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PRACTICAL STABILIZATION OF A QUANTUM PARTICLE IN A ONE-DIMENSIONAL INFINITE SQUARE POTENTIAL WELL*

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Abstract. We consider a nonrelativistic charged particle in a one-dimensional infinite square potential well. This quantum system is subjected to a control, which is a uniform (in space) time-depending electric field. It is represented by a complex probability amplitude solution of a Schrödinger equation on a one-dimensional bounded domain, with Dirichlet boundary conditions. We prove the almost global practical stabilization of the eigenstates by explicit feedback laws.

Key words. control of partial differential equations, bilinear Schrödinger equation, quantum systems, Lyapunov stabilization

AMS subject classifications. 93C20, 35Q40, 93D15

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1. Introduction.

1.1. Main result. As in [23, 5, 6], we consider a nonrelativist charged particle in a one-dimensional space, with a potential $V(x)$, in a uniform electric field $t \mapsto u(t) \in \mathbb{R}$. Under the dipole moment approximation assumption, and after appropriate changes of scales, the evolution of the particle's wave function is given by the following Schrödinger equation:

$$i\frac{\partial \Psi}{\partial t}(t,x) = -\frac{1}{2}\frac{\partial^2 \Psi}{\partial x^2}(t,x) + (V(x) - u(t)x)\Psi(t,x).$$

We study the case of an infinite square potential well: $V(x) = 0$ for $x \in I := (-1/2, 1/2)$ and $V(x) = +\infty$ for x outside I. Therefore our system is

$$i\frac{\partial \Psi}{\partial t}(t,x) = -\frac{1}{2}\frac{\partial^2 \Psi}{\partial x^2}(t,x) - u(t)x\Psi(t,x), \qquad x \in I, \tag{1.1}$$

$$\Psi(0,x) = \Psi_0(x), \tag{1.2}$$

$$\Psi(t, \pm 1/2) = 0. \tag{1.3}$$

It is a nonlinear control system, denoted by (Σ), in which

- the control is the electric field $u : \mathbb{R}_+ \to \mathbb{R}$;
- the state is the wave function $\Psi : \mathbb{R}_+ \times I \to \mathbb{C}$ with $\Psi(t) \in \mathbb{S}$ for every $t \geqslant 0$,

where $\mathbb{S} := \{\varphi \in L^2(I;\mathbb{C}); \|\varphi\|_{L^2} = 1\}$.

Let us introduce the operator A defined by

$$D(A) := (H^2 \cap H^1_0)(I, \mathbb{C}), \quad A\varphi := -\frac{1}{2}\frac{d^2\varphi}{dx^2},$$

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and for $s \in \mathbb{R}$ the spaces

$$H^s_{(0)}(I,\mathbb{C}) := D(A^{s/2}).$$

The following proposition recalls classical existence and uniqueness results for the solutions of (1.1)–(1.3). For sake of completeness, a proof of this proposition is given in the appendix.

PROPOSITION 1.1. *Let $\Psi_0 \in \mathbb{S}$, $T > 0$, and $u \in C^0([0,T],\mathbb{R})$. There exists a unique weak solution of* (1.1)–(1.3)*, i.e., a function $\Psi \in C^0([0,T],\mathbb{S}) \cap C^1([0,T], H^{-2}_{(0)}(I,\mathbb{C}))$ such that*

$$\Psi(t) = e^{-iAt}\Psi_0 + i\int_0^t e^{-iA(t-s)}u(s)x\Psi(s)ds \text{ in } L^2(I,\mathbb{C}) \text{ for every } t \in [0,T]. \tag{1.4}$$

Then (1.1) *holds in $H^{-2}_{(0)}(I,\mathbb{C})$ for every $t \in [0,T]$.*

If, moreover, $\Psi_0 \in (H^2 \cap H^1_0)(I,\mathbb{C})$, then Ψ is a strong solution, i.e., $\Psi \in C^0([0,T],(H^2 \cap H^1_0)(I,\mathbb{C})) \cap C^1([0,T],L^2(I,\mathbb{C}))$, (1.1) *holds in $L^2(I,\mathbb{C})$ for every $t \in [0,T]$,* (1.2) *holds in $H^2 \cap H^1_0(I,\mathbb{C})$, and* (1.3) *holds for every $t \in [0,T]$.*

The weak (resp., strong) solutions are continuous with respect to initial conditions for the $C^0([0,T],L^2)$-topology (resp., for the $C^0([0,T],H^2 \cap H^1_0)$-topology.)

The symbol $\langle .,. \rangle$ denotes the usual Hermitian product of $L^2(I,\mathbb{C})$, i.e.,

$$\langle \varphi, \xi \rangle := \int_I \varphi(x)\overline{\xi(x)}dx.$$

For $\sigma \in \mathbb{R}$, we introduce the operator A_σ defined by

$$D(A_\sigma) := (H^2 \cap H^1_0)(I;\mathbb{C}), \quad A_\sigma\varphi := -\frac{1}{2}\frac{\partial^2\varphi}{dx^2} - \sigma x\varphi.$$

It is well known that there exists an orthonormal basis $(\phi_{k,\sigma})_{k\in\mathbb{N}^*}$ of $L^2(I,\mathbb{C})$ of eigenvectors of A_σ:

$$\phi_{k,\sigma} \in H^2 \cap H^1_0(I,\mathbb{C}), \quad A_\sigma\phi_{k,\sigma} = \lambda_{k,\sigma}\phi_{k,\sigma},$$

where $(\lambda_{k,\sigma})_{k\in\mathbb{N}^*}$ is a nondecreasing sequence of real numbers. For $s > 0$ and $\sigma \in \mathbb{R}$, we define

$$H^s_{(\sigma)}(I,\mathbb{C}) := D(A_\sigma^{s/2}),$$

equipped with the norm

$$\|\varphi\|_{H^s_{(\sigma)}} := \left(\sum_{k=1}^\infty \lambda_{k,\sigma}^s |\langle \varphi, \phi_{k,\sigma}\rangle|^2\right)^{1/2}.$$

For $k \in \mathbb{N}^*$ and $\sigma \in \mathbb{R}$, we define

$$\mathcal{C}_{k,\sigma} := \{\phi_{k,\sigma}e^{i\theta}; \theta \in [0,2\pi)\}.$$

In order to simplify the notation, we will write ϕ_k, λ_k, $\mathcal{C}_k$ instead of $\phi_{k,0}$, $\lambda_{k,0}$, $\mathcal{C}_{k,0}$. We have

$$\lambda_k = \frac{k^2\pi^2}{2}, \quad \phi_k = \begin{cases} \sqrt{2}\cos(k\pi x) \text{ when } k \text{ is odd,} \\ \sqrt{2}\sin(k\pi x) \text{ when } k \text{ is even.} \end{cases} \tag{1.5}$$

The goal of this paper is the study of the stabilization of the system (Σ) around the eigenstates $\phi_{k,\sigma}$. More precisely, for $k \in \mathbb{N}^*$ and $\sigma \in \mathbb{R}$ small, we state feedback laws $u = u_{k,\sigma}(\Psi)$ for which the solution of (1.1)–(1.3) with $u(t) = u_{k,\sigma}(\Psi(t))$ is such that

$$\limsup_{t\to+\infty} \operatorname{dist}_{L^2(I,\mathbb{C})}(\Psi(t), \mathcal{C}_{k,\sigma})$$

is arbitrarily small. We consider convergence toward the circle $\mathcal{C}_{k,\sigma}$ because the wave function Ψ is defined up to a phase factor. For simplicity's sake, we will only work with the ground state $\phi_{1,\sigma}$. However, the whole argument remain valid for the general case.

Note that even though the feedback stabilization of a quantum system necessitates more complicated models taking into account the measurement back action on the system (see, e.g., [14, 29, 19]), the kind of strategy considered in this paper can be helpful for the open-loop control of closed quantum systems. Indeed, one can apply the stabilization techniques for the Schrödinger equation in simulation and retrieve the control signal that will then be applied in open-loop on the real physical system. As will be shown in detail below, in the bibliographic overview, this kind of strategy has been widely used in the context of finite-dimensional quantum systems.

The main result of this article is the following.

Theorem 1.2. *Let $\Gamma > 0$, $s > 0$, $\epsilon > 0$, $\gamma \in (0,1)$. There exists $\sigma^{**} = \sigma^{**}(\Gamma, s) > 0$ such that, for every $\sigma \in (-\sigma^{**}, \sigma^{**})$, there exists a feedback law $v_{\sigma,\Gamma,s,\epsilon,\gamma}(\Psi)$ such that, for every $\Psi_0 \in \mathbb{S} \cap (H^2 \cap H^1_0 \cap H^s_{(\sigma)})(I, \mathbb{C})$ with*

$$\|\Psi_0\|_{H^s_{(\sigma)}} \leqslant \Gamma \text{ and } |\langle \Psi_0, \phi_{1,\sigma}\rangle| > \gamma,$$

the Cauchy problem (1.1)–(1.3) with $u(t) = \sigma + v_{\sigma,\Gamma,s,\epsilon,\gamma}(\Psi)$ has a unique strong solution; moreover, this solution satisfies

$$\limsup_{t\to+\infty} dist_{L^2}(\Psi(t), \mathcal{C}_{1,\sigma}) \leqslant \epsilon.$$

Remark 1. Theorem 1.2 provides *almost global practical stabilization*. In fact, as will be seen, the above feedback law may be found through a Lyapunov analysis which ensures the stability of the target; i.e., for any small $\epsilon_1 > 0$, there exists a $0 < \epsilon_2 < \epsilon_1$ such that, if we initialize the system in an ϵ_2-neighborhood, the solution does not get outside the ϵ_1-neighborhood.

Moreover, applying the theorem, any initial condition $\Psi_0 \in \mathbb{S}$ such that $\Psi_0 \in H^s(I,\mathbb{C})$ for some $s > 0$ and $\langle \Psi_0, \phi_{1,\sigma}\rangle \neq 0$ can be moved approximately to the circle $\mathcal{C}_{1,\sigma}$, thanks to an appropriate feedback law.

The stability and the approximate convergence lead to the practical stabilization.

For $\sigma \neq 0$, the feedback law will be given explicitly. For $\sigma = 0$, the feedback law will be given by an implicit formula. We will see that the assumption "$\Psi_0 \in H^s(I,\mathbb{C})$ for some $s > 0$" is not necessary for the result of the theorem. In fact, even for a Ψ_0 only belonging to $\mathbb{S}$, we can find the appropriate feedback law as a function of the initial state Ψ_0.

Notice that, physically, the assumption $\langle \Psi_0, \phi_{1,\sigma}\rangle \neq 0$ is not really restrictive. Indeed, if $\langle \Psi_0, \phi_{1,\sigma}\rangle = 0$, a control field in resonance with the natural frequencies of the system (the difference between the eigenvalues corresponding to an eigenstate whose population in the initial state is nonzero and the ground state) will, instantaneously, ensure a nonzero population of the ground state in the wavefunction. Then one can just apply the feedback law of Theorem 1.2.

1.2. A brief bibliography. The controllability of a finite-dimensional quantum system, $\iota \frac{d}{dt}\Psi = (H_0 + u(t)\ H_1)\Psi$, where $\Psi \in \mathbb{C}^N$ and H_0 and H_1 are $N \times N$ Hermitian matrices with coefficients in $\mathbb{C}$, has been very well explored [26, 22, 1, 2, 28]. However, this does not guarantee the simplicity of the trajectory generation. Very often the chemists formulate the task of the open-loop control as a cost functional to be minimized. Optimal control techniques (see, e.g., [24]) and iterative stochastic techniques (e.g., genetic algorithms [17]) are then two classes of approaches which are most commonly used for this task.

When some nondegeneracy assumptions concerning the linearized system are satisfied, [20] provides another method based on Lyapunov techniques for generating trajectories. The relevance of such a method for the control of chemical models has been studied in [21]. As mentioned above, the closed-loop system is simulated and the retrieved control signal is applied in open-loop. Such a strategy has already been applied widely in this framework [8, 25].

The situation is much more difficult when we consider an infinite-dimensional configuration, and fewer results are available. However, the controllability of the system (1.1)–(1.3) is now well understood. In [27], the author states some noncontrollability results for general Schrödinger systems. These results apply in particular to the system (1.1)–(1.3). However, this negative result is due to the choice of the functional space that does not allow controllability. Indeed, if we consider different functional spaces, one can get positive controllability results. In [5], the local controllability of the system (1.1)–(1.3) around the ground state $\phi_{1,\sigma}$ for σ small is proved. The case $\sigma \neq 0$ is easier because the linearized system around $\phi_{1,\sigma}$ for $\sigma \neq 0$ small is controllable; this case is treated with the moment theory and a Nash–Moser implicit functions theorem. As has been discussed in [23], the case of $\sigma = 0$ is degenerate: the linearized system around ϕ_1 is not controllable. Therefore, in this case, one needs to apply other tools, namely, the return method (introduced in [9]) and the quantum adiabatic theory [12]. In [6], the steady-state controllability of this nonlinear system is proved (i.e., the particle can be moved in finite time from an eigenstate ϕ_k to another one, ϕ_j). The proof relies on many local controllability results (proved with the previous strategies) together with a compactness argument.

Concerning the trajectory generation problem for infinite-dimensional systems, still much fewer results are available. What literature exists is mostly based on the use of the optimal control techniques [4, 3]. The simplicity of the feedback law found by the Lyapunov techniques in [20, 7] suggests the use of the same approach for infinite-dimensional configurations. However, an extension of the convergence analysis to the PDE configuration is not at all a trivial problem. Indeed, it requires the precompactness of the closed-loop trajectories, a property that is difficult to prove in infinite dimension. This strategy is used, for example, in [11].

In [18], one of the authors proposes a Lyapunov-based method for practical stabilization of a particle in an N-dimensional decaying potential under some restrictive assumptions. The author assumes that the system is initialized in the finite-dimensional discrete part of the spectrum. Then the idea consists in proposing a Lyapunov function which encodes both the distance with respect to the target state and the necessity of remaining in the discrete part of the spectrum. In this way, he prevents the possibility of the "mass lost phenomenon" through dispersion at infinity. Applying some dispersive estimates of Strichartz type, he ensures the practical stabilization of an arbitrary eigenstate in the discrete part of the spectrum.

Finally, let us mention that there exists a huge literature on the other strategies for proving the stabilization of infinite-dimensional control systems. We refer to [10] for a rather complete list of references on these techniques.

In this paper, we study the stabilization of the ground state $\phi_{1,\sigma}$ for σ in a neighborhood of 0. Adapting the techniques proposed in [18], we ensure the practical stabilization of the system around $\phi_{1,\sigma}$. Note that the whole argument holds if we replace the target state by any eigenstate $\phi_{k,\sigma}$ of the system.

1.3. Heuristic of the proof. To stabilize the ground state $\phi_{1,\sigma}$, a first approach would be to consider the simple Lyapunov function

$$\widetilde{\mathcal{V}}(\Psi) = 1 - |\langle \Psi \ , \ \phi_{1,\sigma}\rangle|^2.$$

Just as in the finite-dimensional case [7], the feedback law

$$\tilde{u}(\Psi) = \Im(\langle x\Psi \ , \ \phi_{1,\sigma}\rangle \langle \phi_{1,\sigma} \ , \ \Psi\rangle),$$

where $\Im$ denotes the imaginary part of a complex, ensures the decrease of the Lyapunov function. However, trying to adapt the convergence analysis, based on the use of the LaSalle invariance principle, the precompactness of the trajectories in L^2 constitutes a major obstacle. Note that in order to be able to apply the LaSalle principle to an infinite-dimensional system, one certainly needs to prove such a precompactness result. For the particular case of the infinite potential well considered here, the efforts of the authors, applying the classical functional analysis techniques, have failed to prove the precompactness of the closed-loop system applying the above feedback. In fact, as the system evolves on the unit sphere of L^2, the compactness of the trajectories in weaker spaces is ensured. However, we have not been able to strengthen this weak compactness to a strong one. Indeed, it even seems that phenomena such as the loss of L^2-mass in the high energy levels do not allow this property to hold true.

Similarly to [18], the approach of this paper is to prevent the population from going through the very high energy levels, while trying to stabilize the system around $\phi_{1,\sigma}$.

As in Theorem 1.2, let us consider $\Gamma > 0$, $s > 0$, $\epsilon > 0$, $\gamma > 0$, $\sigma \in \mathbb{R}$. First, we consider the case $\sigma \neq 0$. Let $\Psi_0 \in H^s_{(0)}(I, \mathbb{C})$ with

$$\|\Psi_0\|_{H^s_{(0)}} \leqslant \Gamma \text{ and } |\langle \Psi_0, \phi_{1,\sigma}\rangle| \geqslant \gamma.$$

We claim that there exists $N = N(\Gamma, s, \epsilon, \gamma) \in \mathbb{N}^*$ large enough so that

$$\sum_{k=N+1}^{\infty} |\langle \Psi_0, \phi_{k,\sigma}\rangle|^2 < \frac{\epsilon\gamma^2}{1-\epsilon}. \tag{1.6}$$

Then we consider the Lyapunov function

$$\mathcal{V}(\Psi) = 1 - |\langle \Psi \ , \ \phi_{1,\sigma}\rangle|^2 - (1-\epsilon)\sum_{k=2}^{N} |\langle \Psi \ , \ \phi_{k,\sigma}\rangle|^2. \tag{1.7}$$

Note that this Lyapunov function depends on the constants Γ, s, ϵ, γ through the choice of the cut-off dimension, N. Just like [18], it encodes two tasks: (1) it prevents the loss of L^2-mass through the high-energy eigenstates; and (2) it privileges the increase of the population in the first eigenstate.

When Ψ solves (Σ) with some control $u = \sigma + v$, we have

$$\frac{d\mathcal{V}}{dt} = -2v(t)\Im\Big(\sum_{k=1}^{N} a_k \left\langle x\Psi \ , \ \phi_{k,\sigma}\right\rangle \left\langle \phi_{k,\sigma} \ , \ \Psi \right\rangle \Big),$$

where

$$a_1 := 1 \ \text{ and } \ a_k := 1-\epsilon \text{ for } k = 2, \dots, N. \tag{1.8}$$

Thus, the feedback law

$$v(\Psi) := \varsigma\Im\Big(\sum_{k=1}^{N} a_k \left\langle x\Psi \ , \ \phi_{k,\sigma}\right\rangle \left\langle \phi_{k,\sigma} \ , \ \Psi \right\rangle \Big), \tag{1.9}$$

where $\varsigma > 0$ is a positive constant, trivially ensures the decrease of the Lyapunov function (1.7). We claim that the solution of (1.1)–(1.3) with initial condition Ψ_0 and control $u = \sigma + v(\Psi)$ satisfies

$$\limsup_{t\to+\infty} \operatorname{dist}_{L^2}(\Psi(t), \mathcal{C}_{1,\sigma})^2 \leqslant \epsilon. \tag{1.10}$$

Note that the claimed result here is much stronger than the one provided in [18] for the finite potential well problem. In fact, here we claim the almost global practical stabilization of the system round the eigenstate $\phi_{1,\sigma}$.

The limit (1.10) will be proved by studying the $L^2(I,\mathbb{C})$-weak limits of $\Psi(t)$ when $t \to +\infty$. Namely, let $(t_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive real numbers such that $t_n \to +\infty$. Since $\|\Psi(t_n)\|_{L^2(I,\mathbb{C})} \equiv 1$, there exists $\Psi_\infty \in L^2(I,\mathbb{C})$ such that, up to a subsequence, $\Psi(t_n) \to \Psi_\infty$ weakly in $L^2(I,\mathbb{C})$. Using the controllability of the linearized system around $\phi_{1,\sigma}$ (which is equivalent to $\langle \phi_{1,\sigma}, x\phi_{k,\sigma}\rangle \neq 0$ for every $k \in \mathbb{N}^*$), we will be able to prove that $\Psi_\infty = \beta\phi_{1,\sigma}$, where $\beta \in \mathbb{C}$ and $|\beta|^2 \geqslant 1-\epsilon$. This will imply (1.10).

Therefore, by weakening the stabilization property (i.e., looking for practical stabilization instead of stabilization) we avoid the compactness problem evoked at the beginning of this section.

Note that the controllability of the linearized system around the trajectory $\phi_{1,\sigma}$ plays a crucial role here. This is why the developed techniques for $\sigma \neq 0$ cannot be applied, directly, to the case of $\sigma = 0$.

Now, let us study the case $\sigma = 0$. As emphasized above, the previous strategy does not work for the practical stabilization of ϕ_1 because the linearized system around ϕ_1 is not controllable. The idea is thus to use the above feedback design (1.9) with a dynamic $\sigma = \sigma(t)$ that converges to zero as $t \to +\infty$. Formally, the convergence of Ψ toward $\mathcal{C}_{1,\sigma(t)}$ must happen at a faster rate than that of σ toward zero (see Figure 1.1).

In this aim, we consider the Lyapunov function

$$\mathcal{V}(\Psi) = 1 - (1-\epsilon)\sum_{k=1}^{N} |\left\langle \Psi \ , \ \phi_{k,\sigma(\Psi)}\right\rangle|^2 - \epsilon|\left\langle \Psi \ , \ \phi_{1,\sigma(\Psi)}\right\rangle|^2, \tag{1.11}$$

where the function $\Psi \mapsto \sigma(\Psi)$ is implicitly defined as

$$\sigma(\Psi) = \theta\left(\mathcal{V}(\Psi)\right) \tag{1.12}$$

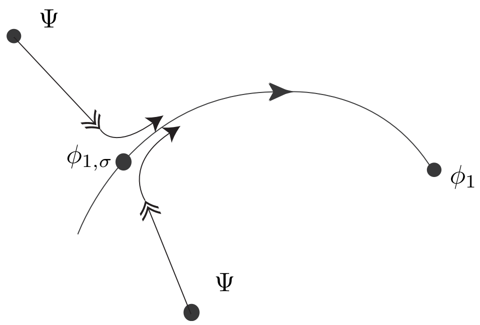

FIG. 1.1.

for a slowly varying real function θ. We claim that such a function $\sigma(\Psi)$ exists. When Ψ solves (Σ), we have

$$\begin{aligned}\frac{d\mathcal{V}}{dt} &= -2v(\Psi)\Im\Big(\sum_{k=1}^{N} a_k \left\langle x\Psi \ , \ \phi_{k,\sigma(\Psi)}\right\rangle \left\langle \phi_{k,\sigma(\Psi)} \ , \ \Psi\right\rangle\Big) \\ &\quad -\frac{d\sigma(\Psi)}{dt}2\Re\Big(\sum_{k=1}^{N} a_k \langle\Psi, \phi_{k,\sigma(\Psi)}\rangle\Big\langle\frac{d\phi_{k,\sigma(\Psi)}}{d\sigma}, \Psi\Big\rangle\Big),\end{aligned}$$

where $\Re$ denotes the real part of a complex number, $(a_k)_{1\leqslant k\leqslant N}$ is defined by (1.8), and the notation $\frac{d\phi_{k,\sigma(\Psi)}}{d\sigma}$ means the derivative of the map $\sigma \mapsto \phi_{k,\sigma}$ taken at the point $\sigma = \sigma(\Psi)$. By definition of $\sigma(\Psi)$, we have

$$\frac{d\sigma(\Psi)}{dt} = \theta'(\mathcal{V}(\Psi))\frac{d\mathcal{V}}{dt}.$$

Thus, the feedback law $u(\Psi) := \sigma(\Psi) + v(\Psi)$, where

$$v(\Psi) := \varsigma\Im\Big(\sum_{k=1}^{N} a_k \left\langle x\Psi \ , \ \phi_{k,\sigma(\Psi)}\right\rangle \left\langle \phi_{k,\sigma(\Psi)} \ , \ \Psi\right\rangle\Big)$$

with $\varsigma > 0$, ensures

$$\frac{d\mathcal{V}}{dt} = -2\varsigma\mu v(\Psi)^2,$$

where

$$\frac{1}{\mu} = 1 + 2\theta'(\mathcal{V}(\Psi))\Re\Big(\sum_{k=1}^{N} a_k \langle\Psi, \phi_{k,\sigma(\Psi)}\rangle\Big\langle\frac{d\phi_{k,\sigma(\Psi)}}{d\sigma}, \Psi\Big\rangle\Big)$$

is a positive constant, when $\|\theta'\|_{L^\infty}$ is small enough. Thus $t \mapsto \mathcal{V}(\Psi(t))$ is not increasing.

We claim that the solution of (1.1)–(1.3) with initial condition Ψ_0 and control $u = \sigma(\Psi) + v(\Psi)$ satisfies

$$\limsup_{t\to+\infty} \mathrm{dist}_{L^2}(\Psi(t), \mathcal{C}_1)^2 \leqslant \epsilon. \tag{1.13}$$

Again, this will be proved by studying the $L^2(I, \mathbb{C})$-weak limits of $\Psi(t)$ when $t \to +\infty$.

1.4. Structure of the article. The rest of the paper is organized as follows.

Section 2 is dedicated to the proof of Theorem 1.2 when $\sigma \neq 0$. We derive this theorem as a consequence of a stronger result stated in Theorem 2.1.

This theorem and a straightforward corollary (Corollary 2.2), leading to Theorem 1.2 in the case $\sigma \neq 0$, will be stated in subsection 2.1. Subsection 2.2 is dedicated to some preliminary study needed for the proofs of Theorem 2.1 and Corollary 2.2. The proofs will be detailed in subsection 2.3.

Section 3 is devoted to the proof of Theorem 1.2 in the case $\sigma = 0$. Again, this theorem will be derived as a consequence of a stronger result stated in Theorem 3.2.

In subsection 3.1, we state a proposition (Proposition 3.1) ensuring the existence of the implicit function $\sigma = \sigma(\Psi)$. Then we state Theorem 3.2 and a straightforward corollary (Corollary 3.3), leading to Theorem 1.2 in the case $\sigma = 0$. A preliminary study, in preparation of the proofs of Theorem 3.2 and Corollary 3.3, will be performed in subsection 3.2. The proofs will be detailed in subsection 3.3.

Finally, in section 4, we provide some numerical simulations to check out the performance of the control design on a rather hard test case.

2. Stabilization of $\mathcal{C}_{1,\sigma}$ with $\sigma \neq 0$.

2.1. Main result. The main result of section 2 is the following theorem.

THEOREM 2.1. *Let $N \in \mathbb{N}^*$. There exists $\sigma^\sharp = \sigma^\sharp(N) > 0$ such that, for every $\sigma \in (-\sigma^\sharp, \sigma^\sharp) - \{0\}$, $\gamma \in (0,1)$, $\epsilon > 0$, and $\Psi_0 \in \mathbb{S}$ verifying*

$$\sum_{k=N+1}^{\infty} |\langle \Psi_0, \phi_{k,\sigma}\rangle|^2 < \frac{\epsilon\gamma^2}{1-\epsilon} \quad \textit{and} \quad |\langle \Psi_0, \phi_{1,\sigma}\rangle| \geqslant \gamma, \tag{2.1}$$

the Cauchy problem (1.1)–(1.3) *with* $u(t) = \sigma + v_{\sigma,N,\epsilon}(\Psi(t))$,

$$v_{\sigma,N,\epsilon}(\Psi) := -\Im\left((1-\epsilon)\sum_{k=1}^{N}\langle x\Psi, \phi_{k,\sigma}\rangle\overline{\langle \Psi, \phi_{k,\sigma}\rangle} + \epsilon\langle x\Psi, \phi_{1,\sigma}\rangle\overline{\langle\Psi, \phi_{1,\sigma}\rangle}\right), \tag{2.2}$$

has a unique weak solution Ψ. Moreover, this solution satisfies

$$\liminf_{t\to+\infty} |\langle \Psi(t), \phi_{1,\sigma}\rangle|^2 \geqslant 1-\epsilon. \tag{2.3}$$

Theorem 2.1 provides an almost global practical stabilization. Indeed, any initial condition $\Psi_0 \in \mathbb{S}$ such that $\langle \Psi_0, \phi_{1,\sigma}\rangle \neq 0$ can be approximately moved to $\mathcal{C}_{1,\sigma}$. Notice that the regularity assumption $\Psi_0 \in H^s_{(\sigma)}(I, \mathbb{C})$ stated in Theorem 1.2 is not necessary for this purpose. Indeed, the feedback law depends on the initial state through the choice of the cut-off dimension N.

The following corollary states that the quantity N appearing in the feedback law may be uniform when Ψ_0 is in a given bounded subset of $H^s_{(\sigma)}(I,\mathbb{C})$.

COROLLARY 2.2. *Let $s > 0$, $\epsilon > 0$, $\Gamma > 0$, and $\gamma \in (0,1)$. There exist $\sigma^{**} = \sigma^{**}(\Gamma, s, \epsilon, \gamma) > 0$ and $N = N(\Gamma, s, \epsilon, \gamma) \in \mathbb{N}^*$ such that, for every $\sigma \in (-\sigma^{**}, \sigma^{**}) - \{0\}$, and $\Psi_0 \in H^s_{(\sigma)}(I,\mathbb{C}) \cap \mathbb{S}$ verifying*

$$\|\Psi_0\|_{H^s_{(\sigma)}} \leqslant \Gamma \ \textit{and}\ |\langle \Psi_0, \phi_{1,\sigma}\rangle| \geqslant \gamma, \tag{2.4}$$

the Cauchy problem (1.1)–(1.3) *with $u = \sigma + v_{\sigma,N,\epsilon}(\Psi)$ has a unique weak solution Ψ. Moreover, this solution satisfies* (2.3).

Remark 2. Theorem 1.2 in the case $\sigma \neq 0$ is a direct consequence of the previous corollary. The feedback law mentioned in Theorem 1.2 is explicitly given via Corollary 2.2 and Theorem 2.1.

Notice that, in the particular case $\sigma \neq 0$, Corollary 2.2 is slightly more general than Theorem 1.2. In fact, the assumption "$\Psi_0 \in H^2 \cap H_0^1(I,\mathbb{C})$" is not needed as we deal with weak solutions instead of strong ones. Trivially, this solution will be a strong solution for $\Psi_0 \in H^2 \cap H_0^1(I,\mathbb{C})$.

For $\sigma = 0$, this will no longer be the case: we will need solutions in $C^1(\mathbb{R}, L^2)$ (for which the assumption $\Psi_0 \in H^2 \cap H_0^1(I,\mathbb{C})$ is needed; see Proposition 1.1).

2.2. Preliminaries. This section is devoted to the preliminary results that will be applied in the proof of Theorem 2.1.

2.2.1. Eigenvalues and eigenvectors of A_σ.

PROPOSITION 2.3. *For every $k \in \mathbb{N}^*$, the eigenvalue $\sigma \mapsto \lambda_{k,\sigma} \in \mathbb{R}$ and the eigenstate $\sigma \mapsto \phi_{k,\sigma} \in (H^2 \cap H_0^1)(I,\mathbb{C})$ are analytic functions of $\sigma \in \mathbb{R}$ around $\sigma = 0$, and the expansion $\lambda_{k,\sigma} = \lambda_k + \sigma^2 \lambda_k^{(2)} + o(\sigma^2)$ holds with*

$$\lambda_k^{(2)} = \frac{1}{24\pi^2 k^2} - \frac{5}{8\pi^4 k^4}. \tag{2.5}$$

There exist $\sigma^* > 0$, $C^* > 0$ such that, for every $\sigma_0, \sigma_1 \in (-\sigma^*, \sigma^*) - \{0\}$, for every $k \in \mathbb{N}^*$,

$$\langle x\phi_{1,\sigma_0}, \phi_{k,\sigma_0} \rangle \neq 0, \tag{2.6}$$

$$|\lambda_{k,\sigma_0} - \lambda_k| \leqslant \frac{C^* \sigma^2}{k}, \tag{2.7}$$

$$\left\| \frac{d\phi_{k,\sigma_0}}{d\sigma} \right\|_{L^2} \leqslant \frac{C^*}{k}, \tag{2.8}$$

$$\left\| \frac{d\phi_{k,\sigma_0}}{d\sigma} \right\|_{H_0^1} \leqslant C^*, \tag{2.9}$$

$$\|\phi_{k,\sigma_0} - \phi_{k,\sigma_1}\|_{L^2} \leqslant \frac{C^* |\sigma_0 - \sigma_1|}{k}. \tag{2.10}$$

In the previous proposition, the notation $\frac{d\phi_{k,\sigma_0}}{d\sigma}$ means the derivative of the map $\sigma \mapsto \phi_{k\sigma}$ taken at the point $\sigma = \sigma_0$. In the same way, we will use the notation $\frac{d\lambda_{k,\sigma_0}}{d\sigma}$ for the derivative of the map $\sigma \mapsto \lambda_{k,\sigma}$ at $\sigma = \sigma_0$.

Proof of Proposition 2.3. We consider the family of self-adjoint operators $A_\sigma = A - \sigma x$ in the space $(H^2 \cap H_0^1)(I,\mathbb{C})$. In this Banach space, the operator x (as a multiplication operator) is relatively bounded with respect to A with relative bound 0 (in the sense of [15, p. 190]). Therefore A_σ is a self-adjoint holomorphic family of type (A) (see [15, p. 375]). Thus the eigenvalues and the eigenstates of A_σ are holomorphic functions of σ.

Thanks to the Rayleigh–Schrödinger perturbation theory, we compute the first terms of the expansions

$$\lambda_{k,\sigma} = \lambda_k + \sigma\lambda_k^{(1)} + \sigma^2\lambda_k^{(2)} + \cdots, \quad \phi_{k,\sigma} = \phi_k + \sigma\phi_k^{(1)} + \sigma^2\phi_k^{(2)} + \cdots.$$

Considering the first and second order terms of the equalities $A_\sigma \phi_{k,\sigma} = \lambda_{k,\sigma}\phi_{k,\sigma}$, $\|\phi_{k,\sigma}\|^2_{L^2} = 1$, we get

$$-\frac{1}{2}\frac{d^2}{dx^2}\phi_k^{(1)} - x\phi_k = \lambda_k\phi_k^{(1)} + \lambda_k^{(1)}\phi_k, \qquad \langle \phi_k^{(1)}, \phi_k\rangle = 0, \tag{2.11}$$

$$-\frac{1}{2}\frac{d^2}{dx^2}\phi_k^{(2)} - x\phi_k^{(1)} = \lambda_k\phi_k^{(2)} + \lambda_k^{(1)}\phi_k^{(1)} + \lambda_k^{(2)}\phi_k, \quad 2\Re\langle \phi_k^{(2)}, \phi_k\rangle + \|\phi_k^{(1)}\|^2_{L^2} = 0. \tag{2.12}$$

Taking the Hermitian product of the first equality of (2.11) with ϕ_k and applying the parity properties of ϕ_k, we get $\lambda_k^{(1)} = 0$. Considering the Hermitian product of the first equality of (2.11) with ϕ_j, we get

$$\phi_k^{(1)} = \sum_{j\in\mathbb{N}^*, P(j)\neq P(k)} \frac{\langle x\phi_j, \phi_k\rangle}{\lambda_j - \lambda_k}\phi_j, \tag{2.13}$$

where the sum is taken over $j \in \mathbb{N}^*$ such that the parity of j is different from the parity of k. Taking the Hermitian product of the first equality of (2.12) with ϕ_k, we get $\lambda_k^{(2)} = -\langle x\phi_k^{(1)}, \phi_k\rangle$. Using (2.13) and the explicit expression of $\langle x\phi_k, \phi_j\rangle$ computed thanks to (1.5), we get

$$\lambda_k^{(2)} = \frac{2^7}{\pi^6}\sum_{j\in\mathbb{N}^*, P(j)\neq P(k)} \frac{k^2 j^2}{(k^2 - j^2)^5}. \tag{2.14}$$

In order to simplify the above sum, we decompose the fraction

$$F(X) := \frac{X^2}{(X-q)^5(X+q)^5}$$

in the form

$$\begin{aligned} F(X) = {} & \frac{1}{2^5 q^3}\left(\frac{1}{(X-q)^5} - \frac{1}{(X+q)^5}\right) - \frac{1}{2^6 q^4}\left(\frac{1}{(X-q)^4} + \frac{1}{(X+q)^4}\right) \\ & - \frac{1}{2^7 q^5}\left(\frac{1}{(X-q)^3} - \frac{1}{(X+q)^3}\right) + \frac{5}{2^8 q^6}\left(\frac{1}{(X-q)^2} + \frac{1}{(X+q)^2}\right) \\ & - \frac{5}{2^8 q^7}\left(\frac{1}{(X-q)} - \frac{1}{(X+q)}\right). \end{aligned}$$

Inserting this relation in the sum (2.14) and simplifying, we find

$$\lambda_k^{(2)} = \frac{1}{\pi^6}\left(\frac{5}{2k^5}S_k^1 - \frac{5}{2k^4}S_k^2 + \frac{1}{k^3}S_k^3 + \frac{2}{k^2}S_k^4 - \frac{4}{k}S_k^5\right), \tag{2.15}$$

where

$$S_k^a := \sum_{j\in\mathbb{N}^*, P(j)\neq P(k)} \left(\frac{1}{(j-k)^a} + \frac{(-1)^a}{(j+k)^a}\right) \text{ for } a = 1, \dots, 5.$$

We apply now the following well-known relations for the Riemann ζ-function:

$$\zeta(2) = \sum_{j=1}^\infty \frac{1}{j^2} = \frac{\pi^2}{6} \qquad \text{and} \qquad \zeta(4) = \sum_{j=1}^\infty \frac{1}{j^4} = \frac{\pi^4}{90}.$$

These relations imply

$$\sum_{k=-\infty}^{\infty} \frac{1}{(2j+1)^2} = \frac{\pi^2}{4} \qquad \text{and} \qquad \sum_{k=-\infty}^{\infty} \frac{1}{(2j+1)^4} = \frac{\pi^4}{48};$$

thus

$$S_k^a = \begin{cases} \frac{1}{k^a} & \text{when } k \text{ is odd} \\ 0 & \text{when } k \text{ is even} \end{cases} \quad \text{for } a = 1, 3, 5,$$

$$S_k^2 = \begin{cases} \frac{\pi^2}{4} - \frac{1}{k^2} & \text{when } k \text{ is odd,} \\ \frac{\pi^2}{4} & \text{when } k \text{ is even,} \end{cases}$$

$$S_k^4 = \begin{cases} \frac{\pi^4}{48} - \frac{1}{k^4} & \text{when } k \text{ is odd,} \\ \frac{\pi^4}{48} & \text{when } k \text{ is even.} \end{cases}$$

Inserting this in (2.15), we get (2.5).

The relation (2.6) is proved in [5, Proposition 1]. The bound (2.7) is given in [15, Chapter 17, Example 2.14, Chapter 2, Problem 3.7]. Inequality (2.8) is proved in [5, Proposition 42]. The bound (2.9) is a consequence of (2.8). Indeed, considering the Hermitian product in $L^2(I, \mathbb{C})$ of $\frac{d\phi_{k,\sigma_0}}{d\sigma}$ with the equation

$$A_{\sigma_0} \frac{d\phi_{k,\sigma_0}}{d\sigma} - x\phi_{k,\sigma_0} = \lambda_{k,\sigma_0} \frac{d\phi_{k,\sigma_0}}{d\sigma} + \frac{d\lambda_{k,\sigma_0}}{d\sigma} \phi_{k,\sigma_0},$$

and using (2.8) together with the orthogonality between ϕ_{k,σ_0} and $\frac{d\phi_{k,\sigma_0}}{d\sigma}$ (which is a consequence of $\|\phi_{k,\sigma}\|_{L^2}^2 \equiv 1$), we get

$$\left\| \frac{d\phi_{k,\sigma_0}}{d\sigma} \right\|_{H_0^1}^2 \leqslant |\sigma_0| \left(\frac{C^*}{k} \right)^2 + \frac{C^*}{k} + \left(\frac{\pi^2 k^2}{2} + C^* \sigma_0^2 \right) \left(\frac{C^*}{k} \right)^2,$$

which gives (2.9). Finally, (2.10) is a consequence of (2.8). ∎

PROPOSITION 2.4. *Let $N \in \mathbb{N}^*$. There exists $\sigma^\sharp = \sigma^\sharp(N) > 0$ such that, for every $\sigma \in (-\sigma^\sharp, \sigma^\sharp) - \{0\}$, $j_2, k_2 \in \mathbb{N}^*$, and $j_1, k_1 \in \{1, \dots, N\}$, verifying $j_1 \neq j_2$ and $k_1 \neq k_2$,*

$$\lambda_{k_1,\sigma} - \lambda_{k_2,\sigma} = \lambda_{j_1,\sigma} - \lambda_{j_2,\sigma} \tag{2.16}$$

implies $(j_1, j_2) = (k_1, k_2)$.

Proof of Proposition 2.4. Let C^* be as in Proposition 2.3 and $\sigma \in (-\sigma_0^\sharp, \sigma_0^\sharp)$ where

$$\sigma_0^\sharp := \frac{\pi}{4\sqrt{C^*}}. \tag{2.17}$$

First, we prove (2.16) to be impossible when $j_2 \neq k_2$ and

$$\max\{j_2, k_2\} > \frac{N^2 + 1}{2}. \tag{2.18}$$

We argue by contradiction. Let us assume the existence of $j_2, k_2 \in \mathbb{N}^*$, $j_1, k_1 \in \{1, \dots, N\}$, with $j_1 \neq j_2$, $k_1 \neq k_2$, $j_2 \neq k_2$, such that (2.18) and (2.16) hold. Without loss of generality, we may assume that $\max\{j_2, k_2\} = j_2 > \frac{N^2+1}{2}$. Using (2.7), we get

$$\begin{aligned}\lambda_{j_2,\sigma} - \lambda_{k_2,\sigma} &\geqslant \frac{\pi^2}{2}(j_2^2 - k_2^2) - 2C^*\sigma^2 \\ &\geqslant \frac{\pi^2}{2}(j_2^2 - (j_2-1)^2) - 2C^*\sigma^2 \\ &\geqslant \frac{\pi^2}{2}(2j_2 - 1) - 2C^*\sigma^2, \\ \lambda_{j_1,\sigma} - \lambda_{k_1,\sigma} &\leqslant \frac{\pi^2}{2}(N^2 - 1) + 2C^*\sigma^2.\end{aligned}$$

Using the equality of the left-hand sides of these inequalities, together with (2.17), we get

$$j_2 \leqslant \frac{N^2}{2} + \frac{8C^*\sigma_0^{\sharp 2}}{\pi^2} \leqslant \frac{N^2+1}{2},$$

which is a contradiction.

Therefore, it is sufficient to prove Proposition 2.4 for $j_2, k_2 \in \{1, \dots, [(N^2 + 1)/2]\}$. Moreover, it is sufficient to prove that, for every $j_1, k_1 \in \{1, \dots, N\}$ and $j_2, k_2 \in \{1, \dots, [(N^2+1)/2]\}$, with $j_1 \neq j_2$, $k_1 \neq k_2$, $(j_1, j_2) \neq (k_1, k_2)$, there exists $\sigma^\sharp_{j_1,k_1,j_2,k_2} \in (0, \sigma_0^\sharp)$ such that, for every $\sigma \in (-\sigma^\sharp_{j_1,k_1,j_2,k_2}, \sigma^\sharp_{j_1,k_1,j_2,k_2})$, (2.16) does not hold. Indeed, then, the following choice of $\sigma^\sharp(N)$ concludes the proof of Proposition 2.4:

$$\begin{aligned}\sigma^\sharp(N) := \min\{\sigma^\sharp_{j_1,k_1,j_2,k_2};\quad & j_1, k_1 \in \{1, \dots, N\}, j_2, k_2 \in \{1, \dots, (N^2+1)/2\}, \\ & (j_1, j_2) \neq (k_1, k_2), j_1 \neq j_2, k_1 \neq k_2\}.\end{aligned}$$

Let $j_1, k_1 \in \{1, \dots, N\}$, $j_2, k_2 \in \{1, \dots, (N^2+1)/2\}$ be such that $j_1 \neq j_2$, $k_1 \neq k_2$, $(j_1, j_2) \neq (k_1, k_2)$. We argue by contradiction. Let us assume that, for every $\sigma_1^\sharp > 0$, there exists $\sigma \in (-\sigma_1^\sharp, \sigma_1^\sharp)$ such that (2.16) holds. Using the analyticity of both sides in (2.16) with respect to σ, at $\sigma = 0$, this assumption implies that

$$\lambda_{k_1}^{(2)} - \lambda_{k_2}^{(2)} = \lambda_{j_1}^{(2)} - \lambda_{j_2}^{(2)}.$$

Using (2.5) together with a rationality argument, we get

$$\frac{1}{k_1^2} - \frac{1}{k_2^2} = \frac{1}{j_1^2} - \frac{1}{j_2^2}, \quad \frac{1}{k_1^4} - \frac{1}{k_2^4} = \frac{1}{j_1^4} - \frac{1}{j_2^4}.$$

Since $k_1 \neq k_2$ and $j_1 \neq j_2$, we deduce from the previous equalities that

$$\frac{1}{k_1^2} - \frac{1}{k_2^2} = \frac{1}{j_1^2} - \frac{1}{j_2^2}, \quad \frac{1}{k_1^2} + \frac{1}{k_2^2} = \frac{1}{j_1^2} + \frac{1}{j_2^2}.$$

Therefore $k_1 = j_1$ and $k_2 = j_2$, which is a contradiction. $\square$

2.2.2. Solutions of the Cauchy problem.

PROPOSITION 2.5. *Let $\sigma \in \mathbb{R}$, $N \in \mathbb{N}^*$, $\epsilon > 0$. For every $\Psi_0 \in \mathbb{S}$, there exists a unique weak solution Ψ of* (1.1)–(1.3) *with $u(t) = \sigma + v_{\sigma,N,\epsilon}(\Psi(t))$, i.e., $\Psi \in C^0(\mathbb{R}, \mathbb{S}) \cap C^1(\mathbb{R}, H^{-2}_{(0)}(I, \mathbb{C}))$,* (1.1) *holds in $H^{-2}_{(0)}(I, \mathbb{C})$ for every $t \in \mathbb{R}$, and* (1.2) *holds in $\mathbb{S}$.*

Proof of Proposition 2.5. Let $\sigma \in \mathbb{R}$, $N \in \mathbb{N}$, $\epsilon > 0$, $\Psi_0 \in \mathbb{S}$, and $T > 0$ be such that

$$TNe^{NT} < 1. \tag{2.19}$$

In order to build solutions on $[0, T]$, we apply the Banach fixed point theorem to the map

$$\begin{array}{rccc} \Theta: & C^0([0,T],\mathbb{S}) & \rightarrow & C^0([0,T],\mathbb{S}) \\ & \xi & \mapsto & \Psi, \end{array}$$

where Ψ is the solution of (1.1)–(1.3) with $u(t) = \sigma + v_{\sigma,N,\epsilon}(\xi(t))$.

The map Θ is well defined and maps $C^0([0,T],\mathbb{S})$ into itself. Indeed, when $\xi \in C^0([0,T],\mathbb{S})$, $u : t \mapsto \sigma + v_{\sigma,N,\epsilon}(\xi(t))$ is continuous and thus Proposition 1.1 ensures the existence of a unique weak solution Ψ. Notice that the map Θ takes values in $C^0([0,T],\mathbb{S}) \cap C^1([0,T], H^{-2}_{(0)})$.

Let us prove that Θ is a contraction of $C^0([0,T],\mathbb{S})$. Let $\xi_j \in C^0([0,T],\mathbb{S})$, $v_j := v_{\sigma,N,\epsilon}(\xi_j)$, $\Psi_j := \Theta(\xi_j)$ for $j = 1, 2$ and $\Delta := \Psi_1 - \Psi_2$. We have

$$\Delta(t) = i \int_0^t e^{-iA_\sigma(t-s)}[v_1 x \Delta(s) + (v_1 - v_2) x \Psi_2(s)] ds.$$

Thanks to (2.2), we have $\|v_j\|_{L^\infty(0,T)} \leqslant N$ for $j = 1, 2$ and $\|v_1 - v_2\|_{L^\infty(0,T)} \leqslant 2N \|\xi_1 - \xi_2\|_{C^0([0,T],L^2)}$. Thus

$$\|\Delta(t)\|_{L^2} \leqslant \int_0^t N \|\Delta(s)\|_{L^2} + N \|\xi_1 - \xi_2\|_{C^0([0,T],L^2)} ds. \tag{2.20}$$

Therefore, the Gronwall lemma implies

$$\|\Delta(t)\|_{C^0([0,T],L^2)} \leqslant \|\xi_1 - \xi_2\|_{C^0([0,T],L^2)} NTe^{NT},$$

and so (2.19) ensures that Θ is a contraction of the Banach space $C^0([0,T],\mathbb{S})$. Therefore, there exists a fixed point $\Psi \in C^0([0,T],\mathbb{S})$ such that $\Theta(\Psi) = \Psi$. Since Θ takes values in $C^0([0,T],\mathbb{S}) \cap C^1([0,T], H^{-2}_{(0)})$, necessarily Ψ belongs to this space, and thus it is a weak solution of (1.1)–(1.3) on $[0,T]$.

Finally, we have introduced a time $T > 0$ and, for every $\Psi_0 \in \mathbb{S}$, we have built a weak solution $\Psi \in C^0([0,T],\mathbb{S})$ of (1.1)–(1.3) on $[0,T]$. Thus, for a given initial condition $\Psi_0 \in \mathbb{S}$, we can apply this result on $[0,T]$, $[T,2T]$, $[2T,3T]$, etc. This proves the existence and uniqueness of a global solution for the closed-loop system. ∎

PROPOSITION 2.6. *Let $\sigma > 0$, $N \in \mathbb{N}$, $\epsilon > 0$, $(\Psi_0^n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{S}$, and let $\Psi_0^\infty \in L^2$ with $\|\Psi_0^\infty\|_{L^2} \leqslant 1$ be such that*

$$\lim_{n\to+\infty} \Psi_0^n = \Psi_0^\infty \text{ strongly in } H^{-1}(I,\mathbb{C}).$$

Let Ψ^n (resp., Ψ^∞) be the weak solution of (1.1)–(1.3) with $u(t) = \sigma + v_{\sigma,N,\epsilon}(\Psi^n)$ (resp., with $u(t) = \sigma + v_{\sigma,N,\epsilon}(\Psi^\infty(t))$). Then, for every $\tau > 0$,

$$\lim_{n\to+\infty} \Psi^n(\tau) = \Psi^\infty(\tau) \text{ strongly in } H^{-1}(I,\mathbb{C}).$$

Proof of Proposition 2.6. Let us recall that the space $H^{-1}(I,\mathbb{C})$ (dual space of $H^1_0(I,\mathbb{C})$ for the $L^2(I,\mathbb{C})$-Hermitian product) coincides with $H^{-1}_{(0)}(I,\mathbb{C})$ and that $\sqrt{2}\|.\|_{H^{-1}} = \|.\|_{H^{-1}_{(0)}}$ (because $\|.\|_{H^1_0} = \sqrt{2}\|.\|_{H^1_{(0)}}$). We introduce $\mathcal{C} > 0$ such that

$$\|x\varphi\|_{H^{-1}} \leqslant \mathcal{C}\|\varphi\|_{H^{-1}} \quad \forall \varphi \in H^{-1}(I,\mathbb{C}). \tag{2.21}$$

Such a constant does exist. Indeed, for every $\xi \in H^1_0(I,\mathbb{C})$, $x\xi \in H^1_0(I,\mathbb{C})$, and

$$\|x\xi\|_{H^1_0} = \left(\int_I |x\xi' + \xi|^2 dx\right)^{1/2} \leqslant \|\xi'\|_{L^2}(1+C_P),$$

where C_P is the Poincaré constant on I. Thus, for $\varphi \in H^{-1}(I,\mathbb{C})$, we have

$$\begin{aligned}\|x\varphi\|_{H^{-1}(I,\mathbb{C})} &= \sup\left\{\langle x\varphi, \xi\rangle; \xi \in H^1_0(I,\mathbb{C}), \|\xi\|_{H^1_0} = 1\right\}\\ &\leqslant \sup\left\{\|\varphi\|_{H^{-1}}\|x\xi\|_{H^1_0}; \xi \in H^1_0(I,\mathbb{C}), \|\xi\|_{H^1_0} = 1\right\}\\ &\leqslant (1+C_P)\|\xi\|_{H^{-1}}.\end{aligned}$$

In order to simplify the notation, in this proof we write $v(\Psi)$ instead of $v_{\sigma,N,\epsilon}(\Psi)$. We have

$$\begin{aligned}(\Psi^n - \Psi^\infty)(t) &= e^{-iAt}(\Psi^n_0 - \Psi^\infty_0) + i\int_0^t e^{-iA(t-s)}\sigma x(\Psi^n - \Psi^\infty)(s)ds\\ &\quad + i\int_0^t e^{-iA(t-s)}[v(\Psi^n(s)) - v(\Psi^\infty(s))]x\Psi^n(s)ds\\ &\quad + i\int_0^t e^{-iA(t-s)}v(\Psi^\infty(s))x[\Psi^n(s) - \Psi^\infty(s)]ds.\end{aligned}$$

Using (2.2), $\|\Psi^n(s)\|_{L^2} = 1$, $\|\Psi^\infty(s)\|_{L^2} \leqslant 1$ and the fact that $\phi_{k,\sigma}, x\phi_{k,\sigma} \in H^1_0(I,\mathbb{C})$ for $k = 1,\dots,N$, we get

$$|v(\Psi^n(s)) - v(\Psi^\infty(s))| \leqslant 2N\mathcal{C}C_\sigma(N)\|(\Psi^n - \Psi^\infty)(s)\|_{H^{-1}}, \tag{2.22}$$

where $C_\sigma(N) := \sup\{\|\phi_{k,\sigma}\|_{H^1_0(I,\mathbb{C})}; k \in \{1,\dots,N\}\}$. The semigroup e^{-iAt} preserves the H^{-1}-norm, and thus, using $|v(\Psi^\infty(s))| \leqslant N$ and (2.22), we get

$$\begin{aligned}\|(\Psi^n - \Psi^\infty)(t)\|_{H^{-1}} &\leqslant \|\Psi^n_0 - \Psi^\infty_0\|_{H^{-1}}\\ &\quad + \mathcal{C}\textstyle\int_0^t (|\sigma| + 2NC_\sigma(N) + N)\|\Psi^n(s) - \Psi^\infty(s)\|_{H^{-1}}ds.\end{aligned}$$

We conclude thanks to the Gronwall lemma. ∎

2.3. Proofs of Theorem 2.1 and Corollary 2.2.

Proof of Theorem 2.1. Let $N \in \mathbb{N}^*$. Let $\sigma^* > 0$ be as in Proposition 2.3 and $\sigma^\sharp = \sigma^\sharp(N)$ be as in Proposition 2.4. Let $\sigma^{**} := \min\{\sigma^*, \sigma^\sharp\}$.

Let $\sigma \in (-\sigma^{**}, \sigma^{**}) - \{0\}$, $\gamma \in (0,1)$, $\epsilon > 0$, $\Psi_0 \in \mathbb{S}$ with (2.1) and let Ψ be the weak solution of (1.1)–(1.3) with $u(t) = \sigma + v_{\sigma,N,\epsilon}(\Psi(t))$ given by Proposition 2.5. For $\varphi \in L^2(I,\mathbb{C})$, we define

$$\mathcal{V}_{\sigma,N,\epsilon}(\varphi) := 1 - |\langle\varphi, \phi_{1,\sigma}\rangle|^2 - (1-\epsilon)\sum_{k=2}^N |\langle\varphi, \phi_{k,\sigma}\rangle|^2. \tag{2.23}$$

Since $\Psi \in C^1(\mathbb{R}, H^{-2}_{(0)}(I,\mathbb{C}))$ and $\phi_{k,\sigma} \in H^2_{(0)}(I,\mathbb{C})$, $t \mapsto \mathcal{V}_{N,\sigma,\epsilon}(\Psi(t))$ is C^1. Using (1.1), integration by parts, and $a_1 := 1$, $a_k := 1-\epsilon$ when $k \geqslant 2$, we get

$$\begin{aligned}\frac{d}{dt}\mathcal{V}_{\sigma,N,\epsilon}(\Psi) &= -2\Re\left(\sum_{k=1}^N a_k\langle -iA_\sigma\Psi + iv_{\sigma,N,\epsilon}(\Psi)x\Psi, \phi_{k,\sigma}\rangle\overline{\langle\Psi, \phi_{k,\sigma}\rangle}\right),\\ &= -2v_{\sigma,N,\epsilon}(\Psi(t))^2.\end{aligned} \tag{2.24}$$

Thus, $t \mapsto \mathcal{V}_{\sigma,N,\epsilon}(\Psi(t))$ is a nonincreasing function. There exists $\alpha \in [0, \mathcal{V}_{\sigma,N,\epsilon}(\Psi_0)]$ such that $\mathcal{V}_{\sigma,N,\epsilon}(\Psi(t)) \to \alpha$ when $t \to +\infty$. Since $\Psi_0 \in \mathbb{S}$ and (2.1) holds we have

$$\begin{aligned}
\mathcal{V}_{\sigma,N,\epsilon}(\Psi_0) &= 1 - (1-\epsilon)\sum_{k=1}^{N} |\langle \Psi, \phi_{k,\sigma}\rangle|^2 - \epsilon|\langle \Psi, \phi_{1,\sigma}\rangle|^2 \\
&= 1 - (1-\epsilon)\left(1 - \sum_{k=N+1}^{\infty} |\langle \Psi, \phi_{k,\sigma}\rangle|^2\right) - \epsilon|\langle \Psi, \phi_{1,\sigma}\rangle|^2 \\
&< 1 - (1-\epsilon)\left(1 - \frac{\epsilon\gamma^2}{1-\epsilon}\right) - \epsilon\gamma^2 \\
&< \epsilon,
\end{aligned}$$

and thus $\alpha \in [0, \epsilon)$.

Let $(t_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive real numbers such that $t_n \to +\infty$ when $n \to +\infty$. Since $\|\Psi(t_n)\|_{L^2} = 1$ for every $n \in \mathbb{N}$, there exists $\Psi_\infty \in L^2(I, \mathbb{C})$ such that, up to an extraction,

$$\Psi(t_n) \to \Psi_\infty \text{ weakly in } L^2(I,\mathbb{C}) \text{ and strongly in } H^{-1}(I,\mathbb{C}).$$

Let ξ be the solution of

$$\begin{cases} i\frac{\partial \xi}{\partial t} = A_\sigma \xi - v_{\sigma,N,\epsilon}(\xi(t))x\xi, x \in I, t \in (0,+\infty), \\ \xi(t, \pm 1/2) = 0, \\ \xi(0) = \Psi_\infty. \end{cases}$$

Thanks to Proposition 2.6, for every $\tau > 0$, $\Psi(t_n + \tau) \to \xi(\tau)$ strongly in $H^{-1}(I,\mathbb{C})$ when $n \to +\infty$. Thus $\mathcal{V}_{\sigma,N,\epsilon}(\Psi(t_n+\tau)) \to \mathcal{V}_{\sigma,N,\epsilon}(\xi(\tau))$ when $n \to +\infty$, because $\mathcal{V}_{\sigma,N,\epsilon}(.)$ is continuous for the L^2-weak topology. Therefore $\mathcal{V}_{\sigma,N,\epsilon}(\xi(\tau)) \equiv \alpha$. Furthermore, relation (2.24) holds when Ψ is replaced by ξ, and thus $v_{\sigma,N,\epsilon}(\xi(\tau)) \equiv 0$ and ξ solves

$$\begin{cases} i\frac{\partial \xi}{\partial t} = A_\sigma \xi, x \in I, t \in (0,+\infty), \\ \xi(t, \pm 1/2) = 0, \\ \xi(0) = \Psi_\infty. \end{cases}$$

Therefore, we have

$$\xi(\tau) = \sum_{k=1}^{\infty} \langle \Psi_\infty, \phi_{k,\sigma}\rangle \phi_{k,\sigma} e^{-i\lambda_{k,\sigma}\tau}.$$

The equality $v_{\sigma,N,\epsilon}(\xi) \equiv 0$, then, gives

(2.25)
$$\Im\left(\sum_{k=1}^{N} \sum_{j\in\mathbb{N}^*, j\neq k} a_k \langle \Psi_\infty, \phi_{j,\sigma}\rangle \langle x\phi_{j,\sigma}, \phi_{k,\sigma}\rangle \overline{\langle \Psi_\infty, \phi_{k,\sigma}\rangle} e^{i(\lambda_{k,\sigma}-\lambda_{j,\sigma})\tau}\right) \equiv 0.$$

Let $\omega_{(k_1,k_2)} := \lambda_{k_1,\sigma} - \lambda_{k_2,\sigma}$ for every $k_1, k_2 \in \mathbb{N}^*$ and $\mathcal{S} := \{(k_1,k_2); k_1 \in \{1,\dots,N\}, k_2 \in \mathbb{N}^*, k_1 \neq k_2\}$. Thanks to Proposition 2.4, all the frequencies ω_K for $K \in \mathcal{S}$ are different. Moreover, there exists a uniform gap $\delta > 0$ such that, for every $\omega, \tilde{\omega} \in \{\pm\omega_K; K \in \mathcal{S}\}$ with $\omega \neq \tilde{\omega}$, then $|\omega - \tilde{\omega}| \geqslant \delta$. Thus, for $T > 0$ large enough, there exists $C = C(T) > 0$ such that the Ingham inequality

$$\sum_{K\in\mathcal{S}} |a_K|^2 \leqslant C \int_0^T \Big| \sum_{K\in\mathcal{S}} a_K e^{i\omega_K t}\Big|^2 dt$$

holds for every $(a_K)_{K\in\mathcal{S}} \in l^2(\mathcal{S}, \mathbb{C})$ (see [16, Theorem 1.2.9]). Equality (2.25) implies, in particular,

$$\langle \Psi_\infty, \phi_{j,\sigma}\rangle \langle x\phi_{j,\sigma}, \phi_{1,\sigma}\rangle \overline{\langle \Psi_\infty, \phi_{1,\sigma}\rangle} = 0 \quad \forall j \geqslant 2.$$

Thanks to (2.6), we get

$$\langle \Psi_\infty, \phi_{j,\sigma}\rangle \overline{\langle \Psi_\infty, \phi_{1,\sigma}\rangle} = 0 \quad \forall j \geqslant 2. \tag{2.26}$$

Let us prove that

$$\langle \Psi_\infty, \phi_{1,\sigma}\rangle \neq 0. \tag{2.27}$$

Since $\|\Psi^\infty\|_{L^2} \leqslant 1$, we have

$$\begin{aligned}\mathcal{V}_{\sigma,N,\epsilon}(\Psi_\infty) &\geqslant 1 - |\langle \Psi^\infty, \phi_{1,\sigma}\rangle|^2 - (1-\epsilon)\sum_{k=2}^{\infty} |\langle \Psi^\infty, \phi_{k,\sigma}\rangle|^2 \\ &= 1 - |\langle \Psi^\infty, \phi_{1,\sigma}\rangle|^2 - (1-\epsilon)[\|\Psi^\infty\|_{L^2}^2 - |\langle \Psi^\infty, \phi_{1,\sigma}\rangle|^2] \\ &\geqslant \epsilon - \epsilon|\langle \Psi^\infty, \phi_{1,\sigma}\rangle|^2.\end{aligned}$$

Moreover, $\mathcal{V}_{\sigma,N,\epsilon}(\Psi_\infty) = \alpha < \epsilon$, and thus

$$\epsilon > \epsilon - \epsilon|\langle \Psi_\infty, \phi_{1,\sigma}|^2,$$

which gives (2.27). Therefore (2.26) justifies the existence of $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$ such that $\Psi_\infty = \beta\phi_{1,\sigma}$. Then $\epsilon > \alpha = \mathcal{V}_{N,\sigma,\epsilon}(\Psi_\infty) = 1 - |\beta|^2$, and thus $|\beta|^2 > 1 - \epsilon$. Finally, we have

$$\lim_{n\to+\infty} |\langle \Psi(t_n), \phi_{1,\sigma}\rangle|^2 = |\langle \Psi_\infty, \phi_{1,\sigma}\rangle|^2 = |\beta|^2 > 1 - \epsilon.$$

This holds for every sequence $(t_n)_{n\in\mathbb{N}}$, and thus (2.3) is proved. □

Proof of Corollary 2.2. Let $C^*, \sigma^* > 0$ be as in Proposition 2.3. There exists $N = N(\Gamma, s, \epsilon, \gamma) \in \mathbb{N}^*$ large enough so that

$$\frac{\Gamma^2}{\left(\lambda_{N+1} - \frac{C^*\sigma^{*2}}{N+1}\right)^s} \leqslant \frac{\epsilon\gamma^2}{1-\epsilon}. \tag{2.28}$$

Let $\sigma^{**} = \sigma^{**}(N)$ be as in Theorem 2.1 (notice that $\sigma^{**} \leqslant \sigma^*$) and $\sigma \in (-\sigma^{**}, \sigma^{**}) - \{0\}$. Let $\Psi_0 \in H^s_{(\sigma)}(I, \mathbb{C}) \cap \mathbb{S}$, verifying (2.4). In order to get the conclusion of Corollary 2.2, we prove that (2.1) holds, and we apply Theorem 2.1. Using (2.7), we get

$$\begin{aligned}\sum_{k=N+1}^{\infty} |\langle \Psi_0, \phi_{k,\sigma}\rangle|^2 &\leqslant \frac{1}{\lambda_{N+1,\sigma}^s} \sum_{k=N+1}^{\infty} \lambda_{k,\sigma}^s |\langle \Psi_0, \phi_{k,\sigma}\rangle|^2 \\ &\leqslant \frac{1}{\lambda_{N+1,\sigma}^s} \sum_{k=1}^{\infty} \lambda_{k,\sigma}^s |\langle \Psi_0, \phi_{k,\sigma}\rangle|^2 \\ &\leqslant \frac{\Gamma^2}{\left(\lambda_{N+1} - \frac{C^*\sigma^2}{N+1}\right)^s}.\end{aligned}$$

Thus (2.28) implies (2.1). ∎

3. Stabilization of $\mathcal{C}_1$.

Throughout this section, the constants C^*, σ^* are as in Proposition 2.3.

3.1. Main result.

First, let us state the existence of an implicit function $\sigma(\Psi)$ that will be used in the feedback law. When X is a normed space, $a \in X$ and $r > 0$, we use the notation $B_X(a,r) := \{y \in X; \|y-a\|_X < r\}$.

PROPOSITION 3.1. *Let $N \in \mathbb{N}^*$, $\epsilon > 0$, and $\theta \in C^\infty(\mathbb{R}_+, [0,\sigma^*])$ be such that*

$$\theta(0) = 0, \quad \theta(s) > 0 \quad \forall s > 0, \quad \|\theta'\|_{L^\infty} \leqslant \frac{1}{36NC^*}. \tag{3.1}$$

There exists a unique $\sigma \in C^\infty(B_{L^2}(0,2), [0, \|\theta\|_{L^\infty}])$ such that

$$\sigma(\psi) = \theta(\mathcal{V}_{\sigma(\psi),N,\epsilon}(\psi)) \qquad \forall \psi \in B_{L^2}(0,2),$$

where $\mathcal{V}_{\sigma,N,\epsilon}$ is defined by (2.23).

The proof of this proposition is done in [7]. For the sake of completeness, we repeat it in the appendix. The main result of this section is the following.

THEOREM 3.2. *Let $N \in \mathbb{N}^*$, $\gamma \in (0,1)$, $\epsilon > 0$, $\theta \in C^\infty(\mathbb{R}_+, [0,\sigma^*])$ verifying* (3.1),

$$\|\theta\|_{L^\infty} \leqslant \min\left\{ \frac{1}{C^*}\left(\frac{\epsilon\gamma^2 N}{32(1-\epsilon/2)}\right)^{1/2}, \frac{\gamma}{2C^*}, \sigma^\sharp(N), \frac{1}{C^*}(\sqrt{1-\epsilon/2} - \sqrt{1-\epsilon}) \right\}, \tag{3.2}$$

and

$$\|\theta'\|_{L^\infty} < \frac{1}{3(1+NC^*)}. \tag{3.3}$$

Let $\sigma \in C^\infty(B_{L^2}(0,2), [0, \|\theta\|_{L^\infty}])$ be as in Proposition 3.1. *For every $\Psi_0 \in \mathbb{S} \cap (H^2 \cap H_0^1)(I, \mathbb{C})$ with*

$$\sum_{k=N+1}^{\infty} |\langle \Psi_0, \phi_k\rangle|^2 < \frac{\epsilon\gamma^2}{32(1-\epsilon/2)} \quad \text{and} \quad |\langle \Psi_0, \phi_1\rangle| \geqslant \gamma, \tag{3.4}$$

the Cauchy problem (1.1)–(1.3) *with $u(t) = \sigma(\Psi(t)) + v_{\sigma(\psi(t)),N,\epsilon}(\Psi(t))$ has a unique strong solution ψ. Moreover this solution satisfies*

$$\liminf_{t\to+\infty} |\langle \Psi(t), \phi_1\rangle|^2 \geqslant 1 - \epsilon. \tag{3.5}$$

The following corollary states that the quantity N appearing in the feedback law may be uniform in a fixed bounded subset of H^s for $s > 0$.

COROLLARY 3.3. *Let $s > 0$, $\epsilon > 0$, $\Gamma > 0$, and $\gamma \in (0,1)$. There exists $N = N(\Gamma, s, \epsilon, \gamma) \in \mathbb{N}^*$ such that, for every $\Psi_0 \in \mathbb{S} \cap (H^2 \cap H_0^1)(I, \mathbb{C})$ with $\Psi_0 \in H^s_{(0)}(I, \mathbb{C})$,*

$$\|\Psi_0\|_{H^s_{(0)}} \leqslant \Gamma \text{ and } |\langle \Psi_0, \phi_1 \rangle| \geqslant \gamma, \tag{3.6}$$

the Cauchy problem (1.1)–(1.3) *with $u(t) = \sigma(\Psi(t)) + v_{\sigma(\psi(t)),N,\epsilon}(\Psi(t))$ has a unique strong solution Ψ. Moreover this solution satisfies* (3.5).

Remark 3. Theorem 1.2 with $\sigma = 0$ is a direct consequence of Corollary 3.3. The feedback law, evoked in Theorem 1.2, is implicitly given by Corollary 3.3.

3.2. Preliminaries.

LEMMA 3.4. *Let $N \in \mathbb{N}^*$, $\epsilon > 0$, and θ satisfy* (3.1). *There exist $C(N) > 0$ and $\tilde{C}(N) > 0$ such that, for all $\xi_1, \xi_2 \in B_{L^2}(0,1)$,*

$$|\sigma(\xi_1) - \sigma(\xi_2)| \leqslant 3N\|\theta'\|_{L^\infty}\|\xi_1 - \xi_2\|_{L^2}, \tag{3.7}$$

$$|\sigma(\xi_1) - \sigma(\xi_2)| \leqslant C(N)\|\theta'\|_{L^\infty}\|\xi_1 - \xi_2\|_{H^{-1}}, \tag{3.8}$$

$$|v_{\sigma(\xi_1),N,\epsilon}(\xi_1) - v_{\sigma(\xi_2),N,\epsilon}(\xi_2)| \leqslant N(1 + 3NC^*\|\theta'\|_{L^\infty})\|\xi_1 - \xi_2\|_{L^2}, \tag{3.9}$$

$$|v_{\sigma(\xi_1),N,\epsilon}(\xi_1) - v_{\sigma(\xi_2),N,\epsilon}(\xi_2)| \leqslant \tilde{C}(N)\|\xi_1 - \xi_2\|_{H^{-1}}. \tag{3.10}$$

Proof of Lemma 3.4. Since N and ϵ are fixed, in order to simplify the notation, we remove them from the subscripts of this proof. We have

$$|\sigma(\xi_1) - \sigma(\xi_2)| \leqslant \|\theta'\|_{L^\infty}|\mathcal{V}_{\sigma(\xi_1)}(\xi_1) - \mathcal{V}_{\sigma(\xi_2)}(\xi_2)|. \tag{3.11}$$

Using

$$\begin{aligned} |\langle \xi_1, \phi_{k,\sigma_1}\rangle|^2 - |\langle \xi_2, \phi_{k,\sigma_2}\rangle|^2 = {} & \langle \xi_1 - \xi_2, \phi_{k,\sigma_1}\rangle\overline{\langle \xi_1, \phi_{k,\sigma_1}\rangle} \\ & + \langle \xi_2, \phi_{k,\sigma_1}\rangle\overline{\langle \xi_1 - \xi_2, \phi_{k,\sigma_1}\rangle} \\ & + \langle \xi_2, \phi_{k,\sigma_1} - \phi_{k,\sigma_2}\rangle\overline{\langle \xi_2, \phi_{k,\sigma_1}\rangle} \\ & + \langle \xi_2, \phi_{k,\sigma_2}\rangle\overline{\langle \xi_2, \phi_{k,\sigma_1} - \phi_{k,\sigma_2}\rangle} \end{aligned} \tag{3.12}$$

and (2.10), we get

$$|\mathcal{V}_{\sigma(\xi_1)}(\xi_1) - \mathcal{V}_{\sigma(\xi_2)}(\xi_2)| \leqslant 2N\|\xi_1 - \xi_2\|_{L^2} + 2NC^*|\sigma(\xi_1) - \sigma(\xi_2)|,$$

$$|\mathcal{V}_{\sigma(\xi_1)}(\xi_1) - \mathcal{V}_{\sigma(\xi_2)}(\xi_2)| \leqslant 2NC_1(N)\|\xi_1 - \xi_2\|_{H^{-1}} + 2NC^*|\sigma(\xi_1) - \sigma(\xi_2)|,$$

where $C_1(N) := \max\{\|\varphi_{k,\sigma}\|_{H_0^1}; k \in \{1, \dots, N\}, \sigma \in [0, \sigma^*]\}$. Using the previous inequalities and (3.1), we get

$$\frac{17}{18}|\sigma(\xi_1) - \sigma(\xi_2)| \leqslant 2N\|\theta'\|_\infty\|\xi_1 - \xi_2\|_{L^2},$$

$$\frac{17}{18}|\sigma_1 - \sigma_2| \leqslant 2NC_1(N)\mathcal{C}\|\theta'\|_\infty\|\xi_1 - \xi_2\|_{H^{-1}},$$

which implies (3.7) and (3.8) with $C(N) = 3N\mathcal{C}C_1(N)$.

Let us write v_j instead of $v_{\sigma(\xi_j)}(\xi_j)$. Using, for the term

$$\langle x\xi_1, \phi_{j,\sigma(\xi_1)}\rangle\overline{\langle \xi_1, \phi_{j,\sigma(\xi_1)}\rangle} - \langle x\xi_2, \phi_{j,\sigma(\xi_2)}\rangle\overline{\langle \xi_2, \phi_{j,\sigma(\xi_2)}\rangle},$$

the same kind of decomposition as in (3.12), together with (2.10), we get

$$|v_1 - v_2| \leqslant N\|\xi_1 - \xi_2\|_{L^2} + NC^*|\sigma(\xi_1) - \sigma(\xi_2)|,$$
$$|v_1 - v_2| \leqslant 2N\mathcal{C}C_1(N)\|\xi_1 - \xi_2\|_{H^{-1}} + 2NC^*|\sigma(\xi_1) - \sigma(\xi_2)|,$$

where $\mathcal{C}$ is defined by (2.21). Thus, using (3.7) and (3.8), we get (3.9) and (3.10) with $\tilde{C}(N) := 2N[\mathcal{C}C_1(N) + C^*C(N)\|\theta'\|_\infty]$. ☐

Proposition 3.5. *Let $N \in \mathbb{N}^*$, $\epsilon > 0$, and θ verify* (3.1) *and* (3.3). *For every $\Psi_0 \in \mathbb{S}$ the Cauchy problem* (1.1)–(1.3) *with $u(t) = \sigma(\Psi(t)) + v_{\sigma(\psi(t)),N,\epsilon}(\Psi(t))$ has a unique weak solution, i.e., $\Psi \in C^0(\mathbb{R},\mathbb{S})\cap C^1((0,+\infty), H^{-2}_{(0)})$. If, moreover, $\Psi \in (H^2\cap H^1_0)(I,\mathbb{C})$, then Ψ is a strong solution, i.e., $\Psi \in C^0(\mathbb{R}, H^2\cap H^1_0)\cap C^1((0,+\infty), L^2)$.*

Proof of Proposition 3.5. The strategy is the same as in the proof of Proposition 2.5. Let $T > 0$ be such that

$$NTe^{(N+\|\theta\|_{L^\infty})T} < \frac{1}{2}.$$

Let $\Psi_0 \in \mathbb{S}$. In order to build solutions on $[0, T]$, we apply the Banach fixed-point theorem to the map

$$\begin{array}{cccc} \Theta: & C^0([0,T],\mathbb{S}) & \to & C^0([0,T],\mathbb{S}) \\ & \xi & \mapsto & \Psi, \end{array}$$

where Ψ is the weak solution of (1.1)–(1.3) with $u(t) = \sigma(\xi(t)) + v_{\sigma(\xi(t)),N,\epsilon}(\xi(t))$.

The map Θ is well defined and maps $C^0([0,T],\mathbb{S})$ into itself; moreover, it takes values in $C^0([0,T],\mathbb{S})\cap C^1((0,T), H^{-2}_{(0)})$ (see Proposition 1.1). Let us prove that Θ is a contraction of $C^0([0,T],\mathbb{S})$. Let $\xi_j \in C^0([0,T],\mathbb{S})$, $v_j := v_{\sigma(\xi_j),N,\epsilon}(\xi_j)$, $\Psi_j := \Theta(\xi_j)$ for $j = 1, 2$ and $\Delta := \Psi_1 - \Psi_2$. We have

$$\Delta(t) = i\int_0^t e^{-iA(t-s)}[(\sigma(\xi_1) + v_1)x\Delta(s) + (\sigma(\xi_1) - \sigma(\xi_2) + v_1 - v_2)x\Psi_2(s)]ds.$$

Using (3.7) and (3.9), we get

$$\begin{aligned}\|\Delta(t)\|_{L^2} \leqslant & \int_0^t \Big(\|\theta'\|_{L^\infty} + N\Big)\|\Delta(s)\|_{L^2}ds \\ & + \int_0^t \Big(3N\|\theta'\|_{L^\infty} + N[1 + 3NC^*\|\theta'\|_{L^\infty}]\Big)\|\xi_1 - \xi_2\|_{L^2}ds.\end{aligned}$$

Thus, the Gronwall lemma implies

$$\|\Delta\|_{C^0([0,T],L^2)} \leqslant \|\xi_1 - \xi_2\|_{C^0([0,T],L^2)}[1 + 3(1 + NC^*)\|\theta'\|_{L^\infty}]NTe^{T[N+\|\theta\|_{L^\infty}]}.$$

The choice of T and (3.3) ensures that Θ is a contraction of $C^0([0,T],\mathbb{S})$. Therefore, there exists a fixed point $\Psi \in C^0([0,T],\mathbb{S})$ such that $\Theta(\Psi) = \Psi$. Since Θ takes values in $C^0([0,T],\mathbb{S})\cap C^1([0,T], H^{-2}_{(0)})$, necessarily Ψ belongs to this space; thus, it is a weak solution of (1.1)–(1.3) on $[0, T]$.

If, moreover, $\Psi_0 \in (H^2 \cap H_0^1)(I, \mathcal{C})$, then the map Θ takes values in $C^0([0,T], H^2 \cap H_0^1) \cap C^1([0,T], L^2)$, and thus Ψ belongs to this space and it is a strong solution.

Since the time T does not depend on Ψ_0, the solution can be continued globally in time. We, therefore, have the existence of global solutions to the closed-loop system. ∎

Proposition 3.6. *Let $\sigma > 0$, $N \in \mathbb{N}$, $\epsilon > 0$, θ as in* (3.1)*, $(\Psi_0^n)_{n\in\mathbb{N}}$ a sequence of $\mathbb{S}$, and $\Psi_0^\infty \in L^2$ with $\|\Psi_0^\infty\|_{L^2} \leqslant 1$ such that*

$$\lim_{n\to+\infty} \Psi_0^n = \Psi_0^\infty \textit{ strongly in } H^{-1}(I,\mathbb{C}).$$

Let Ψ^n (resp., Ψ^∞) be the weak solution of (1.1)–(1.3) *with $u(t) = \sigma(\Psi^n(t)) + v_{\sigma(\Psi^n(t)),N,\epsilon}(\Psi^n(t))$ (resp., with $u(t) = \sigma(\Psi^\infty) + v_{\sigma(\Psi^\infty),N,\epsilon}(\Psi^\infty(t))$). Then, for every $\tau > 0$,*

$$\lim_{n\to+\infty} \Psi^n(\tau) = \Psi^\infty(\tau) \textit{ strongly in } H^{-1}(I,\mathbb{C}).$$

Proof of Proposition 3.6. The proof exactly follows that of Proposition 2.6. In order to simplify the notation, we write $v(\Psi)$ instead of $v_{\sigma(\Psi),N,\epsilon}(\Psi)$. We have

$$\begin{aligned}(\Psi^n - \Psi^\infty)(t) = e^{-iAt}(\Psi_0^n - \Psi_0^\infty) &+ i\int_0^t e^{-iA(t-s)}[\sigma(\Psi^n) - \sigma(\Psi^\infty)]x\Psi^n ds \\ &+ i\int_0^t e^{-iA(t-s)}[v(\Psi^n) - v(\Psi^\infty)]x\Psi^n ds \\ &+ i\int_0^t e^{-iA(t-s)}[\sigma(\Psi^\infty) + v(\Psi^\infty)]x(\Psi^n - \Psi^\infty) ds.\end{aligned}$$

Using (3.8), (3.10), and $\|x\Psi\|_{H^{-1}} \leqslant \|x\Psi\|_{L^2} \leqslant 1$, we get

$$\begin{aligned}&\|(\Psi^n - \Psi^\infty)(t)\|_{H^{-1}} \leqslant \|\Psi_0^n - \Psi_0^\infty\|_{H^{-1}} \\ &+ \int_0^t \Big(C(N)\|\theta'\|_{L^\infty} + \tilde{C}(N) + \mathcal{C}(\|\theta\|_{L^\infty} + N)\Big)\|\Psi^n - \Psi^\infty\|_{H^{-1}} ds,\end{aligned}$$

where $\mathcal{C}$ is given by (2.21). The Gronwall lemma concludes the proof. ∎

3.3. Proofs of Theorem 3.2 and Corollary 3.3.

Proof of Theorem 3.2. For $\varphi \in B_{L^2}(0,2)$, we define

$$\mathcal{V}_{N,\epsilon}(\varphi) := \mathcal{V}_{\sigma(\varphi),N,\epsilon}(\varphi),$$

where $\mathcal{V}_{\sigma,N,\epsilon}$ is defined by (2.23). Since N and ϵ are fixed, in order to simplify the notation, we omit them in the subscripts of this proof and write $v(\Psi)$ instead of $v_{\sigma(\Psi),N,\epsilon}(\Psi)$.

Let $\Psi_0 \in \mathbb{S} \cap (H^2 \cap H_0^1)(I,\mathbb{C})$ and let Ψ be the strong solution of (1.1)–(1.3) with $u(t) = \sigma(\Psi(t)) + v_{\sigma(\psi(t)),N,\epsilon}(\Psi(t))$ given by Proposition 3.5. Since $\Psi \in C^1(\mathbb{R}, L^2)$ and $\sigma \in C^\infty(B_{L^2}(0,2))$, the map $t \mapsto \mathcal{V}(\psi(t))$ is C^1. We have

$$\frac{d}{dt}\mathcal{V}(\Psi) = -2v(\Psi)^2 - \frac{d}{dt}\Big[\sigma(\Psi)\Big]\Re\left(\sum_{k=1}^N a_k \left\langle \Psi, \frac{d\phi_{k,\sigma}}{d\sigma}\Big|_{\sigma(\Psi)}\right\rangle \overline{\langle \Psi, \phi_{k,\sigma(\Psi)}\rangle}\right),$$

where $a_1 := 1$ and $a_k := 1 - \epsilon$ for $k = 2, \ldots, N$. Moreover,

$$\frac{d}{dt}\Big[\sigma(\Psi)\Big] = \theta'(\mathcal{V}(\psi))\frac{d}{dt}\mathcal{V}(\Psi),$$

and thus

(3.13)
$$\left[1+2\theta'(\mathcal{V}(\psi))\Re\left(\sum_{k=1}^{N} a_k \left\langle \Psi, \frac{d\phi_{k,\sigma}}{d\sigma}\Big|_{\sigma(\Psi)}\right\rangle \overline{\langle \Psi, \phi_{k,\sigma(\Psi)}\rangle}\right)\right]\frac{d}{dt}\mathcal{V}(\Psi) = -2v(\Psi)^2.$$

Using (2.8) and (3.1), we get

$$1+2\theta'(\mathcal{V}(\psi))\Re\left(\sum_{k=1}^{N} a_k \left\langle \Psi, \frac{d\phi_{k,\sigma}}{d\sigma}\Big|_{\sigma(\Psi)}\right\rangle \overline{\langle \Psi, \phi_{k,\sigma(\Psi)}\rangle}\right) \geqslant 1-2\|\theta'\|_{L^\infty} NC^* > 0;$$

thus, $t \mapsto \mathcal{V}(\Psi(t))$ is a nonincreasing function. There exists $\alpha \in [0, \mathcal{V}(\Psi_0)]$ such that

$$\lim_{t\to+\infty} \mathcal{V}(\Psi(t)) = \alpha.$$

Using (2.10), (3.2), and (3.4), we get

$$\begin{aligned}
|\langle \Psi_0, \phi_{1,\sigma(\Psi_0)}\rangle| &\geqslant |\langle \Psi_0, \phi_1\rangle| - |\langle \Psi_0, \phi_1 - \phi_{1,\sigma(\Psi_0)}\rangle| \\
&\geqslant \gamma - C^*\|\theta\|_\infty \\
&\geqslant \tilde{\gamma} := \frac{\gamma}{2},
\end{aligned}$$

$$\begin{aligned}
\sum_{k=N+1}^{\infty} |\langle \Psi_0, \phi_{k,\sigma(\Psi_0)}\rangle|^2 &\leqslant 2\sum_{k=N+1}^{\infty}\left(|\langle \Psi_0, \phi_k\rangle|^2 + |\langle \Psi_0, \phi_{k,\sigma(\Psi_0)} - \phi_k\rangle|^2\right) \\
&\leqslant \frac{\epsilon\gamma^2}{16(1-\epsilon/2)} + 2(C^*\|\theta\|_{L^\infty})^2 \sum_{k=N+1}^{\infty}\frac{1}{k^2} \\
&\leqslant \frac{\epsilon\gamma^2}{16(1-\epsilon/2)} + \frac{2(C^*\|\theta\|_{L^\infty})^2}{N} \\
&\leqslant \frac{\tilde{\epsilon}\tilde{\gamma}^2}{(1-\tilde{\epsilon})},
\end{aligned}$$

where $\tilde{\epsilon} := \epsilon/2$. Thus, as in the proof of Theorem 2.1, $\mathcal{V}(\Psi_0) < \tilde{\epsilon}$, so $\alpha \in (0, \tilde{\epsilon})$.

Let $(t_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive real numbers such that $t_n \to +\infty$ when $n \to +\infty$. Since $\|\Psi(t_n)\|_{L^2} = 1$ for every $n \in \mathbb{N}$, there exists $\Psi_\infty \in L^2(I, \mathbb{C})$ such that, up to an extraction,

$$\Psi(t_n) \to \Psi_\infty \text{ weakly in } L^2(I,\mathbb{C}) \text{ and strongly in } H^{-1}(I,\mathbb{C}).$$

Let ξ be the weak solution of

$$\begin{cases} i\frac{\partial\xi}{\partial t} = A_\sigma\xi - v_{\sigma(\xi),N,\epsilon}(\xi(t))x\xi, \\ \xi(t, \pm 1/2) = 0, \\ \xi(0) = \Psi_\infty. \end{cases}$$

Thanks to Proposition 3.6, for every $\tau > 0$, $\Psi(t_n+\tau) \to \xi(\tau)$ strongly in $H^{-1}(I,\mathbb{C})$ when $n \to +\infty$, and thus $\sigma(\Psi(t_n+\tau)) \to \sigma(\xi(\tau))$ when $n \to +\infty$ (see Lemma 3.4). Therefore, $\mathcal{V}(\Psi(t_n+\tau)) \to \mathcal{V}(\xi(\tau))$ when $n \to +\infty$, so $\mathcal{V}(\xi) \equiv \alpha$. Thus, $\sigma(\xi) \equiv \overline{\sigma} := \theta(\alpha)$ and we have, for every $t \in \mathbb{R}_+$,

$$\mathcal{V}(\xi(t)) = 1 - |\langle \xi(t), \phi_{1,\overline{\sigma}}\rangle|^2 - (1-\epsilon)\sum_{k=2}^{N} |\langle \xi(t), \phi_{k,\overline{\sigma}}\rangle|^2.$$

Since $\xi \in C^1(\mathbb{R}_+, H^{-2}_{(0)})$, the previous equality implies

$$\frac{d\mathcal{V}(\xi)}{dt} = -2v(\xi)^2.$$

Since $\mathcal{V}(\xi) \equiv \alpha$, then $v(\xi) \equiv 0$.

First case: $\alpha = 0$. Then $\mathcal{V}(\Psi(t)) \to 0$ when $t \to +\infty$ and $\overline{\sigma} = 0$. Moreover, for every $t \in (0, \infty)$,

$$\begin{aligned}\mathcal{V}(\Psi(t)) &\geqslant 1 - |\langle \Psi, \phi_{1,\sigma(\Psi)}\rangle|^2 - (1-\epsilon)\sum_{k=2}^{\infty} |\langle \Psi, \phi_{k,\sigma(\Psi)}\rangle|^2 \\ &\geqslant \epsilon(1 - |\langle \Psi, \phi_{1,\sigma(\Psi)}\rangle|^2).\end{aligned}$$

Thus,

$$|\langle \Psi(t_n), \phi_{1,\sigma(\Psi(t_n))}\rangle| \to 1,$$

which leads to

$$|\langle \Psi(t_n), \phi_1\rangle| \to 1$$

because $\sigma(\Psi(t_n)) \to 0$.

Second case: $\alpha \neq 0$. Then $\overline{\sigma} = \theta(\alpha) > 0$. Exactly as in the first analysis, done in the proof of Theorem 2.1, we get

$$\Psi_\infty = \beta \phi_{1,\overline{\sigma}},$$

where $\beta \in \mathbb{C}$ and $|\beta|^2 > 1 - \tilde{\epsilon}$. Thus

$$\lim_{n\to+\infty} |\langle \Psi(t_n), \phi_1\rangle| = |\langle \Psi_\infty, \phi_1\rangle| \geqslant |\beta| - |\langle \Psi_\infty, \phi_{1,\overline{\sigma}} - \phi_1\rangle| \geqslant \sqrt{1-\epsilon/2} - C^*\overline{\sigma},$$

where we used (2.7) in the last inequality. Finally, thanks to $0 < \overline{\sigma} \leqslant \|\theta\|_\infty$ and (3.2), we get (3.5). $\square$

Proof of Corollary 3.3. It can be done in a very similar way to the proof of Corollary 2.2. $\square$

4. Numerical simulations. In this section, we check out the performance of the techniques on some numerical simulations. We consider, as a test case, the stabilization of the initial state $\Psi_0 = \frac{1}{\sqrt{2}}(\phi_{1,\sigma} + \phi_{3,\sigma})$ around the ground state $\phi_{1,\sigma}$. Therefore, the cut-off dimension N is 3. Note that such a test case is a particularly hard one in a near-degenerate situation. Indeed, considering the feedback law (1.9) for $\sigma = 0$, one can easily see that for parity reasons $v(\Psi(t)) \equiv 0$.

In a first simulation, we consider the nondegenerate case of $\sigma \neq 0$. As mentioned above, the constant σ needs to be small. In fact, one should choose σ, such that the perturbation σx is small compared to the operator $-\frac{1}{2}\frac{\partial^2}{\partial x^2}$. We choose it here to be $\sigma = 2e+01$. Figure 4.1 illustrates the simulation of the closed-loop system when $u = \sigma + v_\epsilon$ with $\varsigma = 1e+03$ and $\epsilon = 5e-02$. The simulations have been done applying a third order split-operator method (see, e.g., [13]), where instead of computing $\exp(-i\ dt\ (A_\sigma - v_\epsilon x))$ at each time step, we compute

$$\exp(-i\ dt\ A_\sigma/2)\exp(i\ dt\ v_\epsilon x)\exp(-i\ dt\ A_\sigma/2).$$

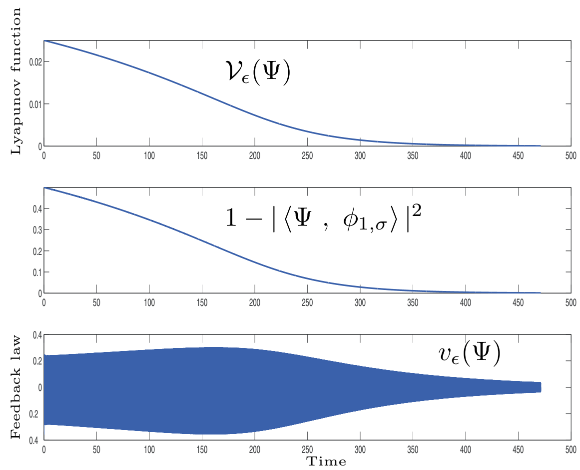

FIG. 4.1. *The practical stabilization of $\mathcal{C}_{1,\sigma}$, where $\Psi_0 = \frac{1}{\sqrt{2}}(\phi_{1,\sigma} + \phi_{3,\sigma})$ and therefore the cut-off dimension is* 3*; as can be seen, the closed-loop system reaches the* .05*-neighborhood of $\phi_{1,\sigma}$ in a time $T = 150\pi$ corresponding to about* 200 *periods of the longest natural period corresponding to the ground to the first excited state.*

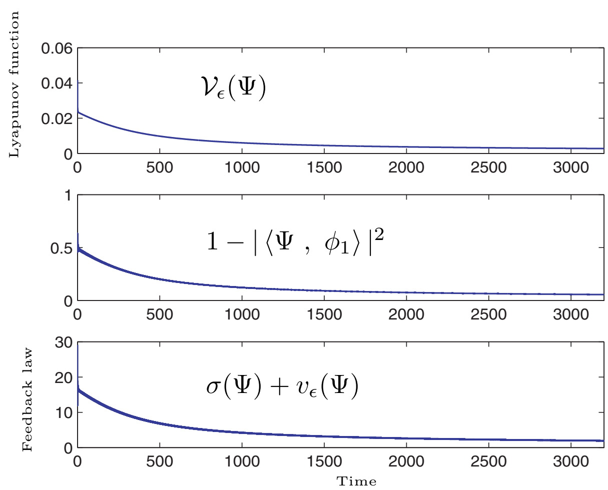

FIG. 4.2. *The practical stabilization of $\mathcal{C}_1$, where $\Psi_0 = \frac{1}{\sqrt{2}}(\phi_1 + \phi_3)$ and therefore the cut-off dimension is* 3*; as can be seen, the closed-loop system reaches the* .05*-neighborhood of ϕ_1 in a time $T = 1000\pi$ corresponding to about* 1300 *periods of the longest natural period corresponding to the ground to the first excited state.*

Moreover, we consider a Galerkin discretization over the first 20 modes of the system (it turns out, by considering higher modal approximations, that 20 modes are completely sufficient to get a reliable result).

Now, let us consider the degenerate case of $\sigma = 0$. As mentioned above, such a case is not treatable with the explicit feedback design of (1.9). However, the simulations of Figure 4.2 show that the implicit Lyapunov design provided in subsection 1.3 removes the degeneracy problem and ensures the practical stabilization of the initial state $\frac{1}{\sqrt{2}}(\phi_1 + \phi_3)$ around the ground state ϕ_1.

We consider the function $\theta(r) = \eta r$ with $\eta = 7e{+}02$. Furthermore, in the feedback design v_ϵ, we consider $\varsigma = 1e+03$ and $\epsilon = 5e-02$. The numerical scheme is similar to the simulations of Figure 4.1. In order to calculate the implicit part of the feedback design $\sigma(\Psi)$, we apply a fixed-point algorithm.

5. Appendix. This appendix is devoted to the proofs of Propositions 1.1 and 3.1.

5.1. Proof of Proposition 1.1. Let $\Psi_0 \in \mathbb{S}$, $T_1 > 0$ and $u \in C^0([0,T_1],\mathbb{R})$. Let $T \in (0,T_1)$ be such that

$$\|u\|_{L^1(0,T)} < 1. \tag{5.1}$$

We prove the existence of $\Psi \in C^0([0,T],L^2(I,\mathbb{C}))$ such that (1.4) holds by applying the Banach fixed-point theorem to the map

$$\begin{array}{cccc} \Theta: & C^0([0,T],L^2) & \to & C^0([0,T],L^2) \\ & \xi & \mapsto & \Psi, \end{array}$$

where Ψ is the weak solution of

$$\begin{cases} i\frac{\partial \Psi}{\partial t} = A\Psi - u(t)x\xi, \\ \Psi(0,x) = \Psi_0(x), \\ \Psi(t,\pm 1/2) = 0, \end{cases}$$

i.e., $\Psi \in C^0([0,T],L^2)$ and satisfies, for every $t \in [0,T]$,

$$\Psi(t) = e^{-iAt}\Psi_0 + i\int_0^t e^{-iA(t-s)}u(s)x\xi(s)ds \text{ in } L^2(I,\mathbb{C}).$$

Notice that Θ takes values in $C^1([0,T],H^{-2}_{(0)}(I,\mathbb{C}))$.

For $\xi_1,\xi_2 \in C^0([0,T],L^2(I,\mathbb{C}))$, $\Psi_1 := \Theta(\xi_1)$, $\Psi_2 := \Theta(\xi_2)$ we have

$$(\Psi_1 - \Psi_2)(t) = i\int_0^t e^{-iA(t-s)}u(s)x(\xi_1-\xi_2)(s)ds,$$

and thus

$$\|(\Psi_1-\Psi_2)(t)\|_{L^2} \leqslant \int_0^t |u(s)|ds\|\xi_1-\xi_2\|_{C^0([0,T],L^2)}.$$

The assumption (5.1) guarantees that Θ is a contraction of $C^0([0,T],L^2)$, and thus Θ has a fixed point $\Psi \in C^0([0,T],L^2)$. Since Θ takes values in $C^1([0,T],H^{-2}_{(0)})$, then Ψ belongs to this space. Moreover, this function satisfies (1.4).

Finally, we have built weak solutions on $[0,T]$ for every Ψ_0, and the time T does not depend on Ψ_0. Thus, this gives solutions on $[0,T_1]$.

Let us prove that this solution is continuous with respect to the the initial condition Ψ_0 for the $L^2(I,\mathbb{C})$-topology. Let $\Psi_0,\Phi_0 \in \mathbb{S}$ and let Ψ, Φ be the associated weak solutions. We have

$$\|(\Psi-\Phi)(t)\|_{L^2} \leqslant \|\Psi_0-\Phi_0\|_{L^2} + \int_0^t |u(s)|\|(\Psi-\Phi)(s)\|_{L^2}ds,$$

and thus the Gronwall lemma gives

$$\|(\Psi - \Phi)(t)\|_{L^2} \leqslant \|\Psi_0 - \Phi_0\|_{L^2} e^{\|u\|_{L^1(0,T_1)}}.$$

This gives the continuity of the weak solutions with respect to the initial conditions.

Now, let us assume that $\Psi_0 \in H^2 \cap H^1_0(I, \mathbb{C})$. Take C to be a positive constant such that for every $\varphi \in H^2 \cap H^1_0(I, \mathbb{C})$, $\|x\varphi\|_{H^2 \cap H^1_0} \leqslant C\|\varphi\|_{H^2 \cap H^1_0}$. We consider, then, $T > 0$ such that $C\|u\|_{L^1(0,T)} < 1$. By applying the fixed-point theorem on

$$\Theta_2 : C^0([0,T], H^2 \cap H^1_0) \to C^0([0,T], H^2 \cap H^1_0)$$

defined by the same expression as Θ, and using the uniqueness of the fixed point of Θ, we get that the weak solution is a strong solution. The continuity with respect to the initial condition of the strong solution can also be proved by applying the same arguments as in above.

Finally, let us justify that the weak solutions take values in $\mathbb{S}$. For $\Psi_0 \in H^2 \cap H^1_0$, the solution belongs to $C^1([0,T], L^2) \cap C^0([0,T], H^2 \cap H^1_0)$ and thus the following computations are justified:

$$\frac{d}{dt}\|\Psi(t)\|^2_{L^2} = 2\Re\left\langle \frac{\partial \Psi}{\partial t}, \Psi \right\rangle = 0.$$

Thus $\Psi(t) \in \mathbb{S}$ for every $t \in [0,T]$.

For $\Psi_0 \in \mathbb{S}$, we get the same conclusion thanks to a density argument and the continuity for the $C^0([0,T], L^2)$-topology of the weak solutions with respect to the initial condition. ∎

5.2. Proof of Proposition 3.1. Let $\Psi \in B_{L^2}(0,2)$. We prove the existence of $\sigma(\Psi)$ by applying the Banach fixed-point theorem to the map

$$\begin{array}{rccc} \Pi : & [0, \|\theta\|_{L^\infty}] & \to & [0, \|\theta\|_{L^\infty}] \\ & \sigma & \mapsto & \theta(\mathcal{V}_{\sigma,N,\epsilon}(\Psi)). \end{array}$$

For $\sigma_1, \sigma_2 \in [0, \|\theta\|_{L^\infty}]$, we have

$$|\Pi(\sigma_1) - \Pi(\sigma_2)| \leqslant \|\theta'\|_{L^\infty} |\mathcal{V}_{\sigma_1,N,\epsilon}(\Psi) - \mathcal{V}_{\sigma_2,N,\epsilon}(\Psi)|.$$

Using the inequality

$$\begin{aligned} \left| |\langle \Psi, \phi_{j,\sigma_1}\rangle|^2 - |\langle \Psi, \phi_{j,\sigma_2}\rangle|^2 \right| &\leqslant \left| \langle \Psi, \phi_{j,\sigma_1} - \phi_{j,\sigma_2}\rangle \overline{\langle \Psi, \phi_{j,\sigma_1}\rangle} \right| \\ &\quad + \left| \langle \Psi, \phi_{j,\sigma_2}\rangle \overline{\langle \Psi, \phi_{j,\sigma_1} - \phi_{j,\sigma_2}\rangle} \right| \\ &\leqslant 8\|\phi_{j,\sigma_1} - \phi_{j,\sigma_2}\|_{L^2}, \end{aligned}$$

together with (2.10), we get

$$|\Pi(\sigma_1) - \Pi(\sigma_2)| \leqslant 8NC^* \|\theta'\|_{L^\infty} |\sigma_1 - \sigma_2|.$$

Thus, the assumption (3.1) ensures that Π is a contraction of $[0, \|\theta\|_{L^\infty}]$. Therefore, Π has a unique fixed point $\sigma(\Psi)$.

Now, let us prove that σ is C^∞. The map

$$\begin{array}{rccc} F : & [0, \|\theta\|_{L^\infty}] \times B_{L^2}(0,2) & \to & \mathbb{R} \\ & (\sigma, \Psi) & \mapsto & \sigma - \theta(\mathcal{V}_{\sigma,N,\epsilon}(\Psi)) \end{array}$$

is regular with respect to σ and Ψ, $F(\sigma(\Psi),\Psi)=0$ for every $\Psi\in B_{L^2}(0,2)$, and

$$\frac{\partial F}{\partial\sigma}(\sigma(\Psi),\Psi)=1-2\theta'(\mathcal{V}_{\sigma(\Psi),N,\epsilon}(\Psi))\frac{\partial}{\partial\sigma}\Big[\mathcal{V}_{\sigma,N,\epsilon}(\Psi)\Big]_{\sigma(\psi)}\geqslant\frac{1}{2}. \tag{5.2}$$

Indeed, for $\sigma_0\in[0,\|\theta\|_{L^\infty}]$ and $\Psi\in B_{L^2}(0,2)$, we have

$$\frac{\partial}{\partial\sigma}\Big[\mathcal{V}_{\sigma,N,\epsilon}(\Psi)\Big]_{\sigma_0}=-2\sum_{k=1}^{N}a_k\Re\left(\left\langle\Psi,\frac{d\phi_{k,\sigma}}{d\sigma}\Big|_{\sigma_0}\right\rangle\overline{\langle\Psi,\phi_{k,\sigma_0}\rangle}\right),$$

where $a_1:=1$ and $a_k:=1-\epsilon$ for $k=2,\dots,N$. Thus, using (2.8), we get

$$\left|\frac{\partial}{\partial\sigma}\Big[\mathcal{V}_{\sigma,N,\epsilon}(\Psi)\Big]_{\sigma_0}\right|\leqslant 8NC^*.$$

We get the inequality in (5.2) thanks to the previous inequality and (3.1).

For every $\Psi\in B_{L^2}(0,2)$, the implicit function theorem provides the existence of a local C^∞ parameterization $\tilde\sigma(\xi)$ for the solutions of $F(\sigma(\xi),\xi)=0$ in a neighborhood of Ψ. The uniqueness of the fixed point $\sigma(\xi)$ justifies that σ and $\tilde\sigma$ coincide, and thus σ is C^∞. $\square$

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