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Partial Differential Equations/Optimal Control

Semi-global weak stabilization of bilinear Schrödinger equations

Stabilisation faible semi-globale d'équations de Schrödinger bilinéaires

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A R T I C L E   I N F O

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A B S T R A C T

We consider a linear Schrödinger equation, on a bounded domain, with bilinear control, representing a quantum particle in an electric field (the control). Recently, Nersesyan proposed explicit feedback laws and proved the existence of a sequence of times $(t_n)_{n\in\mathbb{N}}$ for which the values of the solution of the closed loop system converge weakly in H^2 to the ground state. Here, we prove the convergence of the whole solution, as $t \to +\infty$. The proof relies on control Lyapunov functions and an adaptation of the LaSalle invariance principle to PDEs.

R É S U M É

Nous considérons une équation de Schrödinger linéaire, sur un domaine borné, avec un contrôle bilinéaire, modélisant une particule quantique dans un champ électrique (la commande). Récemment, Nersesyan a proposé des lois de rétroaction explicites et démontré l'existence d'une suite de temps $(t_n)_{n\in\mathbb{N}}$ auxquels les valeurs de la solution du système bouclé convergent faiblement dans H^2 vers l'état fondamental. Ici, nous démontrons la convergence de toute la solution, quand $t \to +\infty$. La preuve repose sur des fonctions de Lyapunov et une adaptation du principe d'invariance de LaSalle aux EDP.

Version française abrégée

On considère le système

$$i\dot{z} = -\Delta z + V(x)z + u(t)Q(x)z, \quad x \in D, \tag{1}$$

$$z|_{\partial D} = 0, \tag{2}$$

$$z(0,x) = z_0(x), \tag{3}$$

où $D \subset \mathbb{R}^m$ est un domaine borné à bord lisse, $V, Q \in C^\infty(\overline{D}, \mathbb{R})$ sont des fonctions données, u est le contrôle et z est l'état. Il modélise une particule quantique dans un potentiel V et un champ électrique u.

Notons $(e_{k,V})_{k\in\mathbb{N}^*}$ les vecteurs propres de l'opérateur $(-\Delta + V)$, $(-\Delta + V)e_{k,V} = \lambda_{k,V} e_{k,V}$, $P_{1,V} z := z - \langle z, e_{1,V}\rangle e_{1,V}$ la projection orthogonale de $L^2(D,\mathbb{C})$ sur $\mathrm{Vect}\{e_{k,V}, k \geqslant 2\}$ et S la sphère unité de $L^2(D,\mathbb{C})$.

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Les lois de rétroaction explicites suivantes sont introduites dans [15] :

$$u(z) := -\delta\,\mathrm{Im}\big[\big\langle \alpha(-\Delta+V)P_{1,V}(Qz), (-\Delta+V)P_{1,V}z\big\rangle - \langle Qz, e_{1,V}\rangle\langle e_{1,V}, z\rangle\big], \tag{4}$$

où $\delta, \alpha > 0$. Elles permettent de considérer le système bouclé

$$i\dot{z} = -\Delta z + V(x)z + u(z)Q(x)z, \quad x \in D. \tag{5}$$

Rappelons la condition suivante, introduite également dans [15] :

Condition 0.1. Les fonctions $V, Q \in C^\infty(\overline{D}, \mathbb{R})$ vérifient

(i) $\langle Qe_{1,V}, e_{j,V}\rangle \neq 0$ pour tout $j \geqslant 2$,
(ii) $\lambda_{1,V} - \lambda_{j,V} \neq \lambda_{p,V} - \lambda_{q,V}$ pour tout $j, p, q \geqslant 1$ tels que $\{1, j\} \neq \{p, q\}$ et $j \neq 1$.

Dans [15], Nersesyan démontre que, sous la Condition 0.1, il existe une suite de temps $(t_n)_{n\in\mathbb{N}}$ auxquels la solution du système bouclé converge faiblement dans H^2 vers l'état fondamental : $z(t_n) \rightharpoonup e_{1,V}$ dans H^2 quand $n \to +\infty$. Dans cet article, nous démontrons que toute la solution converge : $z(t) \rightharpoonup e_{1,V}$ (à phase près) dans H^2 quand $t \to +\infty$.

Théorème 0.2. *On suppose la Condition 0.1 vérifiée. Soit $\mathcal{U}_t$ la résolvante du système bouclé (5), (2). Il existe un ensemble fini ou dénombrable $J \subset \mathbb{R}_+^*$ tel que, pour tout $z_0 \in S \cap H_0^1 \cap H^2$ n'appartenant pas à $\mathcal{C} := \{ce_{1,V} \colon c \in \mathbb{C}, |c| = 1\}$, il existe $\alpha^* := \alpha^*(\|z_0\|_2) > 0$ tel que, pour tout $\alpha \in (0, \alpha^*) - J$, $\mathcal{U}_t(z_0) \rightharpoonup \mathcal{C}$, dans H^2, quand $t \to \infty$.*

La preuve du Théorème 0.2 repose sur le principe d'invariance de LaSalle et se fait en deux étapes. Dans un premier temps, on vérifie que l'ensemble invariant coincide localement avec $\mathcal{C}$. Dans un deuxième temps, on démontre la convergence. Pour cela, on montre que les seules valeurs d'adhérence possibles, pour la topologie faible H^2, de la solution du système bouclé, sont dans $\mathcal{C}$. Considérant une valeur d'adhérence faible H^2, $\mathcal{U}_{t_n}(z_0) \rightharpoonup z_\infty$, on démontre qu'elle appartient à $\mathcal{C}$, en montrant qu'elle engendre une solution invariante, $u[\mathcal{U}_t(z_\infty)] \equiv 0$. Pour cela, on démontre que $u[\mathcal{U}_{t_n+t}(z_0)] \to 0$ quand $n \to +\infty$ pour presque tout $t \in [0, +\infty)$ et on justifie le passage à la limite $[n \to +\infty]$ dans ce feedback.

1. Introduction

1.1. The system

We consider the system (1)–(3) where $D \subset \mathbb{R}^m$ is a bounded domain with smooth boundary, $V, Q \in C^\infty(\overline{D}, \mathbb{R})$ are given functions, u is the control, and z is the state. It represents a quantum particle in a potential V, in an electric field u. The following proposition establishes the well-posedness of system (1)–(3) (see [6] for a proof):

Proposition 1.1. *For any $z_0 \in H_0^1 \cap H^2$ (resp. $z_0 \in L^2$) and for any $u \in L^1_{loc}([0, \infty), \mathbb{R})$ problem (1)–(3) has a unique solution $z \in C([0, \infty), H_0^1 \cap H^2)$ (resp. $z \in C([0, \infty), L^2)$). Furthermore, the resolving operator $\mathcal{U}_t(\cdot, u) : L^2 \to L^2$ taking z_0 to $z(t)$ satisfies the relation*

$$\big\|\mathcal{U}_t(z_0, u)\big\| = \|z_0\|, \quad \forall t \geqslant 0. \tag{6}$$

In all this article, $\|.\|$ (resp. $\|.\|_s$) denotes the usual norm on $L^2(D, \mathbb{C})$ (resp. $H^s(D, \mathbb{C})$, for every $s \in \mathbb{N}^*$). S is the $L^2(D, \mathbb{C})$-sphere and

$$\langle f, g\rangle := \int_D f(x)\overline{g(x)}\,\mathrm{d}x, \quad \forall f, g \in L^2(D, \mathbb{C}).$$

1.2. Bibliography

We refer to [2,4,7,14] for exact or approximate controllability results for the system (1)–(3), with open loop controls. This article is concerned with closed loop controls: we search explicit feedback laws, that asymptotically stabilize the ground state.

In [13], the same question is addressed for ODE models. The control design relies on control Lyapunov functions, and the convergence proof relies on the LaSalle invariance principle. This reference deals with the situation where the linearized system around the ground state is controllable. A degenerate case is studied in [3].

The goal of this article is to adapt the result of [13] to PDE models. Indeed, the LaSalle invariance principle is a powerful tool to prove the asymptotic stability of an equilibrium for a finite-dimensional dynamic system. However, using it for infinite-dimensional systems is more difficult (because closed and bounded subsets are not necessarily compact).

A first possible adaptation consists in proving *approximate* convergence results, as for example in [5,12]. A second possible adaptation consists in proving a *weak* convergence, as, for example, in [1] and in this article. A third possible adaptation consists in proving a strong convergence, as for example in [8]. In this case, one needs an additional compactness property for the trajectories of the closed loop system. Another strategy consists in designing strict Lyapunov functions, as for example in [9].

1.3. Stabilization strategy

Let us recall the stabilization strategy proposed in [15]. We introduce the Lyapunov function

$$\mathcal{V}(z) := \alpha \left\| (-\Delta + V) P_{1,V} z \right\|^2 + 1 - \left| \langle z, e_{1,V} \rangle \right|^2, \quad z \in S \cap H_0^1 \cap H^2,$$

where $\alpha > 0$, $(e_{k,V})_{k \in \mathbb{N}^*}$ are the eigenvectors of the operator $-\Delta + V$, $(-\Delta + V)e_{k,V} = \lambda_{k,V} e_{k,V}$ and $P_{1,V} z := z - \langle z, e_{1,V} \rangle e_{1,V}$ is the orthogonal projection in L^2 onto the closure of $\mathrm{Span}\{e_{k,V},\ k \geqslant 2\}$. Notice that $\mathcal{V}(z) \geqslant 0$ for all $z \in S \cap H_0^1 \cap H^2$ and $\mathcal{V}(z) = 0$ if and only if $z = c e_{1,V}$, $|c| = 1$. For any $z \in S \cap H_0^1 \cap H^2$, we have

$$\mathcal{V}(z) \geqslant \alpha \left\| (-\Delta + V) P_{1,V} z \right\|^2 \geqslant \frac{\alpha}{2} \left\| \Delta (P_{1,V} z) \right\|^2 - C_1 \geqslant \frac{\alpha}{4} \| \Delta z \|^2 - C_2,$$

where C_1 and C_2 are positive constants. Thus

$$C\big(1 + \mathcal{V}(z)\big) \geqslant \|z\|_2 \tag{7}$$

for some constant $C > 0$. Following the ideas of [3,15], we wish to choose a feedback law $u(\cdot)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{V}\big(z(t)\big) \leqslant 0$$

for the solution $z(t)$ of (1)–(3). Let us assume that $\Delta z(t) \in H_0^1 \cap H^2$ for all $t \geqslant 0$. Using (1), we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{V}\big(z(t)\big) &= 2\alpha \,\mathrm{Re}\big[\big\langle (-\Delta + V) P_{1,V} \dot{z}, (-\Delta + V) P_{1,V} z \big\rangle\big] - 2 \,\mathrm{Re}\big[\langle \dot{z}, e_{1,V} \rangle \langle e_{1,V}, z \rangle\big] \\ &= 2\alpha \,\mathrm{Re}\big[\big\langle (-\Delta + V) P_{1,V} (i\Delta z - iVz - iuQz), (-\Delta + V) P_{1,V} z \big\rangle\big] \\ &\quad - 2 \,\mathrm{Re}\big[\langle i\Delta z - iVz - iuQz, e_{1,V} \rangle \langle e_{1,V}, z \rangle\big]. \end{aligned}$$

Integrating by parts and using the facts that V is real valued, $P_{1,V}$ commutes with $-\Delta + V$ and

$$(-\Delta + V) P_{1,V} z|_{\partial D} = z|_{\partial D} = e_{1,V}|_{\partial D} = 0,$$

we obtain

$$\begin{aligned} &2\alpha \,\mathrm{Re}\big[\big\langle -i(-\Delta + V)^2 P_{1,V} z, (-\Delta + V) P_{1,V} z \big\rangle\big] - 2 \,\mathrm{Re}\big[\langle i\Delta z - iVz, e_{1,V} \rangle \langle e_{1,V}, z \rangle\big] \\ &\quad = 2\alpha \,\mathrm{Re}\big[\big\langle -i\nabla(-\Delta + V) P_{1,V} z, \nabla(-\Delta + V) P_{1,V} z \big\rangle\big] + 2\alpha \,\mathrm{Re}\big[\big\langle -iV(-\Delta + V) P_{1,V} z, (-\Delta + V) P_{1,V} z \big\rangle\big] \\ &\qquad + 2\lambda_{1,V} \,\mathrm{Re}\big[\langle iz, e_{1,V} \rangle \langle e_{1,V}, z \rangle\big] = 0. \end{aligned}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{V}\big(z(t)\big) = 2u \,\mathrm{Im}\big[\alpha \big\langle (-\Delta + V) P_{1,V} (Qz), (-\Delta + V) P_{1,V} z \big\rangle - \langle Qz, e_{1,V} \rangle \langle e_{1,V}, z \rangle\big].$$

Let us take $u(z)$ defined by (4) where $\delta > 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{V}\big(z(t)\big) = -\frac{2}{\delta} u^2\big(z(t)\big), \tag{8}$$

thus $t \mapsto \mathcal{V}(z(t))$ is not increasing and one may expect that $z(t) \to \mathcal{C} := \{c e_{1,V}, c \in \mathbb{C}, |c| = 1\}$, in some sense, when $t \to +\infty$. We consider the closed loop system (5). The following proposition ensures the well-posedness of this system and the validity of the computations performed above:

Proposition 1.2. *For any $z_0 \in H_0^1 \cap H^2$ problem* (5), (2), (3) *has a unique solution $z \in C([0, \infty), H_0^1 \cap H^2)$. Moreover if $\Delta z_0 \in H_0^1 \cap H^2$, then, $\Delta z \in C([0, \infty), H_0^1 \cap H^2)$.*

The local well-posedness and the regularity of the solution of (5), (2), (3) is standard (see [6]). From the construction of the feedback law u it follows that a finite-time blow-up in $H_0^1 \cap H^2$ is impossible. Hence the solution is global in time.

1.4. Main result

Let us introduce the following condition on the functions V and Q:

Condition 1.3. The functions $V, Q \in C^\infty(\overline{D}, \mathbb{R})$ are such that:

(i) $\langle Q e_{1,V}, e_{j,V}\rangle \neq 0$ for all $j \geqslant 2$,
(ii) $\lambda_{1,V} - \lambda_{j,V} \neq \lambda_{p,V} - \lambda_{q,V}$ for all $j, p, q \geqslant 1$ such that $\{1, j\} \neq \{p, q\}$ and $j \neq 1$.

See the papers [16,14,11] for the proof of genericity of this condition. The theorem below is the main result of this article.

Theorem 1.4. *Let $\mathcal{U}_t$ be the resolving operator of the closed loop system* (5), (2). *Under Condition* 1.3, *there is a finite or countable set $J \subset \mathbb{R}_+^*$ such that for any $\alpha \notin J$ and $z_0 \in S \cap H_0^1 \cap H^2$ with $0 < \mathcal{V}(z_0) < 1$ we have*

$$\mathcal{U}_t(z_0) \rightharpoonup \mathcal{C} \quad \text{in } H^2 \text{ as } t \to \infty, \tag{9}$$

where $\mathcal{C} := \{ce_{1,V}\colon c \in \mathbb{C}, |c| = 1\}$.

Remark 1.5. This theorem proves the semi-global stabilization of the ground state. Indeed, for every $z_0 \in S \cap H_0^1 \cap H^2$ such that $z_0 \notin \mathcal{C}$, one may chose $\alpha = \alpha(\|z_0\|_2) > 0$ small enough so that the condition $0 < \mathcal{V}(z_0) < 1$ is fulfilled.

2. Convergence proof

The first step of the proof consists in checking that the LaSalle invariance set locally coincides with $\mathcal{C}$.

Proposition 2.1. *We assume Condition* 1.3. *There exists a finite or countable set $J \subset \mathbb{R}_+^*$ such that, for every $\alpha \notin J$, for every $z_0 \in S \cap H_0^1 \cap H^2$ with $\langle z_0, e_{1,V}\rangle \neq 0$ and $u(\mathcal{U}_t(z_0)) = 0$ for all $t \geqslant 0$, then $z_0 \in \mathcal{C}$.*

This proposition is proved in [15]. The second step of the proof consists in proving the convergence. First, we need the following preliminary result:

Proposition 2.2. *Let $\mathcal{U}_t$ be the resolving operator of the closed loop system* (5), (2). *Let $z_n \in H_0^1 \cap H^2$ be such that $z_n \rightharpoonup z_\infty$ in H^2 and $z_n \to z_\infty$ in H_0^1. For every $T > 0$, there exists $N \subset (0, T)$ with zero Lebesgue measure such that*

1. *$\mathcal{U}_t(z_n) \rightharpoonup \mathcal{U}_t(z_\infty)$ in H^2 and $\mathcal{U}_t(z_n) \to \mathcal{U}_t(z_\infty)$ in H_0^1, $\forall t \in (0, T) - N$,*
2. *$u[\mathcal{U}_t(z_n)] \to u[\mathcal{U}_t(z_\infty)]$, $\forall t \in (0, T) - N$.*

Proof of Proposition 2.2.
First step: Let us show that, if $z_n \in H_0^1 \cap H^2$, $z_n \rightharpoonup z_\infty$ in H^2 and $z_n \to z_\infty$ in H_0^1, then $u(z_n) \to u(z_\infty)$. Then, the second conclusion of Proposition 2.2 will be a consequence of the first one.

Notice that (4) and the fact that Q is real valued imply that

$$u(z) = -\delta \operatorname{Im}\big[\big\langle \alpha Q(-\Delta + V)z, (-\Delta + V)z\big\rangle\big] + \tilde{u}(z) = \tilde{u}(z),$$

where

$$\begin{aligned}\tilde{u}(z) = -\delta \operatorname{Im}\big[&\big\langle \alpha(-\Delta + V)P_{1,V}(Qz), (-\Delta + V)\big(-\langle z, e_{1,V}\rangle e_{1,V}\big)\big\rangle \\ &+ \big\langle \alpha(-\Delta + V)\big(-\langle Qz, e_{1,V}\rangle e_{1,V}\big), (-\Delta + V)z\big\rangle + \big\langle \alpha(-\nabla Q \cdot \nabla z - z\Delta Q), (-\Delta + V)z\big\rangle \\ &- \langle Qz, e_{1,V}\rangle\langle e_{1,V}, z\rangle\big].\end{aligned} \tag{10}$$

Thus, passing to the limit in the previous equality, we get $u(z_n) \to u(z_\infty)$.

Second step: Let us prove the first conclusion of Proposition 2.2. Let $z_n \in H_0^1 \cap H^2$ be such that $z_n \rightharpoonup z_\infty$ in H^2 and $z_n \to z_\infty$ in H_0^1. For $T > 0$ define the Banach space $W := \{z \in C([0, T], H_0^1 \cap H^2)$ such that $\dot{z} \in L^2([0, T], L^2)\}$ endowed with the norm $\|z\|_W := \|z\|_{C([0,T],H_0^1 \cap H^2)} + \|\dot{z}\|_{L^2([0,T],L^2)}$. The sequence of functions $(t \in [0, T] \mapsto \mathcal{U}_t(z_n))_{n\in\mathbb{N}}$ is bounded in W, and the embedding $W \to L^2((0, T), H_0^1)$ is compact, by Theorem 5.1 in [10]. Let $Y \in L^2((0, T), H_0^1)$ and φ be an extraction such that

$$\mathcal{U}_{.}(z_{\varphi(n)}) \to Y(.) \quad \text{in } L^2\big((0, T), H_0^1\big).$$

Thanks to the Lebesgue reciprocal theorem, one may assume that

$$\mathcal{U}_t(z_{\varphi(n)}) \to Y(t) \quad \text{in } H_0^1,\ \forall t \in (0,T) - N, \tag{11}$$

where $N \subset (0,T)$ has zero Lebesgue measure (otherwise take another extraction).

For every $t^* \in (0,T) - N$, the sequence $(\mathcal{U}_{t^*}(z_{\varphi(n)}))_{n\in\mathbb{N}}$ is bounded in H^2 and its only possible weak H^2 limit is $Y(t^*)$ because of (11). Thus the whole sequence converges: $\mathcal{U}_{t^*}(z_{\varphi(n)}) \rightharpoonup Y(t^*)$ in H^2. Therefore, we have

$$\mathcal{U}_t(z_{\varphi(n)}) \rightharpoonup Y(t) \quad \text{in } H^2 \quad \text{and} \quad \mathcal{U}_t(z_{\varphi(n)}) \to Y(t) \quad \text{in } H_0^1,\ \forall t \in (0,T) - N. \tag{12}$$

We deduce from the first step that

$$u\big[\mathcal{U}_t(z_{\varphi(n)})\big] \to u\big[Y(t)\big], \quad \forall t \in (0,T) - N.$$

Let $A := -\Delta + V$. We fix $t^* \in (0,T) - N$. For every n, we have

$$\mathcal{U}_{t^*}(z_{\varphi(n)}) = e^{-iAt^*} z_{\varphi(n)} + i \int_0^{t^*} e^{-iA(t^*-s)} u\big[\mathcal{U}_s(z_{\varphi(n)})\big] Q\,\mathcal{U}_s(z_{\varphi(n)})\,\mathrm{d}s.$$

Passing to the limit $[n \to +\infty]$ in H_0^1 in this equality, using the dominated convergence theorem and the continuity of $e^{-iAt}z$ with respect to z in H_0^1 norm, we get

$$Y\big(t^*\big) = e^{-iAt^*} z_\infty + i \int_0^{t^*} e^{-iA(t^*-s)} u\big[Y(s)\big] Q\,Y(s)\,\mathrm{d}s.$$

Thus, $Y(t) = \mathcal{U}_t(z_\infty)$, $\forall t \in (0,T) - N$ (uniqueness of the solution of the closed loop system).

This proves that the sequence $(t \in [0,T] \mapsto \mathcal{U}_t(z_n))_{n\in\mathbb{N}}$ has a unique adherence value in $L^2((0,T), H_0^1)$. Therefore the whole sequence converges (i.e. one may take $\varphi = \mathrm{Id}$). We deduce from (12) that

$$\mathcal{U}_t(z_n) \rightharpoonup \mathcal{U}_t(z_\infty) \quad \text{in } H^2 \quad \text{and} \quad \mathcal{U}_t(z_n) \to \mathcal{U}_t(z_\infty) \quad \text{in } H_0^1,\ \forall t \in (0,T) - N. \qquad \square$$

Remark 2.3. The key point of this proof is that the feedback law $u(z)$ is well defined for z strictly less regular than H^2 (see (10): formally, $z \in H^{3/2}$ is sufficient).

Proof of Theorem 1.4. Let J be as in Proposition 2.1 and $\alpha \notin J$. Let $z_0 \in \mathcal{S} \cap H_0^1 \cap H^2$ with $0 < \mathcal{V}(z_0) < 1$. Let us prove that the weak H^2 ω-limit set of $\{\mathcal{U}_t(z_0); t \geqslant 0\}$ is contained in $\mathcal{C}$.

Let $z_\infty \in H_0^1 \cap H^2$ and $t_n \to +\infty$ be such that $\mathcal{U}_{t_n}(z_0) \rightharpoonup z_\infty$ in H^2. Let us show that $z_\infty \in \mathcal{C}$. One may assume that $\mathcal{U}_{t_n}(z_0) \to z_\infty$ in H_0^1.

There exists an extraction φ and a subset $N_1 \subset (0,+\infty)$ with zero Lebesgue measure such that

$$u[\mathcal{U}_{t_{\varphi(n)}+t}] \to 0, \quad \forall t \in (0,+\infty) - N_1.$$

Indeed, the sequence of functions $(t \in (0,+\infty) \mapsto u[\mathcal{U}_{t_n+t}(z_0)])_{n\in\mathbb{N}}$ tends to zero in $L^2(0,+\infty)$ because $t \mapsto u[\mathcal{U}_t(z_0)]$ belongs to $L^2(0,+\infty)$ (see (8)).

Let $T \in [0,+\infty)$. Thanks to Proposition 2.2, there exists $N \subset (0,T)$ with zero Lebesgue measure such that

$$u\big[\mathcal{U}_{t_{\varphi(n)}+t}(z_0)\big] \to u\big[\mathcal{U}_t(z_\infty)\big], \quad \forall t \in (0,T) - N.$$

The uniqueness of the limit ensures that $u[\mathcal{U}_t(z_\infty)] = 0$, $\forall t \in (0,T) - [N \cup N_1]$.

Finally, the function $t \mapsto u[\mathcal{U}_t(z_\infty)]$ is continuous and vanishes on $(0,T) - [N \cup N_1]$, thus it vanishes on $[0,T]$. This holds for every $T > 0$, thus $u[\mathcal{U}_t(z_\infty)] = 0$, $\forall t \in [0,+\infty)$. As $\mathcal{V}(z_\infty) \leqslant \mathcal{V}(z_0) < 1$, we have $\langle z_\infty, e_{1,V}\rangle \neq 0$. Thanks to Proposition 2.1, we get $z_\infty \in \mathcal{C}$. $\square$

3. Conclusion, open problems, perspectives

We have proposed explicit feedback laws, that asymptotically stabilize the ground state, for the system (1)–(3). To design the feedback laws, we have used control Lyapunov functions. The convergence holds semi-globally in H^2 and for the weak H^2-topology. The proof relies on an adaptation of the LaSalle invariance principle to PDEs.

Generalizations with different regularities are possible: with Lyapunov functions inspired by the H^s distance to the target, one may prove weak H^s stabilization. Generalization to other bilinear equations (for instance wave equations) is possible.

Our proof uses compact injections between Sobolev spaces on a bounded domain. Thus, the stabilization question when such compact injections cannot be used is still an open problem.

Another open problem concerns the simultaneous stabilization of N identical Schrödinger equations, around N different eigenstates, with only one closed loop control. Indeed, if we design feedback laws in the same way as in this article, then, the LaSalle invariance set does not coincide with the target. Thus, new ideas need to be introduced to tackle this problem.

The same question for non-linear Schrödinger equations is also an open problem.

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