,\gamma}\rangle e^{i\lambda_{k,\gamma}t}dt = \frac{1}{\lambda_{k,\gamma}^4}\int_0^T \left(\frac{\partial^4 w}{\partial t^4}\langle q\Psi, \varphi_{k,\gamma}\rangle \right. \\ +4\frac{\partial^3 w}{\partial t^3}\langle q\dot{\Psi}, \varphi_{k,\gamma}\rangle + 6\ddot{w}\langle q\ddot{\Psi}, \varphi_{k,\gamma}\rangle \\ \left. + 4\dot{w}\left\langle q\frac{\partial^3\Psi}{\partial t^3}, \varphi_{k,\gamma}\right\rangle + w\left\langle q\frac{\partial^4\Psi}{\partial t^4}, \varphi_{k,\gamma}\right\rangle\right) e^{i\lambda_{k,\gamma}t}\,dt.$$

Moreover, when $v \in L^2((0,T),\mathbb{R})$ and $f \in C^0([0,T], L^2(I,\mathbb{C}))$, we have

$$\left|\int_0^T v(t)\langle f(t), \varphi_{k,\gamma}\rangle e^{i\lambda_{k,\gamma}t}\,dt\right|^2 \leqslant \|v\|^2_{L^2((0,T),\mathbb{R})}\int_0^T |\langle f(t), \varphi_{k,\gamma}\rangle|^2 dt.$$

Therefore, since the family $(\varphi_{k,\gamma})_{k\in\mathbb{N}^*}$ is orthonormal in $L^2(I,\mathbb{C})$, the sequence

$$\left(\int_0^T v(t)\langle f(t), \varphi_{k,\gamma}\rangle e^{i\lambda_{k,\gamma}t}dt\right)_{k\in\mathbb{N}^*}$$

belongs to $l^2(\mathbb{N}^*,\mathbb{C})$. □

5.4. Proof of Theorem 8

In all this section, we fix $\gamma \in (-\gamma_0, \gamma_0)$ different from zero, where γ_0 is as in Theorem 13.

Let $\rho \in \mathbb{R}$, $\psi_0, \psi_f \in H^7_{(\gamma)}(I,\mathbb{C})$, $S_0, D_0, D_f \in \mathbb{R}$. Let us consider, for $t \in [0,T]$,

$$v(t) := \sqrt{|\rho|}w + |\rho|\nu,$$

where $w := w_+$, $\nu := \nu_+$ if $\rho \geqslant 0$ and $w := w_-$, $\nu := \nu_-$ if $\rho < 0$. Let (ψ, S, D) be the solution of (Σ_0) on $[0,T]$ with $u := \gamma + v$. Then,

$$S(T) = S_0 + \gamma T + \rho \text{ and } D(T) = D_0 + S_0 T + \gamma T^2/2.$$

We have $v \in W^{3,1}((0,T),\mathbb{R})$, $v(0) = v(T) = \dot{v}(0) = \dot{v}(T) = 0$, so [1, Appendix B, Proposition 51], $\psi \in C^0([0,T], H^7(I,\mathbb{C}))$ and $\psi(T) \in H^7_{(\gamma)}(I,\mathbb{C})$.

Proposition 23. *There exists a constant $\mathcal{C}$ such that, for every $\rho \in (-1, 1)$, we have*

$$\|(\psi - \psi_{1,\gamma})(T)\|_{H^7(I,\mathbb{C})} \leqslant \mathcal{C}[\|\psi_0 - \psi_{1,\gamma}(0)\|_{H^7(I,\mathbb{C})} + |\rho|^{3/2}].$$

Proof. We have $(\psi - \psi_{1,\gamma})(T) = (\psi - Z)(T)$ where $Z := \psi_{1,\gamma} + \Psi + \xi$ and Ψ, ξ are the solutions of the following systems

$$\begin{cases} i\frac{\partial \Psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \Psi}{\partial q^2} - \gamma q \Psi - \sqrt{|\rho|} w q \psi_{1,\gamma}, \\ \Psi(0) = 0, \\ \Psi(t, -1/2) = \Psi(t, 1/2) = 0, \end{cases}$$

$$\begin{cases} i\frac{\partial \xi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \xi}{\partial q^2} - \gamma q \xi - \sqrt{|\rho|} w q \Psi - |\rho| v q \psi_{1,\gamma}, \\ \xi(0) = 0, \\ \xi(t, -1/2) = \xi(t, 1/2) = 0. \end{cases}$$

The function $\Delta := \psi - Z$ solves

$$\begin{cases} i\frac{\partial \Delta}{\partial t} = -\frac{1}{2}\frac{\partial^2 \Delta}{\partial q^2} - (\gamma + v) q \Delta - \sqrt{|\rho|} w q \xi - |\rho| v q (\Psi + \xi), \\ \Delta(0) = \psi_0 - \psi_{1,\gamma}(0), \\ \Delta(t, -1/2) = \Delta(t, 1/2) = 0. \end{cases} \tag{5.10}$$

We know from [1, Proposition 51, Appendix B], that the following quantities

$$\|\Psi\|_{C^0([0,T],H^7)}, \|\Psi\|_{C^1([0,T],H^5)}, \|\Psi\|_{C^2([0,T],H^3)}, \|\Psi\|_{C^3([0,T],H^1)},$$

are bounded by

$$A_7(\Psi) := C[\|f\|_{C^0([0,T],H^5)} + \|f\|_{C^1([0,T],H^3)} + \|f\|_{W^{2,1}((0,T),H^2)} + \|f\|_{W^{3,1}((0,T),H^1)}],$$

where C is a positive constant and $f := \sqrt{|\rho|} w q \psi_{1,\gamma}$. Thus, there exists a constant C_1 such that

$$A_7(\Psi) \leqslant C_1 \sqrt{|\rho|}.$$

In the same way, we prove that there exists a constant C_2 such that

$$\|\xi\|_{C^0([0,T],H^7)}, \|\xi\|_{C^1([0,T],H^5)}, \|\xi\|_{C^2([0,T],H^3)}, \|\xi\|_{C^3([0,T],H^1)}$$

are bounded by

$$A_7(\xi) \leqslant C_2 |\rho|.$$

Using (5.10) and [1, Appendix B, Proposition 51] we get the existence of a constant $C_3 > 0$ such that

$$\|\Delta(T)\|_{H^7} \leqslant C_3[\|\psi_0 - \psi_{1,\gamma}(0)\|_{H^7} + \sqrt{|\rho|}A_7(\xi) + |\rho|A_7(\Psi)]. \qquad \square$$

Now, we apply the local controllability of (ψ, D) on $[0, T]$ around $(\psi_{1,\gamma}(t), \beta + \alpha t + \gamma t^2/2)$, with

$$\alpha := S_0 \text{ and } \beta := D_0.$$

Let $\delta > 0$ as in Theorem 10. We assume

$$\|\psi_0 - \psi_{1,\gamma}(0)\|_{H^7(I,\mathbb{C})} < \frac{\delta}{2\mathcal{C}}, \tag{5.11}$$

$$\|\psi_f - \psi_{1,\gamma}(2T)\|_{H^7(I,\mathbb{C})} + |D_f - D_0 - 2S_0T - 2\gamma T^2/2| < \delta, \tag{5.12}$$

$$|\rho| < \eta := \left(\frac{\delta}{2\mathcal{C}}\right)^{2/3}.$$

Then we have

$$\|\psi(T) - \psi_{1,\gamma}(T)\|_{H^7(I,\mathbb{C})} + |D(T) - (\beta + \alpha T + \gamma T^2/2)| < \mathcal{C}\left[\frac{\delta}{2\mathcal{C}} + \eta^{3/2}\right] < \delta,$$

$$\|\psi_f - \psi_{1,\gamma}(2T)\|_{H^7(I,\mathbb{C})} + |D_f - (\beta + 2\alpha T + 2\gamma T^2)| < \delta.$$

So there exists $\tilde{v} \in H_0^1((T, 2T), \mathbb{R})$ such that

$$\psi(2T) = \psi_f \text{ and } D(2T) = D_f,$$

where (ψ, S, D) still the solution of (Σ_0) with control $u := \gamma + v$ on $[0, 2T]$, with v extended to $[0, 2T]$ by $v := \tilde{v}$ on $[T, 2T]$. We know that $\tilde{v}$ can be chosen so that there exists a constant K such that

$$\|v\|_{L^2((T,2T),\mathbb{R})} \leqslant K[\|\psi_0 - \psi_{1,\gamma}(0)\|_{H^7(I,\mathbb{C})} + |\rho|^{3/2} + \|\psi_f - \psi_{1,\gamma}(2T)\|_{H^7(I,\mathbb{C})}$$
$$+ |D_f - (\beta + 2\alpha T + 2\gamma T^2/2)|],$$

We used Proposition 23 in order to get this bound.

Moreover, we have

$$S(2T) = S_0 + 2\gamma T + \rho + \int_T^{2T} v(t)\,dt.$$

We define the map

$$F : \begin{array}{rcl} (-\eta, \eta) & \to & \mathbb{R}, \\ \rho & \mapsto & S(2T). \end{array}$$

There exist $\tau \in (0, \eta)$ such that

$$\sqrt{T} K \tau^{3/2} < \tau/3.$$

Let us assume

$$\sqrt{T} K \|\psi_0 - \psi_{1,\gamma}(0)\|_{H^7(I,\mathbb{C})} < \tau/6,$$

$$\sqrt{T} K \left(\|\psi_f - \psi_{1,\gamma}(2T)\|_{H^7(I,\mathbb{C})} + |D_f - (\beta + 2\alpha T + 2\gamma T^2/2)| \right) < \tau/6.$$

Then,

$$F(\tau) - (S_0 + 2\gamma T) > \tau/3 > 0 \text{ and } F(-\tau) - (S_0 + 2\gamma T) < -\tau/3 < 0.$$

The map F is continuous, thus, F is surjective on a neighborhood of $S_0 + 2\gamma T$, this ends the proof of Theorem 2. □

References

[1] K. Beauchard, Local controllability of a 1-D Schrödinger equation, Prépubl. Univ. Paris Sud, (accepted in Journal de Mathématiques Pures et Appliquées), 2004.

[2] J.-M. Coron, Global asymptotic stabilization for controllable systems without drift, Math. Control Signals Systems 5 (1992) 295–312.

[3] J.-M. Coron, Contrôlabilité exacte frontière de l'équation d'Euler des fluides parfaits incompressibles bidimensionnels, C.R. Acad. Sci. Paris 317 (1993) 271–276.

[4] J.-M. Coron, On the controllability of 2-D incompressible perfect fluids, J. Math. Pures Appl. 75 (1996) 155–188.

[5] J.-M. Coron, On the controllability of the 2-D incompressible Navier–Stokes equations with the Navier slip boundary conditions, ESAIM: COCV 1 (1996) 35–75.

[6] J.-M. Coron, Local controllability of a 1-D tank containing a fluid modeled by the shallow water equations, ESAIM: COCV 8 (2002) 513–554.

[7] J.-M. Coron, E. Crépeau, Exact boundary controllability of a nonlinear KdV equation with critical lengths, J. European Math. Soc. 6 (2004) 367–398.

[8] J.-M. Coron, A. Fursikov, Global exact controllability of the 2-D Navier–Stokes equations on a manifold without boundary, Russian Journal of Mathematical Physics 4 (1996) 429–448.
[9] A.V. Fursikov, O.Yu. Imanuvilov, Exact controllability of the Navier–Stokes and Boussinesq equations, Russian Math. Surveys 54 (3) (1999) 565–618; Uspekhi Mat. Nauk 54 (3) (1999) 565–618 (in Russian).
[10] O. Glass, Exact boundary controllability of 3-D Euler equation, ESAIM: COCV 5 (2000) 1–44.
[11] O. Glass, On the controllability of the Vlasov–Poisson system, Journal of Differential Equations 195 (2003) 332–379.
[12] G. Gromov, Partial Differential Relations, Springer, Berlin, New York, London, 1986.
[13] L. Hörmander, On the Nash–Moser implicit function theorem, Ann. Acad. Sci. Fennicae (1985) 255–259.
[14] Th. Horsin, On the controllability of the Burgers equation, ESAIM: COCV 3 (1998) 83–95.
[15] T. Kato, Perturbation Theory for Linear operators, Springer, Berlin, New York, 1966.
[16] P. Rouchon, Control of a quantum particule in a moving potential well, Second IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control, Seville, 2003.
[17] E. Zuazua, Remarks on the controllability of the Schrödinger equation, CRM Proceedings and Lecture Notes, vol. 33, 2003.
[18] G. Turinici, On the controllability of bilinear quantum systems, in: M. Defranceschi, C. Le Bris (Eds.), Mathematical models and methods for ab initio quantum chemistry. Lecture Notes in Chem. Vol. 74, Springer, Berlin, 2000, pp. 75–92.