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Spectral controllability for 2D and 3D linear Schrödinger equations

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Abstract

We consider a quantum particle in an infinite square potential well of $\mathbb{R}^n$, $n = 2, 3$, subjected to a control which is a uniform (in space) electric field. Under the dipolar moment approximation, the wave function solves a PDE of Schrödinger type. We study the spectral controllability in finite time of the linearized system around the ground state. We characterize one necessary condition for spectral controllability in finite time: (*Kal*) if Ω is the bottom of the well, then for every eigenvalue λ of $-\Delta_\Omega^D$, the projections of the dipolar moment onto every (normalized) eigenvector associated to λ are linearly independent in $\mathbb{R}^n$. In 3D, our main result states that spectral controllability in finite time never holds for one-directional dipolar moment. The proof uses classical results from trigonometric moment theory and properties about the set of zeros of entire functions. In 2D, we first prove the existence of a minimal time $T_{\min}(\Omega) > 0$ for spectral controllability, i.e., if $T > T_{\min}(\Omega)$, one has spectral controllability in time T if condition (*Kal*) holds true for (Ω) and, if $T < T_{\min}(\Omega)$ and the dipolar moment is one-directional, then one does not have spectral controllability in time T. We next characterize a necessary and sufficient condition on the dipolar moment

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insuring that spectral controllability in time $T > T_{\min}(\Omega)$ holds generically with respect to the domain. The proof relies on shape differentiation and a careful study of Dirichlet-to-Neumann operators associated to certain Helmholtz equations. We also show that one can recover exact controllability in abstract spaces from this 2D spectral controllability, by adapting a classical variational argument from control theory.
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Contents

1. Introduction

Let us consider a quantum particle in an infinite square potential well of $\mathbb{R}^n$, $n \in \{1,2,3\}$ subjected to a uniform (in space) time dependent electric field $u : t \mapsto u(t) \in \mathbb{R}^n$. Let Ω be the domain of $\mathbb{R}^n$ corresponding to the bottom of the well. This physical system is modeled by a wave function

$$\psi: \quad \begin{array}{rcl} \mathbb{R}_+ \times \Omega & \to & \mathbb{C}, \\ (t,q) & \mapsto & \psi(t,q), \end{array}$$

such that $|\psi(t,q)|^2\,dq$ represents the probability of the particle to be in the volume dq surrounding the point q at time t. Thus, the wave function ψ lives on the $L^2(\Omega,\mathbb{C})$-sphere S as it is well known that the $L^2(\Omega,\mathbb{C})$-norm of the wave function ψ is preserved over time. Under the dipolar moment approximation, this wave function solves the following Schrödinger equation

$$\begin{cases} i\dfrac{\partial\psi}{\partial t}(t,q) = -\Delta\psi(t,q) - \langle u(t), \mu(q)\rangle\psi(t,q), & (t,q) \in \mathbb{R}_+ \times \Omega, \\ \psi(t,q) = 0, & (t,q) \in \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{1}$$

where $\mu \in C^0(\overline{\Omega}, \mathbb{R}^n)$ is the dipolar moment and $\langle .,. \rangle$ denotes the usual scalar product on $\mathbb{R}^n$. The system (1) is a nonlinear control system in which

- the state is the wave function ψ with $\psi(t) \in S$, for every $t \geqslant 0$,
- the control is the electric field $u : t \in \mathbb{R}_+ \mapsto u(t) \in \mathbb{R}^n$.

Studying controllability properties of the control system (1) reveals interesting features. For instance, Turinici proved in [45] that, the system (1) is not controllable in $H^2 \cap H^1_0(\Omega,\mathbb{C})$ with controls u in $L^r_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^n)$, $r \in (1,+\infty)$. This result is a corollary of a more general result about the controllability of bilinear control systems, due to Ball, Marsden and Slemrod in [7]. However, it has been proved in [8] that the system (1) in 1D, with $\Omega = (-1/2, 1/2)$ and $\mu(q) = q$ is locally controllable around the ground state in $H^7((-1/2,1/2),\mathbb{C})$ with $H^1_0((0,T),\mathbb{R})$ controls, when T is large enough. This system is even controllable between eigenstates, as proved in [9]. Therefore the noncontrollability result emphasized in [45] is essentially due to a choice of functional spaces that do not allow the controllability, but this controllability holds in other satisfying functional spaces. At the moment, in 2D or 3D, no positive exact controllability result is known for (1).

We can also consider a similar nonlinear system. The quantum particle is now placed in a moving infinite square potential well of $\mathbb{R}^n$, $n \in \{1,2,3\}$. Let Ω be the domain of $\mathbb{R}^n$ corresponding to the bottom of the well. It is proved by Rouchon in [38] that this physical system is represented by the following Schrödinger equation

$$\begin{cases} i\dfrac{\partial\psi}{\partial t}(t,q) = -\Delta\psi(t,q) - \langle u(t), \mu(q)\rangle\psi(t,q), & (t,q) \in \mathbb{R}_+ \times \Omega, \\ \psi(t,q) = 0, & (t,q) \in \mathbb{R}_+ \times \partial\Omega, \\ \dot{d}(t) = s(t), & \\ \dot{s}(t) = u(t), & \end{cases} \tag{2}$$

where ψ is the wave function of the particle in the moving frame, $u := \ddot{d}$ is the acceleration of the well, s is the speed of the well, d is the position of the well and $\mu(q) = q$ (but in this article, we will study this system for more general functions μ). The system (2) is a nonlinear control system with state, the triple (ψ, s, d) with $\psi(t) \in S$, for every $t \geqslant 0$, and control, the acceleration of the well $u : t \in \mathbb{R}_+ \mapsto \mathbb{R}^n$. In 1D, with $\Omega = (-1/2, 1/2)$, the local controllability around the eigenstates and the controllability between eigenstates of (2) is proved in [9].

A classical approach to prove the local controllability of nonlinear systems such as (1) and (2) around a reference trajectory consists in proving first, the controllability of the linearized system around the reference trajectory and second, the local controllability of the nonlinear system around the reference trajectory, with the help of an inverse mapping theorem. If the linearized system around the reference trajectory is not controllable, one may use the return method advocated by Coron (cf. [14,15] and references therein, and [8,9] for applications to 1D Schrödinger equations). This method relies on the study of another reference trajectory of the nonlinear system admitting a controllable linearized system.

Therefore, it is natural to linearize (1) and (2) along "simple" trajectories, for instance, along the one corresponding to the zero control, $u \equiv 0$ and to study the controllability of the resulting linear system. For $k \in \mathbb{N}^*$, the *eigenstate* $\psi_k(t, q) := \phi_k(q)e^{-i\lambda_k t}$ defines such a trajectory $((\psi = \psi_k, u \equiv 0)$ for (1) and $(\psi = \psi_k,\ s \equiv 0,\ d \equiv 0,\ u \equiv 0)$ for (2)), where $(\phi_k)_{k\in\mathbb{N}^*}$ is a complete orthonormal system of eigenfunctions for $-\Delta_\Omega^D$, the Laplacian operator on Ω with Dirichlet boundary condition, and $(\lambda_k)_{k\in\mathbb{N}^*}$ are the corresponding nondecreasing sequence of eigenvalues counted with their multiplicity. In the particular case $k = 1$, ψ_1 is called the *ground state* and the following systems are the linearized systems respectively of (1) around the ground state,

$$\begin{cases} i\dfrac{\partial \Psi}{\partial t}(t,q) = -\Delta\Psi(t,q) - \langle v(t), \mu(q)\rangle \psi_1(t,q), & (t,q) \in \mathbb{R}_+ \times \Omega, \\ \Psi(t,q) = 0, & (t,q) \in \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{3}$$

and of (2) around the trajectory $((\psi = \psi_1,\ s \equiv 0,\ d \equiv 0),\ u \equiv 0)$,

$$\begin{cases} i\dfrac{\partial \Psi}{\partial t}(t,q) = -\Delta\Psi(t,q) - \langle v(t), \mu(q)\rangle \psi_1(t,q), & (t,q) \in \mathbb{R}_+ \times \Omega, \\ \Psi(t,q) = 0, & (t,q) \in \mathbb{R}_+ \times \partial\Omega, \\ \dot{d}(t) = s(t), \\ \dot{s}(t) = v(t). \end{cases} \tag{4}$$

In this paper, we only study controllability properties of systems (3) and (4).

Let us recall classical results about the controllability of these two systems in 1D, results being the starting point of the strategies developed in [8] and [9] for the nonlinear systems (1) and (2). Their proof will be sketched in Section 2 in order to explain the difficulties arising in their generalization to the 2D and 3D cases. For system (3), $\Omega = (0, 1)$ and, if s is a nonnegative real number, let $H^s_{(0)}((0,1), \mathbb{C})$ be equal to $D(A^{s/2})$ where $D(A) := H^2 \cap H^1_0((0,1), \mathbb{C})$ and $A\varphi := -\varphi''$. Then, up to a condition satisfied by the dipolar moment μ (see Proposition 2.2 for a detailed statement), the system (3) is controllable in $H^3_{(0)}((0,1), \mathbb{C})$ with control functions in $L^2((0,T), \mathbb{R})$ for every $T > 0$. As regards controllability for system (4), we show that it is not exact controllable in finite time for the 1D problem and we describe the reachable set. The crucial technical reason for that lies in the fact that the eigenvalues of Δ_Ω^D verify a uniform gap condition, i.e., there exists $\rho > 0$ such that, for every positive integer, we have $\lambda_{k+1} - \lambda_k \geqslant \rho$. However, in 2D, the existence of a regular domain Ω of $\mathbb{R}^2$ such that the eigenvalues of Δ_Ω^D present a uniform gap is still an open problem and in 3D, no uniform gap is possible because of the Weyl formula. Therefore, exact controllability of (3) and (4) in 2D and 3D is not a trivial question and it is thus natural to study a weaker controllability property for this system. This is why we investigate, in this article, the *spectral controllability* of systems (3) and (4). To define that concept of controllability, let us denote $\mathcal{D}$, the linear span of the eigenvectors ϕ_k, $k \in \mathbb{N}^*$, and $T_S\varphi$, the tangent space to the sphere S at the point $\varphi \in S$. We say that system (3) is *spectral controllable in time T* if, for every $\Psi_0 \in \mathcal{D} \cap T_S\psi_1(0)$, $\Psi_f \in \mathcal{D} \cap T_S\psi_1(T)$, there exists $v \in L^2((0,T),\mathbb{R}^n)$ such that the trajectory $\Psi(\cdot)$ of (3) starting at Ψ_0 satisfies $\Psi(T) = \Psi_f$. For system (4), that definition must be adapted as follows. Let $\langle .,. \rangle_{L^2}$ denote the $L^2(\Omega,\mathbb{C})$-scalar product. Then, system (4) is *spectral controllable in time T* if, for every $\Psi_0 \in \mathcal{D} \cap T_S\psi_1(0)$, $\Psi_f \in \mathcal{D} \cap T_S\psi_1(T)$ with $\Im\langle \Psi_f, \psi_1(T)\rangle = \Im\langle \Psi_0, \psi_1(0)\rangle$ and for every $d_0 \in \mathbb{R}^n$, there exists $v \in L^2((0,T),\mathbb{R}^n)$ such that the trajectory $(\Psi, s, d)(\cdot)$ of (4) starting at $(\Psi_0, 0, d_0)$ satisfies $(\Psi, s, d)(T) = (\Psi_f, 0, 0)$.

Our main results deal with the spectral controllability of (3) and (4). Before describing them, let us make a general remark. Since we are dealing with controls only depending on time, the control systems under consideration can be put into the general form $\dot{x} = Ax + B(x)u$ where the state belongs to some $\mathbb{C}$-valued functional space X, the control u is $\mathbb{R}^n$-valued, the drift A is an (unbounded) linear operator admitting a complete orthonormal system of eigenfunctions and the controlled vector field $B(\cdot)$ has rank one. Using the classical moment theory, it is easy to characterize two necessary conditions for spectral controllability in some finite time $T > 0$.

The first one corresponds to the Kalman condition for controllability in finite dimension. In our context, it means that

(*Kal*) for every eigenvalue λ of A, the projections $b_{k_j} := \langle \mu(q)\phi_1, \phi_{k_j}\rangle$, $1 \leqslant j \leqslant m(\lambda)$, of the controlled vector field $B(\cdot)$ on each eigenvector associated to λ are linearly independent in $\mathbb{R}^n$.

The above condition implies that the multiplicity of every eigenvalue λ of A is less than or equal to n. Note also that if A has simple spectrum (this will be referred as condition (*Simp*)), then condition (*Kal*) simply reads: the projections $b_k := \langle \mu(q)\phi_1, \phi_k\rangle$ of the controlled vector field $B(\cdot)$ on each (normalized) eigenvector is nonzero. We refer to the latter condition as (*NonZ*).

The second condition is specific to the infinite dimension (for the state space) and it is related to the minimality of the family $(e^{\pm i(\lambda_k - \lambda_1)t})_{k \in \mathbb{N}}$ in $L^2((0,T),\mathbb{C})$ (see Definition 3.1).

By applying a result of Haraux and Jaffard [19], we show that minimality never occurs in 3D for system (4) and also for system (3) if, in addition, the dipolar moment has a constant direction. In 2D, we show that minimality holds for both systems (3) and (4) if T is larger than a minimal time $T_{\min}(\Omega)$. In turn, if the dipolar moment has a constant direction, spectral controllability in time $T > 0$ for system (4) enables one to define a Hilbert subspace H of $L^2(\Omega,\mathbb{C})$ in which (4) is controllable, with $L^2((0,T),\mathbb{R})$-controls, when $T > T_{\min}(\Omega)$.

In order to get spectral controllability in time $T > T_{\min}(\Omega)$, it therefore amounts, for a 2D domain Ω and a dipolar moment function μ, to check the validity of (*Kal*). Since the latter is difficult to verify for a given 2D domain Ω, we rather investigate conditions on the dipolar moment μ to insure that (*Kal*) holds true generically with respect to domains Ω with C^3 boundary. There is a trivial necessary condition on μ for (*Kal*) to hold true generically with respect to the domain: μ must be nowhere locally constant (*NLC*), i.e., its level sets are all of empty interior. (Indeed, simply consider a 2D domain where μ is constant. Then (*Kal*) does not hold, because of the $L^2(\Omega, \mathbb{C})$-orthogonality of the eigenvectors ϕ_k.) One of our main results says that condition (*NLC*) for a C^1 dipolar moment μ is also sufficient to prove that condition (*Kal*) holds true, generically with respect to domains Ω with C^3 boundary. To do so, we start from the well-known fact that the spectrum of the Laplacian operator on a domain $\Omega \subset \mathbb{R}^2$ with Dirichlet boundary conditions is generically simple. Therefore, it amounts to prove that condition (*NLC*) for a C^1 dipolar moment μ is also sufficient for condition (*NonZ*) to hold true, generically with respect to domains Ω with C^3 boundary. In summary, we can finally show that, in 2D, spectral controllability in finite time, for both systems (4) and (3) holds true, generically with respect to domains with C^3 boundary, if and only if the C^1 dipolar moment μ is nowhere locally constant.

Before giving the plan of the paper, let us sketch the argument showing that (*NLC*) implies (*NonZ*), generically with respect to the domain. First of all, we must consider a topology for domains with C^3 boundary. Following [42], the latter is defined by taking as base of neighborhoods the sets $V(\Omega, \varepsilon)$ defined, for Ω any domain with C^3 boundary and $\varepsilon > 0$ small enough, as the images of Ω by $\mathrm{Id}_2 + u$, $u \in W^{4,\infty}(\Omega, \mathbb{R}^2)$ and $\|u\|_{W^{4,\infty}} < \varepsilon$. We use $\mathbb{D}_3$ to denote the Banach space of domains with C^3 boundary equipped with the topology defined previously. A property is said to be generic in $\mathbb{D}_3$ if the subset of domains in $\mathbb{D}_3$ verifying that property is everywhere dense in $\mathbb{D}_3$.

We now fix a domain Ω with C^3 boundary and a C^1 dipolar moment μ verifying (*NLC*). Without loss of generality we assume that (*Simp*) is verified by Ω and we first reduce the argument to showing, for every positive integer $k \geqslant 2$, the existence of a sequence (Ω_n) of domains with C^3 boundary converging to Ω such that $(NonZ)_k$ (i.e., $b_k \neq 0$ along the sequence (Ω_n)) holds true along the sequence. We proceed with a contradiction argument and we thus assume that there exists $\varepsilon > 0$ such that, for every $u \in W^{4,\infty}(\Omega, \mathbb{R}^2)$ with $\|u\|_{W^{4,\infty}} < \varepsilon$, the corresponding b_k is equal to zero. We compute the shape derivative of the relation $b_k = 0$ at $u \equiv 0$ and we can express it as an integral along the boundary of Ω, i.e.,

$$\int_{\partial\Omega} \langle u(q), \nu(q)\rangle M(q)\, d\sigma(q) = 0,$$

where ν denotes the outer unit normal vector field and $M(\cdot)$ is a $\mathbb{R}^2$-valued function defined on $\partial\Omega$. As we will see below, in order to define M, one must introduce ξ_1 and ξ_k, solutions of inhomogeneous Helmholtz equations (see (31) below). We at once deduce that $M(\cdot) \equiv 0$ on $\partial\Omega$. Reaching a contradiction in our argument amounts to show that the functions ξ_1, ξ_k introduced above actually do not exist. Unfortunately, we are not able to do that. By pushing further the contradiction argument, we compute the shape derivative of $b_k = 0$ at every $u \in W^{4,\infty}(\Omega, \mathbb{R}^2)$ with $\|u\|_{W^{4,\infty}} < \varepsilon$. That translates into the following relation: for $\varepsilon > 0$ small enough and for every $u, v \in W^{4,\infty}(\Omega, \mathbb{R}^2)$ with $\|u\|_{W^{4,\infty}} < \varepsilon$ and $\|v\|_{W^{4,\infty}} < \varepsilon$, one has

$$\int_{\partial(\mathrm{Id}_2+u)\Omega} \langle v(q), \nu(u)(q)\rangle M(u)(q)\, d\sigma(q) = 0,$$

where $\nu(u)$ denotes the outer unit normal vector field defined on $\partial(\mathrm{Id}_2 + u)(\Omega)$ and $M(u)(\cdot)$ is an $\mathbb{R}^2$-valued function defined on $\partial(\mathrm{Id}_2 + u)(\Omega)$. The expression of $M(u)(\cdot)$ requires to define $\xi_1(u), \xi_k(u)$, solutions of inhomogeneous Helmholtz equations. Of course, $M(0), \xi_1(0)$ and $\xi_k(0)$ are equal to M, ξ_1 and ξ_k defined previously and we have that $M(u)(\cdot) \equiv 0$ on $\partial(\mathrm{Id}_2 + u)(\Omega)$ for $\|u\|_{W^{4,\infty}} < \varepsilon$.

At this stage, we are again not able to derive a contradiction. So we again take the shape derivative of $M(u)(\cdot) \equiv 0$ on $\partial\Omega$ and end up with the relation

$$M'(u)(q) = -(u.\nu)(q)\frac{\partial M(0)}{\partial \nu}(q), \quad q \in \partial\Omega, \tag{5}$$

for $\|u\|_{W^{4,\infty}} < \varepsilon$. We now start a strategy first introduced in [11], which consists in defining $M'(u)$ for functions u defined on $\partial\Omega$ which are continuous except at some point $\bar{q}$ of $\partial\Omega$. For instance, we will take $u = u_{\bar{q}}$ as a Heaviside function $\mathcal{H}_0(\bar{q})$ admitting a single jump of discontinuity at an arbitrary point $q \in \partial\Omega$. The key remark is the following: if $(u.\nu)$ belongs to the Sobolev space $H^s(\partial\Omega)$ for some $s > 0$ then, by standard elliptic theory arguments, $M'(u)$ belongs to $H^{s-1}(\partial\Omega)$. In order to take advantage of the gap of regularity between the two sides of Eq. (5), we embark in the computation of the singular part of $M'(u_{\bar{q}})(\cdot)$ at $\bar{q}$ (in the distributional sense) and eventually come up with the following expression,

$$M'(u_{\bar{q}})(\sigma) = M_0 \quad \text{p.v.}\left(\frac{1}{\sigma}\right) + R(\sigma),$$

where σ denotes the arc-length (with $\sigma = 0$ corresponding to $\bar{q}$) and $R(\cdot)$ belongs to $H^{1/2-\varepsilon}(\partial\Omega)$ for every $\varepsilon > 0$. Plugging back the above expression into Eq. (5), one must necessarily have $M_0 = 0$. Recalling that $\bar{q} \in \partial\Omega$ is arbitrary, we end up with $M_0(\cdot) \equiv 0$ on $\partial\Omega$. In [11], the previous relation on M_0 was providing additional information with respect to the relation $M(u) \equiv 0$ on $\partial\Omega$, which allowed to conclude the contradiction argument. However, in the present situation, it turns out that $M_0(\cdot)$ is proportional to $M(0)(\cdot)$ and hence is trivially equal to zero. One must therefore compute the first nontrivial term in the "singular" expansion of $M'(u_{\bar{q}}) + (u_{\bar{q}}.\nu)\frac{\partial M(0)}{\partial \nu}$ at $\bar{q}$, in the distributional sense. That procedure requires a detailed study of Dirichlet-to-Neumann operators associated to several Helmholtz equations. Once the nontrivial term is characterized, we easily conclude.

The rest of the paper is organized as follows. In Section 2, we provide the main notations and precise definitions of the control systems (3) and (4), complete 1D results with their proofs and the statements of the main theorems of this article. Then, in Section 3, we give the proofs for the spectral controllability results in 2D and 3D. As for Section 4, the construction of some abstract spaces where we have 2D exact controllability is described. Section 5 contains the proof of the sufficiency of condition (*NLC*) to get generic controllability in 2D for the quantum box and Section 6 presents some conjectures. Finally, we gather in Appendix A the main results on shape differentiation used in the paper and Appendix B contains material on the Dirichlet-to-Neumann map for the Helmholtz equation with the proof of several technical lemmas which are needed in Section 5.

2. Definition of the control problem, notations and statement of the results

2.1. Definition of the control problem

Let Ω be a domain of $\mathbb{R}^n$ (i.e., a bounded nonempty open subset of $\mathbb{R}^n$), $n \in \{1, 2, 3\}$, with a C^1 boundary. We use $-\Delta_\Omega^D$ to denote the Laplacian operator on Ω with Dirichlet boundary conditions, i.e.,

$$D(\Delta_{\Omega}^{D}) = H^2 \cap H_0^1(\Omega, \mathbb{C}), \qquad -\Delta_{\Omega}^{D}\phi = -\Delta\phi.$$

The space $L^2(\Omega, \mathbb{C})$ has a complete orthonormal system $(\phi_k)_{k\in\mathbb{N}^*}$ of eigenfunctions for $-\Delta_{\Omega}^{D}$,

$$\phi_k \in H^2 \cap H_0^1(\Omega, \mathbb{C}), \qquad -\Delta_{\Omega}^{D}\phi_k = \lambda_k\phi_k,$$

where $(\lambda_k)_{k\in\mathbb{N}^*}$ is a nondecreasing sequence of positive real numbers. With this notation, the eigenvalues λ_k are counted as many times as their multiplicity. For $t \in \mathbb{R}$ and $q \in \Omega$, we define the function ψ_1 by

$$\psi_1(t, q) := \phi_1(q)e^{-i\lambda_1 t}.$$

We recall that $-i\Delta_{\Omega}^{D}$ generates a C^0-group of isometries of $L^2(\Omega, \mathbb{C})$ defined by

$$e^{-i\Delta t}\varphi := \sum_{k\in\mathbb{N}^*} \langle\varphi, \phi_k\rangle e^{-i\lambda_k t}\phi_k, \quad \forall\varphi \in L^2(\Omega, \mathbb{C}).$$

In this paper, we study controllability properties of the linear systems (3) and (4).

In order to consider them as control systems, we first need a concept of trajectories associated to these systems. For that purpose, recall that the unit sphere S of $L^2(\Omega, \mathbb{C})$ is defined as follows,

$$S := \{\varphi \in L^2(\Omega, \mathbb{C});\ \|\varphi\|_{L^2(\Omega)} = 1\},$$

and, for $\varphi \in S$, the tangent space to the sphere S at the point φ is given by

$$T_S\varphi := \left\{\xi \in L^2(\Omega, \mathbb{C});\ \Re\left(\int_{\Omega} \xi(q)\overline{\varphi(q)}\, dq\right) = 0\right\}.$$

Definition 2.1 *(Weak solutions).* Let $T > 0$, $\mu \in C^0(\overline{\Omega}, \mathbb{R}^2)$, $\Psi_0 \in T_S\phi_1$ and $v \in L^1((0, T), \mathbb{R}^n)$. A weak solution to the Cauchy problem

$$\begin{cases} i\dfrac{\partial\Psi}{\partial t}(t, q) = -\Delta\Psi(t, q) - \langle v(t), \mu(q)\rangle\psi_1(t, q), & (t, q) \in \mathbb{R}_+ \times \Omega, \\ \Psi(t, q) = 0, & (t, q) \in \mathbb{R}_+ \times \partial\Omega, \\ \Psi(0) = \Psi_0, \end{cases} \tag{6}$$

is a function $\Psi \in C^0([0, T], L^2(\Omega, \mathbb{C}))$ such that for every $t \in [0, T]$,

$$\Psi(t) = e^{i\Delta t}\Psi_0 + i\int_0^t e^{i\Delta(t-s)}\big[\langle v(s), \mu\rangle\psi_1(s)\big]\, ds \quad \text{in } L^2(\Omega, \mathbb{C}). \tag{7}$$

Then (Ψ, v) is a trajectory of the control system (3) on $[0, T]$.

Let $s_0, d_0 \in \mathbb{R}^n$. A weak solution to the Cauchy problem

$$\begin{cases} i\dfrac{\partial\Psi}{\partial t}(t,q) = -\Delta\Psi(t,q) - \langle v(t), \mu(q)\rangle\psi_1(t,q), & (t,q)\in\mathbb{R}_+\times\Omega, \\ \Psi(t,q) = 0, & (t,q)\in\mathbb{R}_+\times\partial\Omega, \\ \Psi(0) = \Psi_0, \\ \dot{s}(t) = v(t), \qquad s(0) = s_0, \\ \dot{d}(t) = s(t), \qquad d(0) = d_0, \end{cases} \tag{8}$$

is a function (Ψ, s, d) with $s \in W^{1,1}((0,T),\mathbb{R}^n)$, $d \in W^{2,1}((0,T),\mathbb{R}^n)$ solutions of

$$\dot{s}(t) = v(t) \quad \text{in } L^1\big((0,T),\mathbb{R}^n\big), \qquad s(0) = s_0,$$
$$\dot{d}(t) = s(t) \quad \text{in } L^1\big((0,T),\mathbb{R}^n\big), \qquad d(0) = d_0,$$

and $\Psi \in C^0([0,T], L^2(\Omega,\mathbb{C}))$ such that for every $t \in [0,T]$, (7) holds. Then $((\Psi, s, d), v)$ is a trajectory of the control system (4) on $[0,T]$.

The following proposition recalls a classical existence and uniqueness result for the solutions of (6), from which one can deduce the similar result for (8).

Theorem 2.1. *For every $T > 0$, $\Psi_0 \in T_S\phi_1$, $v \in L^1((0,T),\mathbb{R}^n)$, there exists a unique weak solution to the Cauchy problem* (6) *and $\Psi(t) \in T_S\psi_1(t)$ for every $t \geqslant 0$.*

Then, the system (3) is a control system where

- the state is the function Ψ, with $\Psi(t) \in T_S\psi_1(t)$ for every $t \in \mathbb{R}_+$,
- the control is $v : t \in \mathbb{R}_+ \mapsto v(t) \in \mathbb{R}^n$, $L^1_{\text{loc}}(\mathbb{R}_+,\mathbb{R}^n)$ is the set of admissible controls

and the system (4) is a control system where

- the state is the triple (Ψ, s, d), with $\Psi(t) \in T_S\psi_1(t)$ for every $t \in \mathbb{R}_+$,
- the control is $v : t \in \mathbb{R}_+ \mapsto v(t) \in \mathbb{R}^n$ and $L^1_{\text{loc}}(\mathbb{R}_+,\mathbb{R}^n)$ is the set of admissible controls.

More precisely, in this paper, we investigate the following controllability property for (3).

Definition 2.2 *(Spectral controllability for (3)).* The system (3) is spectral controllable in time T if, for every $\Psi_0 \in \mathcal{D} \cap T_S\psi_1(0)$, $\Psi_f \in \mathcal{D} \cap T_S\psi_1(T)$, there exists $v \in L^2((0,T),\mathbb{R}^n)$ such that the solution of (6) satisfies $\Psi(T) = \Psi_f$, where

$$\mathcal{D} := \operatorname{Span}\big\{\phi_k;\ k \in \mathbb{N}^*\big\}.$$

For the system (4), this definition needs to be adapted because of the presence of s and d in the state variable and because the directions $\Im\langle\Psi(t), \psi_1(t)\rangle$ and $s(t)$ are linked. Indeed, any solution of (8) satisfies

$$\Im\big\langle\Psi(t), \psi_1(t)\big\rangle = \Im\big\langle\Psi_0, \psi_1(0)\big\rangle + \sum_{j=1}^{n}\big\langle\mu^{(j)}\phi_1, \phi_1\big\rangle\big[s^{(j)}(t) - s^{(j)}(0)\big], \quad \forall t, \tag{9}$$

where, for $x \in \mathbb{R}^n$, $x^{(j)}$ denotes its components, $x = (x^{(1)}, \dots, x^{(n)})$ and $\langle .,. \rangle$ denotes the $L^2(\Omega, \mathbb{C})$-scalar product. Therefore, we study the following controllability property for (4).

Definition 2.3 *(Spectral controllability for (4)).* The system (4) is spectral controllable in time T if for every $\Psi_0 \in \mathcal{D} \cap T_S\psi_1(0)$, $\Psi_f \in \mathcal{D} \cap T_S\psi_1(T)$ with $\Im\langle \Psi_f, \psi_1(T)\rangle = \Im\langle \Psi_0, \psi_1(0)\rangle$, for every $d_0 \in \mathbb{R}^n$, there exists $v \in L^2((0,T), \mathbb{R}^n)$ such that the solution of (8) with $s_0 = 0$ satisfies $(\Psi, s, d)(T) = (\Psi_f, 0, 0)$.

The notations Ω, $n \in \{1,2,3\}$, ϕ_k, ψ_1, $\langle .,. \rangle$, S, T_S, $\mathcal{D}$, $x = (x^{(1)}, \dots, x^{(n)}) \in \mathbb{R}^n$ introduced in this section are used all along this article. We also denote $(e_j)_{1 \leqslant j \leqslant n}$ the canonical basis of $\mathbb{R}^n$ and $\omega_k := \lambda_k - \lambda_1$, for every $k \in \mathbb{N}^*$. We use the same notation for the $\mathbb{R}^n$-scalar product and the $L^2(\Omega)$-scalar product but if a confusion is possible we precise the space in subscript $\langle .,. \rangle_{L^2(\Omega)}$ or $\langle .,. \rangle_{\mathbb{R}^n}$. When some confusion is possible, we also precise the domain on the eigenvalues and eigenfunctions of the Laplacian: λ_k^Ω, ϕ_k^Ω.

2.2. *Previous 1D results, difficulties of the 2D and 3D generalizations*

In this section, we recall classical results about the controllability of the systems (3) and (4) in 1D, that are the starting point of the strategies developed in [8] and [9] for the nonlinear systems (1) and (2). We also give their proof in order to explain the difficulties arising in their generalization to the 2D and 3D cases.

We take $\Omega = (0,1)$, so

$$\phi_k(q) = \sqrt{2}\sin(k\pi q), \qquad \lambda_k = (k\pi)^2$$

and we use the following notations

$$H^s_{(0)}\big((0,1),\mathbb{C}\big) := D\big(A^{s/2}\big) \quad \text{where } D(A) := H^2 \cap H^1_0\big((0,1),\mathbb{C}\big),\ A\varphi := -\varphi''.$$

2.2.1. *1D controllability of (3)*

For the control system (3), we have the following result.

Proposition 2.2. *Let $\Omega = (0,1)$ and $\mu \in W^{3,\infty}((0,1),\mathbb{R})$.*

(1) *We assume that*

$$\exists c_1, c_2 > 0, \quad \frac{c_1}{k^3} \leqslant \big|\langle \mu\phi_1, \phi_k\rangle\big| \leqslant \frac{c_2}{k^3}, \quad \forall k \in \mathbb{N}^*. \tag{10}$$

Then, for every $T > 0$, the system (3) *is controllable in $H^3_{(0)}((0,1),\mathbb{C})$ with control functions in $L^2((0,T),\mathbb{R})$: for every $T > 0$, $\Psi_0, \Psi_f \in H^3_{(0)}((0,1),\mathbb{C})$ with $\Psi_0 \in T_S\psi_1(0)$ and $\Psi_f \in T_S\psi_1(T)$, there exists $v \in L^2((0,T),\mathbb{R})$ such that the solution of* (6) *satisfies $\Psi(T) = \Psi_f$.*

(2) *We assume that there exists $m \in \mathbb{N}^*$ such that $\langle \mu\phi_1, \phi_m\rangle = 0$ and*

$$\exists c_1, c_2 > 0, \quad \frac{c_1}{k^3} \leqslant \big|\langle \mu\phi_1, \phi_k\rangle\big| \leqslant \frac{c_2}{k^3}, \quad \forall k \in \mathbb{N}^* \text{ such that } \langle \mu\phi_1, \phi_k\rangle \neq 0. \tag{11}$$

Then, the system (3) *is not controllable: for every $T > 0$, $\Psi_0 \in L^2((0,1),\mathbb{C})$ and $v \in L^1((0,T),\mathbb{R})$ the solution of* (6) *satisfies*

$$\langle \Psi(T), \phi_k \rangle = \langle \Psi_0, \phi_k \rangle e^{-i\lambda_k T}, \quad \forall k \in \mathbb{N}^* \text{ such that } \langle \mu\phi_1, \phi_k \rangle = 0.$$

But one can characterize the reachable set: for every $T > 0$, $\Psi_0, \Psi_f \in H^3_{(0)}((0,1),\mathbb{C})$ with $\Psi_0 \in T_S\psi_1(0)$, $\Psi_f \in T_S\psi_1(T)$, $\langle \Psi_f, \phi_k \rangle = \langle \Psi_0, \phi_k \rangle e^{-i\lambda_k T}$ for every $k \in \mathbb{N}^*$ such that $\langle \mu\phi_1, \phi_k \rangle = 0$, there exists $v \in L^2((0,T),\mathbb{R})$ such that the solution of (6) satisfies $\Psi(T) = \Psi_f$.

Remark 2.1. Let us emphasize that the assumption (10) is generic with respect to $\mu \in W^{3,\infty}((0,1),\mathbb{R})$. Indeed, thanks to Baire's Lemma, it is easy to prove that the property "$\langle \mu\phi_1, \phi_k \rangle \neq 0$, $\forall k \in \mathbb{N}^*$" holds generically with respect to $\mu \in W^{3,\infty}((0,1),\mathbb{R})$. Moreover, for such a function μ, integrations by parts lead to

$$\langle \mu\phi_1, \phi_k \rangle = 2\int_0^1 \mu(q)\sin(\pi q)\sin(k\pi q)\,dq = \frac{4k[(-1)^{k+1}\mu'(1) - \mu'(0)]}{(k^2-1)^2} + o\left(\frac{1}{k^3}\right).$$

Thus, the asymptotic behavior in $1/k^3$ of these coefficients is equivalent to the property "$\mu'(1) \pm \mu'(0) \neq 0$," that is also generic in $W^{3,\infty}((0,1),\mathbb{R})$.

The key ingredient for the proof of Proposition 2.2 is the following theorem due to Kahane [26, Theorem III.6.1, p. 114].

Theorem 2.3. *Let $(\mu_k)_{k\in\mathbb{N}^*} \subset \mathbb{R}$ such that $\mu_1 = 0$ and*

$$\mu_{k+1} - \mu_k \geqslant \rho > 0, \quad \forall k \in \mathbb{N}^*. \tag{12}$$

Let $T > 0$ be such that

$$\lim_{x\to+\infty} \frac{N(x)}{x} < \frac{T}{2\pi},$$

where, for $x > 0$, $N(x)$ is the largest number of μ_k's contained in an interval of length x. Then, there exists $C > 0$ such that, for every $c = (c_k)_{k\in\mathbb{N}^*} \in l^2(\mathbb{N}^*,\mathbb{C})$ with $c_1 \in \mathbb{R}$, there exists $w \in L^2((0,T),\mathbb{R})$ such that $\|w\|_{L^2((0,T),\mathbb{R})} \leqslant C\|c\|_{l^2(\mathbb{N}^*,\mathbb{C})}$ and

$$\int_0^T w(t)e^{i\mu_k t}\,dt = c_k, \quad \forall k \in \mathbb{N}^*.$$

Remark 2.2. The proof of Theorem 2.3 relies on an Ingham inequality for the family

$$\left\{1, e^{i\mu_k t}, e^{-i\mu_k t};\ k \in \mathbb{N}^*,\ k \geqslant 2\right\},$$

which corresponds to the Riesz basis property of this family in $L^2((0,T),\mathbb{C})$. For the proof of Theorem 2.3, see, for example Krabs [30, Section 1.2.2], Komornik and Loreti [29, Chapter 9], or Avdonin and Ivanov [5, Chapter II, Section 4]. For the proof of similar results, we also refer to the prior works by Ingham [22], and to Beurling [10, pp. 341–365], Haraux [18], Redheffer [37], Russel [39, Section 3], Schwartz [40]. Improvements of Theorem 2.3 have been obtained by Jaffard, Tucsnak and Zuazua [24,25], Jaffard and Micu [23], Baiocchi, Komornik and Loreti [6], Komornik and Loreti [28], [29, Theorem 9.4, p. 177].

Proof of Proposition 2.2. We assume (10). Let $T > 0$ and $\Psi_0 \in T_S\psi_1(0)$. By definition, the weak solution of (6) with some control $v \in L^2((0,T),\mathbb{R})$ is

$$\Psi(t,q) = \sum_{k=1}^{\infty} x_k(t)\phi_k(q) \quad \text{where } x_k(t)$$
$$= \left(\langle \Psi_0, \phi_k\rangle + i\langle \mu\phi_1, \phi_k\rangle \int_0^t v(\tau)e^{i\omega_k\tau}\,d\tau\right)e^{-i\lambda_k t}, \quad \forall k \in \mathbb{N}^*,$$

with convergence in $L^2((0,1),\mathbb{C})$ for every $t \in [0,T]$, where $\omega_k := \lambda_k - \lambda_1$, for every $k \in \mathbb{N}^*$. Since $\langle \mu\phi_1, \phi_k\rangle \neq 0$, for every $k \in \mathbb{N}^*$, the equality $\Psi(T) = \Psi_f$ in $L^2((0,1),\mathbb{C})$ is equivalent to the following trigonometric moment problem on the control v,

$$\int_0^T v(t)e^{i\omega_k t}\,dt = d_k, \quad \forall k \in \mathbb{N}^*, \tag{13}$$

where

$$d_k := \frac{\langle \Psi_f, \phi_k\rangle e^{i\lambda_k T} - \langle \Psi_0, \phi_k\rangle}{i\langle \mu\phi_1, \phi_k\rangle}, \quad \forall k \in \mathbb{N}^*. \tag{14}$$

Thanks to (10), the right-hand side $(d_k)_{k\in\mathbb{N}^*}$ belongs to $l^2(\mathbb{N}^*,\mathbb{C})$ if and only if $\Psi_f - e^{-iAT}\Psi_0 \in H^3_{(0)}((0,1),\mathbb{C})$, and in that case, (13) has a solution $v \in L^2((0,T),\mathbb{R})$ for every $T > 0$, thanks to Theorem 2.3. The proof of the statement (2) is similar. $\square$

Now, let us discuss the generalization of Proposition 2.2 to the 2D and 3D cases. In 2D and 3D, the equality $\Psi(T) = \Psi_f$ for a solution of (6) is equivalent to

$$i\left\langle \langle \mu\phi_1, \phi_k\rangle_{L^2(\Omega)}, \int_0^T v(t)e^{i\omega_k t}\,dt\right\rangle_{\mathbb{R}^n} = \langle \Psi_f, \phi_k\rangle e^{i\lambda_k T} - \langle \Psi_0, \phi_k\rangle, \quad \forall k \in \mathbb{N}^*. \tag{15}$$

Thus, the property

$$\langle \mu\phi_1, \phi_k\rangle \neq 0, \quad \forall k \in \mathbb{N}^*$$

is still a necessary condition for the controllability of (3). Let us assume that this property holds, then (15) is satisfied in particular when

$$\int_0^T v(t)e^{i\omega_k t}\,dt = -i\frac{\langle \mu\phi_1, \phi_k\rangle}{|\langle \mu\phi_1, \phi_k\rangle|^2}\big(\langle \Psi_f, \phi_k\rangle e^{i\lambda_k T} - \langle \Psi_0, \phi_k\rangle\big), \quad \forall k \in \mathbb{N}^*.$$

Thus, the controllability of (3) can be reduced to the solvability of n trigonometric moment problems on the real valued functions $v^{(1)}, \dots, v^{(n)}$.

In 2D, the existence of a regular domain Ω of $\mathbb{R}^2$ such that the eigenvalues of Δ_Ω^D present a uniform gap (which corresponds to the assumption (12)) is an open problem. For general 2D regular domains, we only have Weyl's formula

$$\exists c = c(\Omega) > 0,\ \exists \alpha = \alpha(\Omega) \in (0,1), \quad \sharp\big\{k \in \mathbb{N}^*;\ \lambda_k \in [0,t]\big\} = ct + O\big(t^\alpha\big) \quad \text{when } t \to +\infty.$$

This formula is not sufficient to ensure the existence of a uniform gap between the frequencies ω_k. Therefore the classical result given in Theorem 2.3 cannot be applied: the controllability of (3) is a more difficult problem in 2D than in 1D.

In 3D, with Weyl's formula,

$$\exists c = c(\Omega) > 0,\ \exists \alpha = \alpha(\Omega) \in (0,3/2), \quad \sharp\big\{k \in \mathbb{N}^*;\ \lambda_k \in [0,t]\big\} = ct^{3/2} + O\big(t^\alpha\big) \quad \text{when } t \to +\infty,$$

no uniform gap is possible. Thus, the noncontrollability of (3) is expected.

The exact controllability of (3) in 2D and 3D being a difficult problem, it is natural to study a weaker controllability property for this system. This is why we investigate its spectral controllability in this article. Notice that the spectral controllability in time T of (3) is equivalent to the existence of a solution $v \in L^2((0,T),\mathbb{R}^n)$ of (15) for any right-hand side with finite support. This remark will be used in the study of the spectral controllability of (3) (see Section 3.2).

2.2.2. 1D controllability of (4)

For the control system (4), we have the following result.

Proposition 2.4. *Let $\Omega = (0,1)$ and $\mu \in W^{3,\infty}((0,1),\mathbb{R})$.*

(1) *The system* (4) *is not controllable: for every $\Psi_0 \in T_S\psi_1(0)$, $s_0, d_0 \in \mathbb{R}$, $v \in L^1_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R})$, the solution of* (8) *satisfies* (9).
(2) *If* (11) *holds, then, one can characterize the reachable set for* (4): *for every $T > 0$, $\Psi_0, \Psi_f \in H^3_{(0)}((0,1),\mathbb{C})$, $s_0, s_f, d_0, d_f \in \mathbb{R}$ with $\langle \Psi_f, \psi_1(T)\rangle = \langle \Psi_0, \psi_1(0)\rangle + i\langle \mu\phi_1, \phi_1\rangle(s_f - s_0)$ and*

$$\big\langle \Psi(T), \phi_k\big\rangle = \langle \Psi_0, \phi_k\rangle e^{-i\lambda_k T}, \quad \forall k \geqslant 2 \text{ such that } \langle \mu\phi_1, \phi_k\rangle = 0, \tag{16}$$

there exists $v \in L^2((0,T),\mathbb{R})$ such that the solution of (8) *satisfies $(\Psi, s, d)(T) = (\Psi_f, s_f, d_f)$.*

The proof of this proposition is similar to the one of Proposition 2.2.

Notice that, in 2D and 3D, the equality $(\Psi, s, d)(T) = (\Psi_f, 0, 0)$ for the solution of (8) with $s_0 = 0$, $\Psi_0 \in T_S\psi_1(0)$, $\Psi_f \in T_S\psi_1(T)$ such that $\Im\langle \Psi_0, \psi_1(0)\rangle = \Im\langle \Psi_f, \psi_1(T)\rangle$ is equivalent to

$$\begin{cases} i\left\langle \langle \mu\phi_1, \phi_k\rangle_{L^2(\Omega)}, \displaystyle\int_0^T v(t)e^{i\omega_k t}\,dt \right\rangle_{\mathbb{R}^n} = \langle \Psi_f, \phi_k\rangle e^{i\lambda_k T} - \langle \Psi_0, \phi_k\rangle, \quad \forall k \geqslant 2, \\ \displaystyle\int_0^T v(t)\,dt = 0, \\ \displaystyle\int_0^T t v(t)\,dt = d_0. \end{cases} \tag{17}$$

Thus, the spectral controllability in time T of (4) is equivalent to the existence of a solution $v \in L^2((0,T),\mathbb{R}^n)$ of (17), for any right-hand side with finite support. This remark will be used in the study of the spectral controllability of (4) (see Section 3.3).

2.3. Statement of the main results

In order to state our results, we first give several definitions relative to the domain and the dipolar moment.

Definition 2.4 *(Kalman condition (Kal)).* Let Ω be a domain of $\mathbb{R}^n$, $n = 2, 3$ with C^1 boundary. Then Ω verifies property (*Kal*) if

(*Kal*) any eigenvalue λ of $-\Delta_\Omega^D$ has a multiplicity $m \leqslant n$ and the vectors $\langle \mu\phi_1, \phi_{k_1}\rangle, \dots, \langle \mu\phi_1, \phi_{k_m}\rangle$ are linearly independent in $\mathbb{R}^n$, where $k_1 < \cdots < k_m$ and $\phi_{k_1}, \dots, \phi_{k_m}$ are the eigenvectors associated to λ.

Definition 2.5 *(Simplicity of the spectrum (Simp)).* Let Ω be a domain of $\mathbb{R}^n$, $n = 2, 3$ with C^1 boundary. Then Ω verifies property (*Simp*) if

(*Simp*) the eigenvalues of $-\Delta_\Omega^D$ are simple.

Definition 2.6 *(Nonzero projection (NonZ)).* Consider $\mu \in C^0(\overline{\Omega}, \mathbb{R}^n)$, $n = 2, 3$ and $(\phi_k)_{k\in\mathbb{N}^*}$ the complete orthonormal system of eigenvectors of $-\Delta_\Omega^D$. Then $\mu\phi_1$ has a nonzero projection on $(\phi_k)_{k\in\mathbb{N}^*}$ if, for every integer $k \geqslant 2$, we have

$$(NonZ)_k \quad \langle \mu\phi_1, \phi_k\rangle \neq 0.$$

In that case, we say that μ verifies property (*NonZ*).

Remark that if a domain Ω satisfies (*Simp*), then condition (*Kal*) reduces to condition (*NonZ*). The next theorem gathers our result regarding the spectral controllability properties for system (3).

Theorem 2.5.

(1) *Let Ω be a domain of $\mathbb{R}^2$ with C^1 boundary and $\mu \in C^0(\overline{\Omega}, \mathbb{R}^2)$ verifying* (*Kal*). *Then, there exists $T_{\min} = T_{\min}(\Omega) > 0$ such that*

(1.a) *for every* $T > T_{\min}$, *system* (3) *is spectral controllable in time* T;
(1.b) *for every* $T < T_{\min}$, *system* (3) *is not spectral controllable in time* T, *under the additional assumption*

$$\mu(x) = \tilde{\mu}(x)e_1 \quad \text{where } \tilde{\mu} \in C^0(\overline{\Omega}, \mathbb{R}). \tag{18}$$

(2) *Let* Ω *be a domain of* $\mathbb{R}^n$, $n = 2, 3$, *with* C^1 *boundary and* $\mu \in C^0(\overline{\Omega}, \mathbb{R}^n)$ *such that* (*Kal*) *is not verified. Then, system* (3) *is not spectral controllable.*
(3) *Let* Ω *be a domain of* $\mathbb{R}^3$ *with* C^1 *boundary and* $\mu \in C^0(\overline{\Omega}, \mathbb{R}^3)$ *of the form* (18). *Then, system* (3) *is not spectral controllable.*

Remark 2.3. Let us emphasize that (*Kal*) holds true generically with respect to the pair (Ω, μ) because conditions (*Simp*) and (*NonZ*) hold true simultaneously generically with respect to the pair (Ω, μ), where Ω is a domain of $\mathbb{R}^2$ with C^1 boundary and $\mu \in C^0(\overline{\Omega}, \mathbb{R}^2)$. Indeed, the genericity of (*Simp*) with respect to the domain Ω is a classical result (see, for instance, [21]). Moreover, for a domain Ω of $\mathbb{R}^2$ with C^1 boundary verifying (*Simp*), the set

$$\left\{\mu \in C^0\left(\overline{\Omega}, \mathbb{R}^2\right); \langle \mu\phi_1, \phi_k \rangle \neq 0, \ \forall k \in \mathbb{N}^*\right\}$$

is dense in $C^0(\overline{\Omega}, \mathbb{R}^2)$ (it can be proved thanks to Baire's Lemma).

As for system (4), we prove the following result.

Theorem 2.6.

(1) *Let* Ω *be a domain of* $\mathbb{R}^2$ *with* C^1 *boundary and* $\mu \in C^0(\overline{\Omega}, \mathbb{R}^2)$ *verifying* (*Kal*). *Let* $T_{\min} = T_{\min}(\Omega)$ *be as in Theorem* 2.5. *Then, system* (4) *is spectral controllable in time* $T > T_{\min}$.
(2) *Let* Ω *be a domain of* $\mathbb{R}^n$, $n = 2, 3$, *with* C^1 *boundary and* $\mu \in C^0(\overline{\Omega}, \mathbb{R}^n)$ *such that* (*Kal*) *is not verified. Then, system* (4) *is not spectral controllable.*
(3) *Let* Ω *be a domain of* $\mathbb{R}^3$ *with* C^1 *boundary and* $\mu \in C^0(\overline{\Omega}, \mathbb{R}^3)$. *Then system* (4) *is not spectral controllable*: *for every* $T > 0$ *and* $m \in \mathbb{N}^*$, *there exists* $d_0 \in \mathbb{R}^3$ *such that* $(i\phi_m, 0, d_0)$ *is not zero controllable in time* T.

Remark 2.4. Notice that in item (3) of Theorem 2.6, the dipolar moment μ is not necessarily one-dimensional. Thus, we prove a stronger noncontrollability result for this 3D system, than the one given in Theorem 2.5(3). This improvement is due to the presence of s and d in the state variable.

The proofs of Theorems 2.5 and 2.6 are given in Section 3.

In Section 4, we prove that, one can recover the exact controllability, in some abstract spaces, for the system (3) in 2D with μ of the form (18) thanks to the previous spectral controllability result. Such abstract spaces may be used for the study of the nonlinear system. This is an open problem.

According to Theorem 2.6, one knows that, in 2D, property (*Kal*) is a necessary and sufficient condition for the spectral controllability of (3) and (4) in time $T > T_{\min}(\Omega)$. We next use that characterization to prove that spectral controllability of (3) and (4) in large time holds true generically with respect to the 2D domain Ω. For that purpose, let us first precise the topology on domains we are using, then define genericity and finally state the condition on the dipolar moment μ that ensures the genericity.

For $l \geqslant 1$, the set $\mathbb{D}_l$ of domains Ω of $\mathbb{R}^2$ with C^l boundary. Following [42], we define next a topology on $\mathbb{D}_l$. Consider the Banach space $W^{l+1,\infty}(\Omega, \mathbb{R}^2)$ equipped with its standard norm. For $\Omega \in \mathbb{D}_l$, $u \in W^{l+1,\infty}(\Omega, \mathbb{R}^2)$, let $\Omega + u := (\mathrm{Id}_{+}u)(\Omega)$ be the subset of points $y \in \mathbb{R}^2$ such that $y = x + u(x)$ for some $x \in \Omega$ and $\partial\Omega + u := (\mathrm{Id}_{+}u)(\partial\Omega)$ its boundary. For $\varepsilon > 0$, let $V(\Omega, \epsilon)$ be the set of all $\Omega + u$ with $u \in W^{l+1,\infty}(\Omega, \mathbb{R}^2)$ and $\|u\|_{W^{l+1,\infty}} \leqslant \varepsilon$. The topology of $\mathbb{D}_l$ is defined by taking the sets $V(\Omega, \varepsilon)$ with ε small enough as a base of neighborhoods of Ω. Then, $\mathbb{D}_l$ is a Banach space.

Definition 2.7. We say that a property (P) is generic in $\mathbb{D}_l$ if the set of domains of $\mathbb{D}_l$ on which this property holds true is dense in $\mathbb{D}_l$: for every $\Omega \in \mathbb{D}_l$, there exists $\rho > 0$ such that the set $\{u \in E_\rho(\Omega);\ \Omega + u \text{ satisfies } (P)\}$ is dense in $E_\rho(\Omega)$, where $E_\rho(\Omega) := \{u \in W^{l+1,\infty}(\Omega, \mathbb{R}^2);\ \|u\|_{W^{l+1,\infty}} < \rho\}$.

Definition 2.8 *(Nonlocally constant (NLC)).* A map $\mu \in C^0(\mathbb{R}^2, \mathbb{R}^2)$ is said to be *nowhere locally constant* if, for every $\mu_0 \in \mathbb{R}^2$, the level set $\{q \in \mathbb{R}^2 \mu(q) = \mu_0\}$ has an empty interior.

Note that if μ is (*NLC*) and continuously differentiable, then the subset of $\mathbb{R}^n$, $n = 2, 3$, where the differential of μ is not zero, must be open and dense.

We now state one of the main results of the paper.

Theorem 2.7. *Let $\mu \in C^1(\mathbb{R}^2, \mathbb{R}^2)$. The spectral controllability in large time for system* (4) *is generic in $\mathbb{D}_3$ if and only if μ is nowhere locally constant.*

According to item (2) of Theorem 2.6, the proof of the previous theorem reduces to establishing the next proposition, since (*Simp*) and (*NonZ*) both verified imply that (*Kal*) holds true.

Proposition 2.8. *Let $\mu \in C^1(\mathbb{R}^2, \mathbb{R}^2)$. If $\Omega \in \mathbb{D}_1$, we say that Ω has property (A) if* (*Simp*) *and* (*NonZ*) *hold true for Ω. Then, property (A) is generic in $\mathbb{D}_3$ if and only if μ is nowhere locally constant.*

Section 5 is devoted to the proof of the above proposition.

3. Spectral controllability in 2D and 3D

The goal of this Section is the proof of Theorems 2.5 and 2.6. This section is organized as follows.

In Section 3.1, we state a sufficient condition for the minimality in $L^2((0, T), \mathbb{C})$ of a family of complex exponentials. This condition, due to Haraux and Jaffard [19], involves Weyl's formula.

In Section 3.2, we prove Theorem 2.5, thanks to Haraux and Jaffard's result.

In Section 3.3, we prove Theorem 2.6. The proofs of the two first statements also rely on Haraux and Jaffard's result. The proof of the third statement involves different ideas, about the set of zeros of holomorphic functions.

3.1. Haraux and Jaffard's result

First, let us recall the definition of the minimality of a family of vectors.

Definition 3.1. Let X be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A family $(z_k)_{k\in\mathbb{Z}}$ of vectors of X is minimal in X if, for every $m \in \mathbb{Z}$, z_m does not belong to the closure in X of the vector space generated by $\{z_k;\ k \in \mathbb{Z} - \{m\}\}$,

$$z_m \notin \mathrm{Cl}_X\big(\mathrm{Span}\big\{z_k;\ k \in \mathbb{Z} - \{m\}\big\}\big), \quad \forall m \in \mathbb{Z}.$$

When X is an Hilbert space, we have the following classical equivalent definitions.

Proposition 3.1. *Let $(X, \langle .\,, .\rangle_X)$ be a Hilbert space and $(z_k)_{k\in\mathbb{Z}}$ be a family of vectors of X. The following statements are equivalent.*

(1) $(z_k)_{k\in\mathbb{Z}}$ *is minimal in X.*
(2) *For every $m \in \mathbb{Z}$, there exists $C_m > 0$ such that, for every $f \in X$ of the form $f = \sum_{k\in K} f_k z_k$ where $K \subset \mathbb{Z}$ is finite,*

$$C_m|f_m| \leqslant \|f\|_X.$$

(3) *There exists a family $(Z_k)_{k\in\mathbb{Z}}$ of vectors of X bi-orthogonal to $(z_k)_{k\in\mathbb{Z}}$, i.e.,*

$$\langle z_m, Z_k\rangle_X = \delta_{m,k}, \quad \forall k, m \in \mathbb{Z}.$$

(4) *For every $(d_k)_{k\in\mathbb{Z}} \subset \mathbb{K}$ with finite support, there exists $v \in X$ solution of the moment problem*

$$\langle v, z_k\rangle_X = d_k, \quad \forall k \in \mathbb{Z}. \tag{19}$$

Proof of Proposition 3.1. For (1) $\Rightarrow$ (2), the largest value for the constant C_m is

$$C_m := \mathrm{dist}\big(z_m, \mathrm{Span}\big\{z_k;\ k \in \mathbb{Z} - \{m\}\big\}\big).$$

The implications (2) $\Rightarrow$ (1), *no* (1) $\Rightarrow$ *no* (3) and (3) $\Leftrightarrow$ (4) are easy. For (1) $\Rightarrow$ (3), one can take

$$Z_m := P_m\left(\frac{z_m}{\|z_m\|_X^2}\right)$$

where P_m is the orthogonal projection from X to $V_m^{\perp}$, the orthogonal supplementary of $V_m := \mathrm{Cl}_X(\mathrm{Span}\{z_k;\ k \in \mathbb{Z} - \{m\}\})$ in X, which is a closed vector subspace of X. $\Box$

Remark 3.1. The statement (4) is particularly important in this article. Indeed, as seen in Section 2.2, the spectral controllability in time T of (3) is equivalent to the solvability of a moment problem of the form (19) with $X = L^2((0,T),\mathbb{R}^n)$, $z_0 := \langle \mu\phi_1, \phi_1\rangle$, $z_k := \langle \mu\phi_1, \phi_{k+1}\rangle \cos(\omega_{k+1}t)$, $z_{-k} := \langle \mu\phi_1, \phi_{k+1}\rangle \sin(\omega_{k+1}t)$, $\forall k \in \mathbb{N}^*$. Thus, the spectral controllability in time T of (3) is equivalent to the minimality of the family $(z_k)_{k\in\mathbb{Z}}$ in $L^2((0,T),\mathbb{R}^n)$.

The following theorem is the key point of Section 3. It has been proved by Haraux and Jaffard in [19, Corollary 2.3.5], as a consequence of the Beurling–Malliavin theorem, thanks to the computation of the Beurling–Malliavin density of a sequence that satisfies Weyl's formula.

Theorem 3.2. *Let* $(\mu_k)_{k\in\mathbb{Z}}$ *be a sequence of real numbers such that*

$$\sharp\{k\in\mathbb{Z};\ 0\leqslant\mu_k\leqslant t\}=dt+O\big(t^\alpha\big),\qquad \sharp\{k\in\mathbb{Z};\ -t\leqslant\mu_k\leqslant 0\}=dt+O\big(t^\alpha\big),$$

for some $d\geqslant 0$ *and* $\alpha\in(0,1)$. *Then,*

(1) *for every* $T>2\pi d$, *the family* $\{e^{i\mu_k t};\ k\in\mathbb{Z}\}$ *is minimal in* $L^2((0,T),\mathbb{C})$,
(2) *for every* $T<2\pi d$, *the family* $\{e^{i\mu_k t};\ k\in\mathbb{Z}\}$ *is not minimal in* $L^2((0,T),\mathbb{C})$.

Remark 3.2. Notice that, when $\mu_0=0$ and $\mu_k=-\mu_{-k}>0$, for every $k\in\mathbb{N}^*$, then the minimality of the family $\{e^{i\mu_k t};\ k\in\mathbb{Z}\}$ in $L^2((0,T),\mathbb{C})$ is equivalent to the minimality of the family $\{1,\cos(\mu_k t),\sin(\mu_k t);\ k\geqslant 0\}$ in $L^2((0,T),\mathbb{R})$.

3.2. Proof of Theorem 2.5

The goal of this section is the proof of Theorem 2.5 thanks to Theorem 3.2.

Proof of Theorem 2.5. (1) Let Ω be a domain of $\mathbb{R}^2$ with C^1 boundary and $\mu\in C^0(\overline{\Omega},\mathbb{R}^2)$ be such that (*Kal*) holds. Thanks to Weyl's formula, there exists $d=d(\Omega)\in(0,+\infty)$ and $\alpha=\alpha(\Omega)\in(0,1)$ such that

$$\sharp\big\{k\in\mathbb{N}^*;\ \omega_k\in[0,t]\big\}=dt+O\big(t^\alpha\big)\quad\text{when } t\to+\infty. \tag{20}$$

Let $T_{\min}=T_{\min}(\Omega):=2\pi d$.

(1.a) Let $T>T_{\min}$, $\Psi_0\in\mathcal{D}\cap T_S\psi_1(0)$, $\Psi_f\in\mathcal{D}\cap T_S\psi_1(T)$ and let us prove that there exists $v\in L^2((0,T),\mathbb{R}^2)$ solution of (15). We introduce

$$\Lambda_1:=\big\{k\in\mathbb{N}^*;\ \lambda_k \text{ is a simple eigenvalue of } \Delta_\Omega^D\big\},\qquad \Lambda_2:=\big\{k\in\mathbb{N}^*;\ \lambda_k=\lambda_{k+1}\big\}.$$

For every $k\in\Lambda_2$, the vectors $\langle\mu\phi_1,\phi_k\rangle$ and $\langle\mu\phi_1,\phi_{k+1}\rangle$ are linearly independent in $\mathbb{R}^2$, thus there exists a unique $Z_k\in\mathbb{C}^2$ such that

$$\big\langle\langle\mu\phi_1,\phi_k\rangle_{L^2(\Omega)},Z_k\big\rangle_{\mathbb{R}^2}=-id_k,\qquad \big\langle\langle\mu\phi_1,\phi_{k+1}\rangle_{L^2(\Omega)},Z_k\big\rangle_{\mathbb{R}^2}=-id_{k+1},$$

where $d_j:=\langle\Psi_f,\phi_j\rangle e^{i\lambda_j T}-\langle\Psi_0,\phi_j\rangle$, for every $j\in\mathbb{N}^*$. For a function $v\in L^2((0,T),\mathbb{R}^2)$, (15) is satisfied in particular when

$$\int_0^T v(t)e^{i\omega_k t}\,dt=-id_k\frac{\langle\mu\phi_1,\phi_k\rangle}{|\langle\mu\phi_1,\phi_k\rangle|^2},\quad \forall k\in\Lambda_1,$$
$$\int_0^T v(t)e^{i\omega_k t}\,dt=Z_k,\quad \forall k\in\Lambda_2, \tag{21}$$

i.e., when $v^{(1)}$ and $v^{(2)}$ solve a trigonometric moment problem with a finite supported right-hand side. The solvability of (21) is equivalent to the minimality of the family

$$\{1, \cos(\omega_k t), \sin(\omega_k t);\ k \geqslant 2\}$$

in $L^2((0,T),\mathbb{R})$ (see Proposition 3.1), which holds true thanks to Theorem 3.2.

For the proof of (1.b) and (3), let us first emphasize that, when (18) and (*Kal*) hold, then the spectral controllability in time T of (3) is *equivalent to* (and not only implied by) the minimality of the family $\{1, \cos(\omega_k t), \sin(\omega_k t);\ k \geqslant 2\}$ in $L^2((0,T),\mathbb{R})$.

(1.b) Let $T < T_{\min}$ and let us assume (18). Theorem 3.2 ensures that the family $\{1, \cos(\omega_k t), \sin(\omega_k t);\ k \geqslant 2\}$ is not minimal in $L^2((0,T),\mathbb{R})$, thus (4) is not spectral controllable in time T.

(2) Let Ω be a domain of $\mathbb{R}^n$ with C^1 boundary, $n = 2, 3$, and $\mu \in C^0(\overline{\Omega}, \mathbb{R}^n)$. We assume that (*Kal*) does not hold. There exists $k \in \mathbb{N}^*$ such that λ_k has multiplicity m and there exists $(\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m - \{0\}$ such that $\alpha_1\langle\mu\phi_1, \phi_{k_1}\rangle + \dots + \alpha_m\langle\mu\phi_1, \phi_{k_m}\rangle = 0$, where $k_1, \dots, k_m$ are all the integers such that $\lambda_k = \lambda_{k_1} = \dots = \lambda_{k_m}$. Let $\Psi_0 \in \mathcal{D} \cap T_S\psi_1(T)$ of the form $\Psi_0 = \beta_1\phi_{k_1} + \dots + \beta_m\phi_{k_m}$ where $\beta_1, \dots, \beta_m \in \mathbb{C}$ and $\alpha_1\beta_1 + \dots + \alpha_m\beta_m \neq 0$. Any solution of (6) satisfies, for $j \in \{1, \dots, m\}$,

$$\langle \Psi(T), \phi_{k_j}\rangle = \left(\langle\Psi_0, \phi_{k_j}\rangle + i\left\langle \langle\mu\phi_1, \phi_{k_j}\rangle, \int_0^T v(t)e^{i\omega_k t}\,dt\right\rangle_{\mathbb{R}^n}\right)e^{-i\lambda_k T}.$$

We then have

$$\alpha_1\langle\Psi(T), \phi_{k_1}\rangle + \dots + \alpha_m\langle\Psi(T), \phi_{k_m}\rangle = (\alpha_1\beta_1 + \dots + \alpha_m\beta_m)e^{-i\lambda_k T} \neq 0,$$

implying that Ψ_0 is not zero controllable in time T.

(3) Let Ω be a domain of $\mathbb{R}^3$ with C^1 boundary and $\mu \in C^0(\overline{\Omega}, \mathbb{R}^3)$ of the form (18) be such that (*Kal*) holds true (otherwise, we already know that (3) is not spectral controllable thanks to (2)). Let $T > 0$. Thanks to Weyl's formula, we have

$$\sharp\{\omega_k \in [0,t]\} = dt^{3/2} + O(t^\alpha), \quad \text{when } t \to +\infty,$$

where $d \in (0, +\infty)$ and $\alpha \in (0, 3/2)$. Thus, there exists a subsequence $(\omega_{\sigma(k)})_{k\in\mathbb{N}^*}$ of $(\omega_k)_{k\in\mathbb{N}^*}$ such that

$$\sharp\{k \in \mathbb{N}^*;\ \omega_{\sigma(k)} \in [0,t]\} = d't + O(t^{\alpha'}) \quad \text{when } t \to +\infty,$$

for some $d' > T/2\pi$ and some $\alpha' \in (0,1)$. Theorem 3.2 ensures that the family

$$\{e^{i\omega_{\sigma(k)}t}, e^{-i\omega_{\sigma(k)}t};\ k \in \mathbb{N}^*\}$$

is not minimal in $L^2((0,T),\mathbb{C})$. Thus, the family $\{1, e^{i\omega_k t}, e^{-i\omega_k t};\ k \geqslant 2\}$ is not minimal in $L^2((0,T),\mathbb{C})$. Therefore, (3) is not spectral controllable. $\Box$

Remark 3.3. When a domain Ω of $\mathbb{R}^2$ with C^1 boundary and $\mu \in C^0(\overline{\Omega}, \mathbb{R}^2)$ are such that (*Kal*) holds but (18) does not hold, then $T_{\min}(\Omega) := 2\pi d(\Omega)$ may not be the minimal time for the spectral controllability of (3). Indeed, let us consider $\mu = (\mu^{(1)}, \mu^{(2)})$ such that

$$\langle \mu^{(1)}\phi_1, \phi_k \rangle \neq 0 \quad \text{if and only if} \quad k \in \mathbb{N}^* \text{ is odd} \quad \text{and}$$
$$\langle \mu^{(2)}\phi_1, \phi_k \rangle \neq 0 \quad \text{if and only if} \quad k \in \mathbb{N}^* \text{ is even.}$$

Then, the minimal time for the spectral controllability of (3) is $T_{\min}(\Omega, \mu) = \pi d(\Omega)$.

Remark 3.4. In order to remove the assumption (18), one could try to adapt Haraux and Jaffard's result to families of vector exponentials of the form

$$\{b_k e^{i\omega_k t};\ k \in \mathbb{Z}\}$$

where $b_k \in \mathbb{R}^n - \{0\}$. Indeed, the spectral controllability of (3) is equivalent to the minimality in $L^2((0,T), \mathbb{C}^n)$ of this family with $b_k = \langle \mu\phi_1, \phi_k \rangle$. This generalization is an open problem.

3.3. Proof of Theorem 2.6

The goal of this section is the proof of Theorem 2.6. The proof of the statement (1) can be deduced from the following lemma in the same way as the proof of Theorem 2.5(1.a) was deduced from Theorem 3.2(1).

Lemma 3.3. *Let $(\mu_k)_{k\in\mathbb{Z}}$ be a sequence of real numbers such that $\mu_0 = 0$ and*

$$\sharp\{k \in \mathbb{Z};\ 0 \leqslant \mu_k \leqslant t\} = dt + O(t^\alpha),$$
$$\sharp\{k \in \mathbb{Z};\ -t \leqslant \mu_k \leqslant 0\} = dt + O(t^\alpha),$$

for some $d > 0$ and $\alpha \in (0,1)$. Then, for every $T > 2\pi d$, the family $\{t, e^{i\mu_k t};\ k \in \mathbb{Z}\}$ is minimal in $L^2((0,T), \mathbb{C})$.

Proof of Lemma 3.3. Let $T > 2\pi d$ and let us assume that the family $\{t, e^{i\mu_k t};\ k \in \mathbb{Z}\}$ is not minimal in $L^2((0,T), \mathbb{C})$. Thanks to Theorem 3.2, the family $\{e^{i\mu_k t};\ k \in \mathbb{Z}\}$ is minimal in $L^2((0,T), \mathbb{C})$ thus, necessarily,

$$t \in \mathrm{Cl}_{L^2((0,T),\mathbb{C})}\big(\mathrm{Span}\{e^{i\mu_k t};\ k \in \mathbb{Z}\}\big). \tag{22}$$

With successive integrations, we see that

$$t^k \in \mathrm{Cl}_{C^0([0,T],\mathbb{C})}\big(\mathrm{Span}\{t, e^{i\mu_k t};\ k \in \mathbb{Z}\}\big), \quad \forall k \in \mathbb{N},\ k \geqslant 2.$$

The Stone Weirstrass theorem ensures that $\mathrm{Span}\{1, t^k;\ k \in \mathbb{N},\ k \geqslant 2\}$ is dense in $C^0([0,T], \mathbb{C})$, thus it is also dense in $L^2((0,T), \mathbb{C})$. From (22), we deduce that the vector space $\mathrm{Span}\{e^{i\mu_k t};\ k \in \mathbb{Z}\}$ is dense in $L^2((0,T), \mathbb{C})$. This is a contradiction, because, thanks to Theorem 3.2, for every $\alpha \in \mathbb{R} - \{\mu_k;\ k \in \mathbb{Z}\}$, the family $\{e^{i\alpha t}, e^{i\mu_k t};\ k \in \mathbb{Z}\}$ is minimal in $L^2((0,T), \mathbb{C})$ i.e.,

$$e^{i\alpha t} \notin \mathrm{Cl}_{L^2((0,T),\mathbb{C})}\big(\mathrm{Span}\{e^{i\mu_k t};\ k \in \mathbb{Z}\}\big). \qquad \square$$

Item (2) of Theorem 2.6 is a direct consequence of Theorem 2.5(2). The proof of the statement (3) of Theorem 2.6 involves different ideas. A useful preliminary result is stated in the next lemma (see [31, Lecture 2, Section 2.3, pp. 10–11]) that has already been used in [11, Lemma 16] for similar purposes.

Lemma 3.4. *Let $f:\mathbb{C}\to\mathbb{C}$ be a holomorphic function such that*

$$\exists C_0>0,\ \textit{such that}\ \forall s\in\mathbb{C},\quad \bigl|f(s)\bigr|\leqslant C_0e^{C_0|s|}.$$

Assume that $f\neq 0$. Let $n:[0,+\infty)\to\mathbb{N}$ be defined by

$$n(R):=\sharp\bigl\{s\in\mathbb{C};\ f(s)=0\ \textit{and}\ |s|\leqslant R\bigr\}.$$

Then,

$$\exists C_1>0,\ \forall R\in(1,+\infty),\quad \int_1^R\frac{n(t)}{t}\,dt\leqslant C_1R.$$

Proof of item (3) of Theorem 2.6. Let Ω be a regular domain of $\mathbb{R}^3$ such that (*Kal*) holds (otherwise, the system (4) is already known to be nonspectral controllable thanks to (2)). Let $T>0$ and $m\in\mathbb{N}^*$. We assume $(i\phi_m,0,e_l)$ is zero controllable in time T for $l=1,2,3$: there exists $v_l\in L^2((0,T),\mathbb{R}^3)$ such that

$$\begin{cases}\displaystyle\Bigl\langle\langle\mu\phi_1,\phi_k\rangle_{L^2(\Omega)},\int_0^T v_l(t)e^{i\omega_k t}\,dt\Bigr\rangle_{\mathbb{R}^3}=-\delta_{k,m},\quad\forall k\geqslant 2,\\ \displaystyle\int_0^T v_l(t)\,dt=0,\\ \displaystyle\int_0^T tv_l(t)\,dt=e_l,\end{cases}\tag{23}$$

for $l=1,2,3$. In particular, for every $k\in\mathbb{N}^*-\{1,m\}$, the vector $\langle\mu\phi_1,\phi_k\rangle_{L^2(\Omega)}\in\mathbb{R}^3-\{0\}$ belongs to the kernel of the matrix $C(i\omega_k)$, where

$$C(\lambda):=\begin{pmatrix}\int_0^T v_1^{(1)}(t)e^{\lambda t}\,dt & \int_0^T v_1^{(2)}(t)e^{\lambda t}\,dt & \int_0^T v_1^{(3)}(t)e^{\lambda t}\,dt\\ \int_0^T v_2^{(1)}(t)e^{\lambda t}\,dt & \int_0^T v_2^{(2)}(t)e^{\lambda t}\,dt & \int_0^T v_2^{(3)}(t)e^{\lambda t}\,dt\\ \int_0^T v_3^{(1)}(t)e^{\lambda t}\,dt & \int_0^T v_3^{(2)}(t)e^{\lambda t}\,dt & \int_0^T v_3^{(3)}(t)e^{\lambda t}\,dt\end{pmatrix}.$$

Thus $G(\lambda):=\det[C(\lambda)]$ satisfies $G(i\omega_k)=0$, for every $k\in\mathbb{N}^*-\{1,m\}$. It is easy to see that G is a holomorphic function verifying the growth condition of Lemma 3.4. Then using Weyl's formula and Lemma 3.4, we deduce that $G\equiv 0$. However, thanks to the last two equalities in (23), we have

$$C(\lambda) = \begin{pmatrix} \lambda + o(\lambda) & o(\lambda) & o(\lambda) \\ o(\lambda) & \lambda + o(\lambda) & o(\lambda) \\ o(\lambda) & o(\lambda) & \lambda + o(\lambda) \end{pmatrix} \quad \text{when } \lambda \to 0,$$

so $G(\lambda) = \lambda^3 + o(\lambda^3) \neq 0$ when $\lambda \to 0$, which is a contradiction. □

4. 2D exact controllability in abstract spaces

The goal of this section is the proof of the following result.

Theorem 4.1. *Let Ω be a domain of $\mathbb{R}^2$ with C^1 boundary and $\mu \in C^0(\overline{\Omega}, \mathbb{R}^2)$ be of the form* (18) *such that condition* (*Kal*) *holds true. Let $d \in (0, +\infty)$ and $\alpha \in (0, 1)$ be such that* (20) *holds, $T > 2\pi d$ and $(x_m)_{m \in \mathbb{N}^*} \subset \mathbb{R}_+^*$ be such that $\sum_{m=1}^{\infty} x_m = 1$.*

For every $m \in \mathbb{N}^*$, there exists $C_m > 0$ such that, for every $\varphi_T \in T_S\psi_1(T)$, the solution of

$$\begin{cases} i\dfrac{\partial \varphi}{\partial t} = -\Delta\varphi, & (t, q) \in \mathbb{R}_+ \times \Omega, \\ \varphi(t, q) = 0, & (t, q) \in \mathbb{R} \times \partial\Omega, \\ \varphi(T) = \varphi_T, \end{cases} \tag{24}$$

satisfies

$$C_m \left| \langle \varphi_T, \phi_m \rangle \right|^2 \leqslant \int_0^T \left| \Im\langle \tilde{\mu}\psi_1(t), \varphi(t) \rangle \right|^2 dt, \quad \forall m \in \mathbb{N}^*. \tag{25}$$

We introduce the Hilbert spaces

$$H^* := \left\{ \varphi\colon\ \Omega \to \mathbb{C};\ \Re\langle \varphi, \psi_1(T) \rangle = 0 \text{ and } \sum_{m=1}^{\infty} C_m x_m \left| \langle \varphi, \phi_m \rangle \right|^2 < +\infty \right\},$$

$$H := \left\{ \varphi : \Omega \to \mathbb{C};\ \Re\langle \varphi, \psi_1(T) \rangle = 0 \text{ and } \sum_{m=1}^{\infty} \frac{1}{C_m x_m} \left| \langle \varphi, \phi_m \rangle \right|^2 < +\infty \right\}.$$

Then, for every $\Psi_f \in H$, there exists $\tilde{v} \in L^2((0, T), \mathbb{R})$ such that the solution of (6) *with $\Psi_0 = 0$ and $v(t) = \tilde{v}(t)e_1$ satisfies $\Psi(T) = \Psi_f$.*

Remark 4.1. Notice that $T_S\psi_1(T) \subset H^*$ and $H \subset T_S\psi_1(T)$ because $C_m x_m \to 0$ when $m \to +\infty$. The space H is a regular space, its regularity depends on the asymptotic behavior of the sequence $(C_m x_m)_{m \in \mathbb{N}^*}$.

Remark 4.2. The spaces H and H^* are defined in order to have an observability inequality in H^*. Indeed, considering the product of the inequality (25) with x_m and summing over $m \in \mathbb{N}^*$, we get

$$\|\varphi_T\|_{H^*}^2 \leqslant \int_0^T \left|\Im\langle\tilde{\mu}\psi_1(t), \varphi(t)\rangle\right|^2 dt, \quad \forall \varphi_T \in H^*. \tag{26}$$

Remark 4.3. Trying to apply the classical approach in order to get the controllability thanks to (26), we introduce the functional

$$\begin{aligned} J:\quad & H^* \to \mathbb{R}, \\ & \varphi_T \mapsto \frac{1}{2}\int_0^T \left|\Im\langle\tilde{\mu}\psi_1(t), \varphi(t)\rangle\right|^2 dt + \Re\langle\varphi_T, \Psi_f\rangle. \end{aligned}$$

In the classical situation, J is continuous, convex and coercive on H^*, thus $\inf\{J(\varphi_T);\ \varphi_T \in H^*\}$ is achieved at some point φ_T. Writing $dJ(\varphi_T) = 0$, we get a control $\tilde{v}(t) := \Im\langle\tilde{\mu}\psi_1(t), \varphi(t)\rangle$ that steers (3) from $\Psi(0) = 0$ to $\Psi(T) = \Psi_f$.

In our situation, this classical approach does not work because the functional J may not be well defined on H^*. Thus, an adaptation of this approach is needed.

Proof of Theorem 4.1. First, let us prove (25). For $\varphi_T \in T_S\psi_1(T)$, the solution of (24) is

$$\varphi(t) = \sum_{k=1}^{\infty} \langle\varphi_T, \phi_k\rangle e^{-i\lambda_k(t-T)}\phi_k$$

so

$$\Im\langle\tilde{\mu}\psi_1(t), \varphi(t)\rangle = \sum_{k=2}^{\infty} \frac{\langle\tilde{\mu}\phi_1, \phi_k\rangle}{2i}\left(\overline{\langle\varphi_T, \phi_k\rangle}e^{i\lambda_k(t-T)} - \langle\varphi_T, \phi_k\rangle e^{-i\lambda_k(t-T)}\right).$$

Applying Theorem 3.2 and Proposition 3.1, there exists a constant $\tilde{C}_m > 0$ such that, for every $\varphi_T \in T_S\psi_1(T)$,

$$\tilde{C}_m\left|\langle\tilde{\mu}\phi_1, \phi_m\rangle\right|^2\left|\langle\varphi_T, \phi_m\rangle\right|^2 \leqslant \int_0^T \left|\Im\langle\tilde{\mu}\psi_1(t), \varphi(t)\rangle\right|^2 dt.$$

We get (25) with $C_m := \tilde{C}_m|\langle\tilde{\mu}\phi_1, \phi_m\rangle|^2$.

Now, let us prove the controllability result. Let $\Psi_f \in H$. For $\epsilon > 0$ we introduce the functional $J_\epsilon : T_S\psi_1(T) \to \mathbb{R}$,

$$J_\epsilon(\varphi_T) := \frac{1}{2}\int_0^T \left|\Im\langle\tilde{\mu}\psi_1(t), \varphi(t)\rangle\right|^2 dt + \Re\langle\Psi_f, \varphi_T\rangle + \epsilon\|\varphi_T\|_{L^2(\Omega)}^2,$$

where φ is the solution of (24). The functional J_ϵ is convex, continuous and coercive because

$$J_\epsilon(\varphi_T) \geqslant \epsilon\|\varphi_T\|_{L^2}^2 - \|\Psi_f\|_{L^2}\|\varphi_T\|_{L^2}.$$

Thus, there exists $\varphi_T^\epsilon \in T_S\psi_1(T)$ such that

$$J_\epsilon\big(\varphi_T^\epsilon\big) = \min\big\{J_\epsilon(\varphi_T);\ \varphi_T \in T_S\psi_1(T)\big\}.$$

Then, φ_T^ϵ solves the Euler equation associated to this optimization problem,

$$\int_0^T \tilde{v}_\epsilon(t)\Im\big\langle\tilde{\mu}\psi_1(t), \xi(t)\big\rangle\,dt + \Re\langle\Psi_f, \xi_T\rangle + 2\epsilon\Re\big\langle\varphi_T^\epsilon, \xi_T\big\rangle = 0, \quad \forall \xi_T \in T_S\psi_1(T), \tag{27}$$

where

$$\tilde{v}_\epsilon(t) := \Im\big\langle\tilde{\mu}\psi_1(t), \varphi^\epsilon(t)\big\rangle,$$

φ^ϵ (resp. ξ) is the solution of (24) with $\varphi_T = \varphi_T^\epsilon$ (resp. $\varphi_T = \xi_T$).

For $0 < \epsilon_1 < \epsilon_2$, we have $J_{\epsilon_1} \leqslant J_{\epsilon_2}$ thus the sequence $(J_\epsilon(\varphi_T^\epsilon))_{\epsilon>0}$ decreases when ϵ decreases to zero. Thus,

$$J_\epsilon\big(\varphi_T^\epsilon\big) \leqslant M_1 := J_1\big(\varphi_T^1\big), \quad \forall \epsilon \in (0,1).$$

There exists $M_2 > 0$ such that,

$$\big\|\varphi_T^\epsilon\big\|_{H^*} \leqslant M_2, \quad \forall \epsilon \in (0,1).$$

Indeed, thanks to (26), we have,

$$M_1 \geqslant J_\epsilon\big(\varphi_T^\epsilon\big) \geqslant \frac{1}{2}\big\|\varphi_T^\epsilon\big\|_{H^*}^2 - \|\Psi_f\|_H\big\|\varphi_T^\epsilon\big\|_{H^*}.$$

The sequence $(\tilde{v}_\epsilon)_{\epsilon\in(0,1)}$ is bounded in $L^2((0,T),\mathbb{R})$. Indeed, we have

$$M_1 \geqslant J_\epsilon\big(\varphi_T^\epsilon\big) \geqslant \frac{1}{2}\|\tilde{v}_\epsilon\|_{L^2}^2 - \|\Psi_f\|_H\big\|\varphi_T^\epsilon\big\|_{H^*},$$

thus

$$\|\tilde{v}_\epsilon\|_{L^2(0,T)}^2 \leqslant 2\big(M_1 + M_2\|\Psi_f\|_H\big).$$

Therefore, there exists $\tilde{v} \in L^2((0,T),\mathbb{R})$ such that $\tilde{v}_\epsilon \to \tilde{v}$ weakly in $L^2((0,T),\mathbb{R})$. Passing to the limit $\epsilon \to 0$ in (27) with $\xi_T \in H$, we get

$$\int_0^T \tilde{v}(t)\Im\big\langle\tilde{\mu}\psi_1(t), \xi(t)\big\rangle\,dt + \Re\langle\Psi_f, \xi_T\rangle = 0, \quad \forall \xi_T \in H,$$

because

$$\big|2\epsilon\Re\big\langle\varphi_T^\epsilon, \xi_T\big\rangle\big| \leqslant 2\epsilon\big\|\varphi_T^\epsilon\big\|_{H^*}\|\xi_T\|_H \leqslant 2\epsilon M_2\|\xi_T\|_H.$$

Since H is dense in $T_S\psi_1(T)$, we have

$$\int_0^T \tilde{v}(t)\Im\langle\tilde{\mu}\psi_1(t),\xi(t)\rangle dt + \Re\langle\Psi_f,\xi_T\rangle = 0, \quad \forall\xi_T \in T_S\psi_1(T). \tag{28}$$

Let Ψ be the solution of (6) with $\Psi_0 = 0$. Using the fact that ξ solves (24) and Ψ solves (6) with $\Psi_0 = 0$, we deduce from (28) that

$$\Re\langle\Psi(T),\xi_T\rangle = \Re\langle\Psi_f,\xi_T\rangle, \quad \forall\xi_T \in T_S\psi_1(T).$$

Thus $\Psi(T) = \Psi_f$. $\square$

Remark 4.4. A uniform gap condition for the eigenvalues of $-\Delta_\Omega^D$, cf. (12), would imply that the constants C_m, $m \in \mathbb{N}^*$ admit a uniform positive lower bound and, in that case, H can be taken as the subset of $T_S\psi_1(T)$ made of the functions ϕ with $H^{1+\epsilon}$ finite norm. As we mentioned before, the existence of a planar domain verifying (12) is not even known. One could maybe define weaker gaps conditions in order to relate H to some Sobolev spaces.

5. Generic spectral controllability for the quantum box

The goal of this Section is the proof of Proposition 2.8.

Consider $\mu \in C^1(\mathbb{R}^2,\mathbb{R}^2)$. If μ is not nowhere constant, then there exists an open ball B where μ is constant. Taking an open neighborhood of domains of $\mathbb{D}_3$ included in B, condition (*NonZ*) will never be satisfied for those domains, thus property (A) is not generic in $\mathbb{D}_3$.

For the rest of the section, we fix $\mu \in C^1(\mathbb{R}^2,\mathbb{R}^2)$ which is nowhere constant.

In Section 5.1, we reduce the proof of the genericity of property (A) (Proposition 2.8) to the proof of the genericity of a weaker property (B_k). In Section 5.2, we present the strategy for the proof of the genericity of property (B_k): it is sufficient to prove a weaker result, stated in Proposition 5.4. In Section 5.3, we present the strategy for the proof of Proposition 5.4. In Section 5.4, we perform some preliminary results for the proof of Proposition 5.4, which is achieved in Section 5.5.

5.1. Reduction of the problem

The goal of this section is to reduce the proof of the genericity of the property (A), (Proposition 2.8) to the proof of the genericity of a weaker property (B_k). For that purpose, we introduce the properties (A_k) and (B_k).

For the rest of the paper, the notations $\lambda_j^{\Omega_0}$ and $\phi_j^{\Omega_0}$ are used to denote respectively the jth eigenvalue and one corresponding normalized eigenvector associated to $-\Delta_{\Omega_0}^D$. If, in the course of a definition or an argument, one domain under consideration is denoted Ω, then we simply use λ_j and ϕ_j instead of λ_j^Ω and ϕ_j^Ω.

Definition 5.1. Let $k \in \mathbb{N}^*$, $k \geqslant 2$ and $\Omega \in \mathbb{D}_3$. We say that Ω satisfies property (A_k) if

$$\int_\Omega \mu(q)\phi_1(q)\phi_k(q)\,dq \neq 0.$$

Definition 5.2. Let $k \in \mathbb{N}^*$, $k \geqslant 2$ and $\Omega \in \mathbb{D}_3$. We say that Ω satisfies property (B_k) if

$$\text{either} \quad \int_\Omega \mu(q)\phi_1(q)\phi_k(q)\,dq \neq 0,$$

$$\text{or} \quad \int_\Omega \mu(q)\phi_1(q)\phi_k(q)\,dq = 0 \quad \text{and } M(\cdot) \text{ is not identically equal to zero,} \tag{29}$$

where $M : \partial\Omega \to \mathbb{R}^2$ is given by

$$M(q) := \frac{\partial\phi_1}{\partial\nu}(q)\frac{\partial\xi_k}{\partial\nu}(q) + \frac{\partial\phi_k}{\partial\nu}(q)\frac{\partial\xi_1}{\partial\nu}(q), \tag{30}$$

ν is the unit outward normal to $\partial\Omega$ and ξ_1, ξ_k are the solutions of the following systems,

$$\begin{cases} -(\Delta+\lambda_1)\xi_k = \mu\phi_k, & \text{in } \Omega, \\ \xi_k = 0, & \text{on } \partial\Omega, \\ \int_\Omega \xi_k\phi_1 = 0, \end{cases} \qquad \begin{cases} -(\Delta+\lambda_k)\xi_1 = \mu\phi_1, & \text{in } \Omega, \\ \xi_1 = 0, & \text{on } \partial\Omega, \\ \int_\Omega \xi_1\phi_k = 0. \end{cases} \tag{31}$$

A first reduction is given in the next proposition. Its proof is standard and relies on Baire Lemma, we write it for sake of completeness.

Proposition 5.1. *If (A_k) is generic in $\mathbb{D}_3$ for every $k \geqslant 2$, then (A) is generic in $\mathbb{D}_3$.*

Proof of Proposition 5.1. Let $\Omega \in \mathbb{D}_3$. We want to prove that the set

$$\mathcal{G} := \{u \in W^{4,\infty}(\Omega, \mathbb{R}^2);\ \Omega + u \text{ satisfies } (A)\}$$

is dense in $W^{4,\infty}(\Omega, \mathbb{R}^2)$. For $k \in \mathbb{N}^*$, we introduce the set $\mathcal{G}_k$ of functions $u \in W^{4,\infty}(\Omega, \mathbb{R}^2)$ such that

$$\lambda_1^{\Omega+u} < \cdots < \lambda_k^{\Omega+u} \leqslant \lambda_{k+1}^{\Omega+u} \leqslant \cdots \quad \text{and}$$

$$\int_{\Omega+u} \mu(q)\phi_1^{\Omega+u}(q)\phi_j^{\Omega+u}(q)\,dq \neq 0, \quad \forall j \in \{2, \ldots, k\}.$$

Then, $\mathcal{G}_1 = W^{4,\infty}(\Omega, \mathbb{R}^2)$, $\mathcal{G}_{k+1}$ is an open subset of $\mathcal{G}_k$ for every $k \in \mathbb{N}^*$ (thanks to the continuity of $u \mapsto \lambda_j^{\Omega+u}$ and $u \mapsto \phi_j^{\Omega+u}$ for $j = 2, \ldots, k+1$) and $\mathcal{G} = \bigcap_{k\in\mathbb{N}^*} \mathcal{G}_k$. Thanks to Baire Lemma, it is sufficient to prove that, for every $k \in \mathbb{N}^*$, $\mathcal{G}_{k+1}$ is a dense in $\mathcal{G}_k$.

Let $k \in \mathbb{N}^*$, $u_0 \in \mathcal{G}_k - \mathcal{G}_{k+1}$ and $\epsilon > 0$. We have

$$\lambda_1^{\Omega_0} < \cdots < \lambda_k^{\Omega_0} \leqslant \lambda_{k+1}^{\Omega_0} \leqslant \cdots, \qquad \int_{\Omega_0} \mu(q)\phi_1^{\Omega_0}(q)\phi_j^{\Omega_0}(q)\,dq \neq 0, \quad \forall j \in \{2, \ldots, k\},$$

and $\lambda_k^{\Omega_0} = \lambda_{k+1}^{\Omega_0}$ or

$$\int_{\Omega_0} \mu(q)\phi_1^{\Omega_0}(q)\phi_{k+1}^{\Omega_0}(q)\,dq = 0,$$

where $\Omega_0 := \Omega + u_0$. Thanks to the generic simplicity of the eigenvalues of the Laplacian and the continuity of $u \mapsto \phi_j^{\Omega_0+u}$ for $2 \leqslant j \leqslant k$ (see [21]), there exists $u_1 \in W^{4,\infty}(\Omega_0, \mathbb{R}^2)$ with $\|u_1\|_{W^{4,\infty}} < \epsilon$ such that

$$\lambda_1^{\Omega_1} < \cdots < \lambda_k^{\Omega_1} < \lambda_{k+1}^{\Omega_1} \leqslant \cdots \quad \text{and} \quad \int_{\Omega_1} \mu(q)\phi_1^{\Omega_1}(q)\phi_j^{\Omega_1}(q)\,dq \neq 0, \quad \forall j \in \{2, \ldots, k\},$$

where $\Omega_1 := \Omega_0 + u_1$. Thanks to the genericity of (A_{k+1}) and the continuity of $u \mapsto \lambda_j^{\Omega_1+u}$ for $2 \leqslant j \leqslant k+1$, $u \mapsto \phi_j^{\Omega_1+u}$ for $2 \leqslant j \leqslant k$ there exists $u_2 \in W^{4,\infty}(\Omega_1, \mathbb{R}^2)$ with $\|u_2\|_{W^{4,\infty}} < \epsilon$, such that

$$\lambda_1^{\Omega_2} < \cdots < \lambda_k^{\Omega_2} < \lambda_{k+1}^{\Omega_2} \leqslant \cdots \quad \text{and} \quad \int_{\Omega_2} \mu(q)\phi_1^{\Omega_2}(q)\phi_j^{\Omega_2}(q)\,dq \neq 0, \quad \forall j \in \{2, \ldots, k+1\}.$$

Then, $u := (I + u_2) \circ (I + u_1) \circ (I + u_0) - I$ is arbitrarily close to u_0 in $W^{4,\infty}(\Omega, \mathbb{R}^2)$ and $u \in \mathcal{G}_{k+1}$. $\square$

A second reduction is given in the next proposition. Its proof is also standard. The argument goes by contradiction and relies on shape differentiation with respect to the domain Ω. It has been introduced by Albert [3] and recently used in [11]. We gathered in Appendix A well-known facts about shape differentiation which will be used in the proof.

Proposition 5.2. *Let $k \geqslant 2$. If (B_k) is generic in $\mathbb{D}_3$, then (A_k) is generic in $\mathbb{D}_3$.*

Proof of Proposition 5.2. Let $\Omega_0 \in \mathbb{D}_3$, $k \in \mathbb{N}^*$, $k \geqslant 2$. We want to prove that the set

$$\mathcal{G} := \{u \in W^{4,\infty}(\Omega_0, \mathbb{R}^2);\ \Omega_0 + u \text{ satisfies } (A_k)\}$$

is dense in $W^{4,\infty}(\Omega_0, \mathbb{R}^2)$. We argue by contradiction. Let us assume the existence of $u_0 \in W^{4,\infty}(\Omega_0, \mathbb{R}^2)$ and $\rho_0 > 0$ such that, for every $u \in W^{4,\infty}(\Omega_0, \mathbb{R}^2)$ with $\|u_0 - u\|_{W^{4,\infty}} < \rho_0$, we have $u \notin \mathcal{G}$. Thanks to the genericity of (B_k), we can assume that $\Omega := \Omega_0 + u_0$ satisfies (B_k). Then, there exists $\rho > 0$ such that, for every $u \in E_\rho(\Omega) := \{v \in W^{4,\infty}(\Omega, \mathbb{R}^2);\ \|v\|_{W^{4,\infty}} < \rho\}$, we have

$$\int_{\Omega+u} \mu(q)\phi_1^{\Omega+u}(q)\phi_k^{\Omega+u}(q)\,dq = 0, \quad \forall u \in E_\rho(\Omega). \tag{32}$$

Thus, the directional derivative of the integral appearing in (32) in the direction u is equal to zero, for every $u \in E_\rho(\Omega)$. By classical results on shape differentiation (cf. [42] or Appendix A below), we get

$$\int_{\Omega} \mu\big(\phi_1'(u)\phi_k + \phi_1\phi_k'(u)\big)\,dq = 0, \quad \forall u \in E_\rho(\Omega), \tag{33}$$

where $\phi_1'(u)$ et $\phi_k'(u)$ are solutions of

$$\begin{cases} -(\Delta+\lambda_1)\phi_1'(u) = \lambda_1'(u)\phi_1, & \text{in } \Omega, \\ \phi_1'(u) = -\langle u, \nabla\phi_1\rangle, & \text{on } \partial\Omega, \\ \int_{\Omega} \phi_1\phi_1'(u) = 0, \end{cases} \qquad \begin{cases} -(\Delta+\lambda_k)\phi_k'(u) = \lambda_k'(u)\phi_k, & \text{in } \Omega, \\ \phi_k'(u) = -\langle u, \nabla\phi_k\rangle, & \text{on } \partial\Omega, \\ \int_{\Omega} \phi_k\phi_k'(u) = 0. \end{cases} \tag{34}$$

In order to transform (33) in a linear form in u, we introduce the dual systems (31). Note that these systems have unique solutions, thanks to (32). Using Green's second formula and systems (34), we have

$$\begin{aligned} -\int_{\Omega} &\mu\big(\phi_1'(u)\phi_k + \phi_1\phi_k'(u)\big)\,dq \\ &= \int_{\Omega} \phi_1'(u)(\Delta+\lambda_1)\xi_k\,dq + \int_{\Omega} \phi_k'(u)(\Delta+\lambda_k)\xi_1\,dq \\ &= \int_{\Omega} (\Delta+\lambda_1)\phi_1'(u)\xi_k\,dq + \int_{\partial\Omega} \left(\phi_1'(u)\frac{\partial\xi_k}{\partial\nu} - \xi_k\frac{\partial\phi_1'(u)}{\partial\nu}\right)d\sigma(q) \\ &\quad + \int_{\Omega} (\Delta+\lambda_k)\phi_k'(u)\xi_1\,dq + \int_{\partial\Omega} \left(\phi_k'(u)\frac{\partial\xi_1}{\partial\nu} - \xi_1\frac{\partial\phi_k'(u)}{\partial\nu}\right)d\sigma(q) \\ &= \int_{\partial\Omega} \left(\phi_1'(u)\frac{\partial\xi_k}{\partial\nu} + \phi_k'(u)\frac{\partial\xi_1}{\partial\nu}\right)d\sigma(q). \end{aligned}$$

Then, (33) is equivalent to

$$\int_{\partial\Omega} \langle u, \nu\rangle \left(\frac{\partial\phi_1}{\partial\nu}\frac{\partial\xi_k}{\partial\nu} + \frac{\partial\phi_k}{\partial\nu}\frac{\partial\xi_1}{\partial\nu}\right)d\sigma(q) = 0, \quad \forall u \in E_\rho(\Omega). \tag{35}$$

This implies that $M \equiv 0$ which is a contradiction because Ω satisfies (B_k). $\square$

5.2. *Proof strategy for the genericity of* (B_k)

According to Propositions 5.1 and 5.2, it remains to show that the property (B_k) is generic in $\mathbb{D}_3$ for every $k \geqslant 2$. To proceed in that direction, fix $k \geqslant 2$ and $\Omega \in \mathbb{D}_3$. Without loss of generality, we assume from now that

1. the spectrum of $-\Delta_D$ is simple on Ω;
2. there exists $\bar{q} \in \partial\Omega$ such that

$$d\mu(\bar{q}) \cdot \tau_{\bar{q}} \neq 0, \tag{36}$$

where $\tau_{\bar{q}}$ is the unit tangent vector on $\partial\Omega$ at the point $\bar{q}$.

Indeed, the second condition is generic and open. Therefore, for a given domain $\Omega \in \mathbb{D}_3$, one can choose an arbitrarily close domain $\Omega' \in \mathbb{D}_3$ verifying condition 2. The latter holding in an open neighborhood of Ω', one can pick a domain $\Omega'' \in \mathbb{D}_3$ arbitrarily close to Ω verifying both conditions 1 and 2.

Arguing by contradiction, we assume there exists $\rho > 0$ such that

$$\int\limits_{\Omega+u} \mu(q)\phi_1^{\Omega+u}(q)\phi_k^{\Omega+u}(q)\,dq = 0, \quad \forall u \in E_\rho(\Omega), \tag{37}$$

and

$$M(u) \equiv 0 \quad \text{on } \partial\Omega + u,\ \forall u \in E_\rho(\Omega), \tag{38}$$

where $E_\rho(\Omega) := \{v \in W^{4,\infty}(\Omega, \mathbb{R}^2);\ \|v\|_{W^{4,\infty}} < \rho\}$ and $M(u) : \partial(\Omega + u) \to \mathbb{R}^2$ is defined by

$$M(u)(q) = \frac{\partial\phi_k(u)}{\partial\nu}(q)\frac{\partial\xi_1(u)}{\partial\nu}(q) + \frac{\partial\phi_1(u)}{\partial\nu}(q)\frac{\partial\xi_k(u)}{\partial\nu}(q), \tag{39}$$

where $\phi_1(u)$, and $\phi_k(u)$ are normalized eigenvectors of $\Delta^D_{\Omega+u}$ associated to $\lambda_1(u)$ and $\lambda_k(u)$ respectively and $\xi_1(u)$ and $\xi_k(u)$ are the solutions of (31) associated to $\Omega + u$. (Such systems have solutions since (37) holds true.) In the sequel, we (sometimes) drop the variable (u) when it corresponds to $u = 0$.

The next step consists in shape differentiating the condition $M(u) \equiv 0$ for $u \in E_\rho(\Omega)$. Applying the classical shape differentiation formula regarding Dirichlet boundary condition (see Theorem A.2), we get

$$M'(u) = -\langle u, \nu\rangle \frac{\partial M(0)}{\partial\nu} \quad \text{on } \partial\Omega. \tag{40}$$

Remark 5.1. For technical details on regular extension of outward normal vector, we refer to [42, Théorème 4.1, Chapitre IV, p. 69].

After computations, we get

$$\begin{aligned}
&\left(\frac{\partial\phi_k}{\partial\nu}\right)'(u)\frac{\partial\xi_1}{\partial\nu} + \frac{\partial\phi_k}{\partial\nu}\left(\frac{\partial\xi_1}{\partial\nu}\right)'(u) + \left(\frac{\partial\phi_1}{\partial\nu}\right)'(u)\frac{\partial\xi_k}{\partial\nu} + \frac{\partial\phi_1}{\partial\nu}\left(\frac{\partial\xi_k}{\partial\nu}\right)'(u) \\
&\quad = -\langle u, \nu\rangle\left(\frac{\partial}{\partial\nu}\left(\frac{\partial\phi_k}{\partial\nu}\right)\frac{\partial\xi_1}{\partial\nu} + \frac{\partial\phi_k}{\partial\nu}\frac{\partial}{\partial\nu}\left(\frac{\partial\xi_1}{\partial\nu}\right) + \frac{\partial}{\partial\nu}\left(\frac{\partial\phi_1}{\partial\nu}\right)\frac{\partial\xi_k}{\partial\nu}\right. \\
&\qquad \left. + \frac{\partial\phi_1}{\partial\nu}\frac{\partial}{\partial\nu}\left(\frac{\partial\xi_k}{\partial\nu}\right)\right) \quad \text{on } \partial\Omega.
\end{aligned} \tag{41}$$

The relation between the first shape derivative of a normal derivative $(\frac{\partial\phi}{\partial\nu})'(u)$ and the normal derivative of a first shape derivative $\frac{\partial\phi'}{\partial\nu}$ is given in [20, Théorème 5.5.2, formula (5.74), p. 205] and reads as follows.

Lemma 5.3. *With the notations above, We have*

$$\left(\frac{\partial\phi}{\partial\nu}\right)' = \frac{\partial\phi'}{\partial\nu} - \langle u,\nu\rangle\left(\frac{\partial}{\partial\nu}\left(\frac{\partial\phi}{\partial\nu}\right) - \frac{\partial^2\phi}{\partial\nu^2}\right) - \big\langle\nabla\phi, \nabla_\Gamma\big(\langle u,\nu\rangle\big)\big\rangle \quad \text{on } \partial\Omega, \tag{42}$$

where ∇_Γ is the tangential gradient and $\frac{\partial^2\phi}{\partial\nu^2}$ is understood as the image of the second derivative of ϕ (a bilinear form) applied to (ν,ν).

Using the above lemma and the fact that the involved functions vanish on $\partial\Omega$, (41) is rewritten as follows

$$\begin{aligned}&\frac{\partial\phi_k'(u)}{\partial\nu}\frac{\partial\xi_1}{\partial\nu} + \frac{\partial\phi_k}{\partial\nu}\frac{\partial\xi_1'(u)}{\partial\nu} + \frac{\partial\phi_1'(u)}{\partial\nu}\frac{\partial\xi_k}{\partial\nu} + \frac{\partial\phi_1}{\partial\nu}\frac{\partial\xi_k'(u)}{\partial\nu}\\ &\quad= -\langle u,\nu\rangle\left(\frac{\partial^2\phi_k}{\partial\nu^2}\frac{\partial\xi_1}{\partial\nu} + \frac{\partial\phi_k}{\partial\nu}\frac{\partial^2\xi_1}{\partial\nu^2} + \frac{\partial^2\phi_1}{\partial\nu^2}\frac{\partial\xi_k}{\partial\nu} + \frac{\partial\phi_1}{\partial\nu}\frac{\partial^2\xi_k}{\partial\nu^2}\right) \quad \text{on } \partial\Omega,\end{aligned} \tag{43}$$

where $\phi_1'(u)$ and $\phi_k'(u)$ are defined in (34) and $\xi_1'(u)$ and $\xi_k'(u)$ are solutions of

$$\begin{cases} -(\Delta+\lambda_1)\xi_k'(u) = \lambda_1'(u)\xi_k + \mu\phi_k'(u), & \text{in } \Omega,\\ \xi_k'(u) = -\langle u,\nu\rangle\dfrac{\partial\xi_k}{\partial\nu}, & \text{on } \partial\Omega,\\ \displaystyle\int_\Omega \big(\phi_1\xi_k'(u) + \phi_1'(u)\xi_k\big)\,dq = 0, \end{cases}$$

and

$$\begin{cases} -(\Delta+\lambda_k)\xi_1'(u) = \lambda_k'(u)\xi_1 + \mu\phi_1'(u), & \text{in } \Omega,\\ \xi_1'(u) = -\langle u,\nu\rangle\dfrac{\partial\xi_1}{\partial\nu}, & \text{on } \partial\Omega,\\ \displaystyle\int_\Omega \big(\phi_k\xi_1'(u) + \phi_k'(u)\xi_1\big)\,dq = 0. \end{cases} \tag{44}$$

As a consequence of the previous computations, the genericity of (B_k) in $\mathbb{D}_3$ results from the next proposition.

Proposition 5.4. *Let $k \geqslant 2$ and $\Omega \in \mathbb{D}_3$. Assume that* (37) *and* (38) *hold true. Then, there does not exist $\rho' > 0$ such that* (34) *and* (44) *admit solutions satisfying* (43) *for every $u \in E_{\rho'}(\Omega)$.*

Remark 5.2. Let $J(\Omega)$ be a smooth functional depending on the domain Ω and u a variation belonging to $W^{k,\infty}(\Omega,\mathbb{R}^2)$. As pointed out in [42], we have

$$J''(\Omega)(u,u) = (J')'(\Omega)(u,u) - J'(\Omega)(u \cdot \nabla u). \tag{45}$$

This equation says that $J''(\Omega)$, the second derivative with respect to the domain at the point Ω, applied to the function u is not in general equal to the first derivative of the function $J'(\Omega)(u)$ at the point Ω applied to u. The difference between them is equal to the first shape derivative of the function $J(\Omega)$ applied to $u \cdot \nabla u$. However, in our case, they are equal because the first shape derivative is equal to zero by assumption. Thus, (43) exactly corresponds to the second shape derivative of (37).

5.3. *Proof strategy for Proposition 5.4*

To prove Proposition 5.4, our strategy is similar to that developed in [11] and, in order to describe it, we first need information on the regularity of the solutions of (34) and (44). For that purpose, we consider the following standard definitions of Sobolev spaces and distributions on Ω (cf. [32]). If m is a positive integer, we use $H^m(\Omega)$ to denote the Sobolev space of order m on Ω defined by

$$H^m(\Omega) := \{\Psi \mid D^\alpha \Psi \in L^2(\Omega),\ |\alpha| \leqslant m\},$$

where $D^\alpha = \frac{\partial^{\alpha_1+\alpha_2}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$, and $|\alpha| = \alpha_1 + \alpha_2$. Here the differential operators D^α are defined in the distributional sense on Ω, with $\mathcal{D}'(\Omega)$ the space of distributions on Ω being dual to $\mathcal{D}(\Omega)$, the set of smooth functions with compact support in Ω (cf. [32]). Let $\rho : \overline{\Omega} \to \mathbb{R}^+$ be a function of class $C^2(\bar{\Omega})$ equal to the distance function to $\partial\Omega$ ($\rho(x) = d(x, \partial\Omega)$) for $d(x, \partial\Omega)$ small enough. Such a function exists as noted in [32, Chapter 1, §11.2, p. 62].

According to [32], for $s \in \mathbb{N}$, we set

$$\Xi^s(\Omega) := \{\Psi \mid \rho^{|\alpha|} D^\alpha \Psi \in L^2(\Omega),\ |\alpha| \leqslant s\},$$

equipped with the norm

$$\|\Psi\|_{\Xi^s(\Omega)} = \left(\sum_{|\alpha| \leqslant s} \|\rho^{|\alpha|} D^\alpha \Psi\|^2_{L^2(\Omega)}\right)^{1/2}.$$

Then $\Xi^s(\Omega)$ is a Hilbert space so that $H^s(\Omega) \subset \Xi^s(\Omega) \subset \Xi^0(\Omega) = L^2(\Omega)$ with a continuous embedding. Let $\Xi^{-s}(\Omega) := (\Xi^s(\Omega))'$ be the dual space of $\Xi^s(\Omega)$ for the $L^2(\Omega)$ scalar-product. Then, $\Xi^{-s}(\Omega)$ is a distribution space as proved in [32].

Remark 5.3. By interpolations techniques, we can also define the spaces $\Xi^s(\Omega)$ for all real positive number s. Then, we have $H^s(\Omega) \subset \Xi^s(\Omega) \subset \Xi^{s'}(\Omega) \subset L^2(\Omega)$ if $0 < s' < s$ (see [32, p. 184] for more details).

We can now apply the general theorems stated in [32] to the present situation. Let $A := \Delta + \lambda$ and B_0 be the Dirichlet trace operator. We set

$$D_A^s(\Omega) = \{\Psi \mid \Psi \in H^s(\Omega),\ A\Psi \in \Xi^{s-2m}(\Omega)\}, \quad 0 < s < 2m,$$

with the norm defined by $\|\Psi\| = (\|\Psi\|^2_{H^s(\Omega)} + \|A\Psi\|^2_{\Xi^{s-2m}})^{1/2}$. Then, $D^s_A(\Omega)$ is a Hilbert space.

We write system (34) with new notations

$$A\Psi = f \quad \text{in } \Omega \quad \text{and} \quad B_0\Psi = g \quad \text{on } \partial\Omega, \tag{46}$$

where

$$f = -\lambda'(u)\phi_j \quad \text{and} \quad g = -\langle u, \nu\rangle \frac{\partial \phi_j}{\partial \nu}, \quad \text{with } j = 1, k. \tag{47}$$

We apply [32, Theorem 7.4, p. 202] for $m = 1$ (one boundary condition) and $m_0 = 0$ (there is not derivation in the trace operator). As ϕ_1 is an eigenfunction, f is in every distribution space, in particular it is an element of every $\Xi^{s'}(\Omega)$ for $s' < 0$. Then, if $0 < s < 2$, we have $f \in \Xi^{s-2}(\Omega)$. If $g \in H^{s-1/2}(\Omega)$, by [32, Theorem 7.4, p. 202], $\Psi \in D^s_A(\Omega)$. We now apply [32, Theorem 7.3, p. 201] with $B_1 = \frac{\partial}{\partial \nu}$ and $m_1 = 1$. Then, we have $\frac{\partial \Psi}{\partial \nu} \in H^{s-3/2}(\Omega)$. We summarize these results in the following lemma.

Lemma 5.5. *Let $s \in (0, 2)$ and $j \in \{1, k\}$. With the notations above, if the Dirichlet boundary condition $g = \langle u, \nu\rangle \frac{\partial \phi_j}{\partial \nu} \in H^{s-1/2}(\partial\Omega)$, then we have $\phi'_j(u) \in H^s(\Omega)$ and $\frac{\partial \phi'_j(u)}{\partial \nu} \in H^{s-3/2}(\partial\Omega)$.*

As already mentioned in the introduction, the starting remark for the argument of Proposition 5.4 goes as follows. By taking into account Lemma 5.5, the right-hand side of (43) is in $H^{s-1/2}(\partial\Omega)$ and, at the same time, the left-hand side in $H^{s-3/2}(\partial\Omega)$, for $s \in (0, 2)$. To take advantage of that gap of regularity between the two sides of (43), we first consider variations exhibiting just one jump of discontinuity on Ω, let say at some point $q_* \in \partial\Omega$, so that, for all the quantities involved in (43), an irregular part only occurs at the point q_*. If we are able to compute exactly this irregular part, we would infer that it has to be equal to zero by using (43). It would provide some extra information at the point q_*, of the type $F(q_*) = 0$ where F is an $\mathbb{R}^2$-valued map defined on $\partial\Omega$. Since the point q_* is arbitrary, we would end with the relation $F \equiv 0$ on $\partial\Omega$, similar to (38). Using this new information together with $M(0) \equiv 0$, one hopes to derive a contradiction.

Let us provide more details. Fix $q_* \in \partial\Omega$ and a parametrization σ of C_1, the connected component of $\partial\Omega$ containing q_*, so that $\sigma \in [-L, L)$ and q_* corresponds to $\sigma = 0$. Fix an open neighborhood V_α of q_* in C_1 parameterized by $(-\alpha, \alpha)$ with $\alpha < L$. We consider an admissible variation u_{q_*} (see Definition 5.3 below) defined as follows: on $(-\alpha, 0)$, $\langle u_{q_*}, \nu\rangle = 0$, on $(0, \alpha)$, $\langle u_{q_*}, \nu\rangle = 1$ and $\langle u_{q_*}, \nu\rangle$ is smooth in C_1 except at $\sigma = 0$. According to Remark 5.4 below, we can extend the definition of $M'(u)$ to functions u which are not regular enough to perform shape differentiation (such as u_{q_*}). We then show that $M'(u_{q_*})$ admits, in the distributional sense, the following Taylor expansion valid in $(-\alpha, \alpha)$,

$$M'(u_{q_*})(\sigma) = M_0 \quad \text{p.v.}\left(\frac{1}{\sigma}\right) + M_1 \ln(|\sigma|) + M_3\sigma \ln(|\sigma|) + \mathcal{R}(\sigma), \tag{48}$$

and we also have, according to (40),

$$M'(u_{q_*})(\sigma) = M_2 H_0(\sigma) + \mathcal{R}(\sigma).$$

In the above equations, the coefficients M_i, $0 \leqslant i \leqslant 3$, are $\mathbb{R}^2$-valued, $\mathcal{R}$ denotes a (generic) C^1 function over $(-\alpha, \alpha)$ and H_0 belongs to $H^{1/2-\epsilon}(\partial\Omega)$ for every $\epsilon > 0$.

We will then prove that M_i, $0 \leqslant i \leqslant 2$, are always equal to zero and, from the relation $M \equiv 0$ on $\partial\Omega$, we will therefore be left with the relation

$$M_3 \sigma \ln\bigl(|\sigma|\bigr) + \mathcal{R}(\sigma) = 0 \quad \text{on } (-\alpha, \alpha). \tag{49}$$

It would immediately yield $M_3 = 0$. Moreover, we will compute M_3 as a function of the values of ϕ_1, ϕ_k, ξ_1, ξ_k and their normal derivatives at $\sigma = 0$ (i.e., at q_*). Therefore, M_3 can be seen as a function defined on $\partial\Omega$ and, since q_* is arbitrary, we will get from (49) that $M_3(\cdot) \equiv 0$ on $\partial\Omega$. It will provide us with a new nontrivial relationship between ϕ_1, ϕ_k, ξ_1, ξ_k and their normal derivatives and we will reach shortly after a contradiction, hence concluding the proof of Proposition 5.4.

In order now to access to (48) and get a hold on the M_i's, we split $M'(u_{q_*})$ as follows,

$$M'(u_{q_*}) = M'_b(u_{q_*}) + M'_d(u_{q_*}), \tag{50}$$

where

$$M'_b(u_{q_*}) = \frac{\partial \phi'_k(u_{q_*})}{\partial \nu} \frac{\partial \xi_1}{\partial \nu} + \frac{\partial \phi_k}{\partial \nu} \frac{\partial \xi'_{1,b}(u_{q_*})}{\partial \nu} + \frac{\partial \phi'_1(u_{q_*})}{\partial \nu} \frac{\partial \xi_k}{\partial \nu} + \frac{\partial \phi_1}{\partial \nu} \frac{\partial \xi'_{k,b}(u_{q_*})}{\partial \nu}, \tag{51}$$

$$M'_d(u_{q_*}) = \frac{\partial \phi_k}{\partial \nu} \frac{\partial \xi'_{1,d}(u_{q_*})}{\partial \nu} + \frac{\partial \phi_1}{\partial \nu} \frac{\partial \xi'_{k,d}(u_{q_*})}{\partial \nu}, \tag{52}$$

where $M'_b(u_{q_*})$ and $M'_d(u_{q_*})$ are the contributions of respectively the boundary $\partial\Omega$ and the domain Ω to $M'(u_{q_*})$. In (51) and (52), we choose the variation u_{q_*} (see Definition 5.3) such that $\phi'_1(u_{q_*})$ and $\phi'_k(u_{q_*})$ are solutions of

$$\begin{cases} -(\Delta + \lambda_1)\phi'_1(u_{q_*}) = \lambda'_1(u_{q_*})\phi_1, & \text{in } \Omega, \\ \phi'_1(u_{q_*}) = -\langle u_{q_*}, \nu\rangle \dfrac{\partial \phi_1}{\partial \nu}, & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega \phi_1 \phi'_1(u_{q_*}) = 0, \end{cases}$$

$$\begin{cases} -(\Delta + \lambda_k)\phi'_k(u_{q_*}) = \lambda'_k(u_{q_*})\phi_k, & \text{in } \Omega, \\ \phi'_k(u_{q_*}) = -\langle u_{q_*}, \nu\rangle \dfrac{\partial \phi_k}{\partial \nu}, & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega \phi_k \phi'_k(u_{q_*}) = 0, \end{cases} \tag{53}$$

and the $\xi'_{1,b}(u_{q_*})$, $\xi'_{1,d}(u_{q_*})$, $\xi'_{k,b}(u_{q_*})$ and $\xi'_{k,d}$ are defined as the solutions of the following Helmholtz equations,

$$\begin{cases} -(\Delta+\lambda_1)\xi'_{k,b}(u_{q_*})=0, & \text{in } \Omega,\\ \xi'_{k,b}(u_{q_*})=-\langle u_{q_*},\nu\rangle\dfrac{\partial\xi_k}{\partial\nu}, & \text{on } \partial\Omega,\\ \displaystyle\int_\Omega \phi_1\xi'_{k,b}(u_{q_*})=0, \end{cases} \qquad \begin{cases} -(\Delta+\lambda_k)\xi'_{1,b}(u_{q_*})=0, & \text{in } \Omega,\\ \xi'_{1,b}(u_{q_*})=-\langle u_{q_*},\nu\rangle\dfrac{\partial\xi_1}{\partial\nu}, & \text{on } \partial\Omega,\\ \displaystyle\int_\Omega \phi_k\xi'_{1,b}(u_{q_*})=0, \end{cases} \tag{54}$$

$$\begin{cases} -(\Delta+\lambda_1)\xi'_{k,d}(u_{q_*})=\lambda'_1(u)\xi_k+\mu\phi'_k(u_{q_*}), & \text{in } \Omega,\\ \xi'_{k,d}(u_{q_*})=0, & \text{on } \partial\Omega,\\ \displaystyle\int_\Omega \phi_1\xi'_{k,d}(u_{q_*})=0, \end{cases} \tag{55}$$

$$\begin{cases} -(\Delta+\lambda_k)\xi'_{1,d}(u_{q_*})=\lambda'_k(u)\xi_1+\mu\phi'_1(u_{q_*}), & \text{in } \Omega,\\ \xi'_{1,d}(u_{q_*})=0, & \text{on } \partial\Omega,\\ \displaystyle\int_\Omega \phi_k\xi'_{1,d}(u_{q_*})=0. \end{cases} \tag{56}$$

By linearity, $\xi'_1(u_{q_*})=\xi'_{1,b}(u_{q_*})+\xi'_{1,d}(u_{q_*})+c_1\phi_k$ and $\xi'_k(u_{q_*})=\xi'_{k,b}(u_{q_*})+\xi'_{k,d}(u_{q_*})+c_2\phi_1$, where $c_1=-\int_\Omega\phi'_k(u)\xi_1 dq$ and $c_2=-\int_\Omega\phi'_1(u)\xi_k dq$. We simply intend here to compute $\xi'_j(u_{q_*})$, $j=1,k$, as the sum of two terms, one coming from the boundary condition and the second from the inhomogeneous part of the PDE. Each of these terms requires the study of a Dirichlet-to-Neumann operator associated to a Helmholtz equation. In the next section, we develop in details these computations.

5.4. Evaluations of the singular parts of $M'_b(u_{q_*})$ and $M'_d(u_{q_*})$

In what follows, p and q denote points of $\mathbb{R}^2$ and x, y denotes respectively the first and second coordinates of a point in $\mathbb{R}^2$.

For the rest of the paper, we fix a point $q_*\in\partial\Omega$ and, with no loss of generality, we assume that $\partial\Omega$ has only one connected component.

We next choose a parametrization of $\partial\Omega$ by arc-length $\sigma\in[-L,L)$ so that q_* corresponds to $(x(0),y(0))$. The initial control problem (4) is clearly invariant by rotation and thus we can assume that the tangent vector at $\sigma=0$ is equal to $(-1,0)^T$. We finally proceed to a translation of vector q_* which implies that $(x(0),y(0))=(0,0)$. That transformation only modifies the PDEs governing ξ_j, $\xi'_{j,d}$ and $\xi'_{j,b}$, $j=1,k$, replacing q by $q+q_*$ in (32), (31), (54) and (55).

Since Ω is of class C^3, there exists a neighborhood $\mathcal{N}_0$ of $0\in\mathbb{R}$ such that for every $\sigma\in\mathcal{N}_0$, we have

$$x(\sigma)=-\sigma+O\big(\sigma^3\big), \tag{57}$$

$$y(\sigma)=\frac{\kappa(0)}{2}\sigma^2+O\big(\sigma^3\big), \tag{58}$$

where κ is the curvature function of $\partial\Omega$. Let $\mathcal{N}_a$ be the subset of $\partial\Omega$ made of points $q(\sigma)=(x(\sigma),y(\sigma))$ with $\sigma\in\mathcal{N}_0$ and $\nu(\cdot)$ be the unit outward normal along $\partial\Omega$, which is of class C^2, and has direction $(y'(\cdot),-x'(\cdot))$.

We now consider a variation u_{q_*} which exhibits a unique jump of discontinuity at q_*, i.e., u_{q_*} is only defined through its normal part $\langle u_{q_*}, \nu\rangle$ given next

$$\langle u_{q_*}, \nu\rangle(\sigma) = \begin{cases} 0, & \sigma \in [-\alpha, 0), \\ 1, & \sigma \in [0, \alpha), \\ \eta(\sigma), & \sigma \in [-L, -\alpha) \cup [\alpha, L), \end{cases} \tag{59}$$

where $0 < \alpha$ is small enough so that $[-\alpha, \alpha] \subset \mathcal{N}_0$ and η is smooth so that $\langle u_{q_*}, \nu\rangle$ is $2L$-periodic and smooth except at $\sigma = 0$. We sometimes refer to $\langle u_{q_*}, \nu\rangle$ as the Heaviside function on $\partial\Omega$ and use $\mathcal{H}_0$ to denote it.

Remark 5.4. Strictly speaking, u_{q_*} cannot be considered as a variation of domain since it is not in $W^{4,\infty}(\Omega, \mathbb{R}^2)$. However, it is rather easy to see that solutions of the differential systems obtained after shape differentiation can be defined by standard density arguments for function spaces containing $W^{4,\infty}(\Omega, \mathbb{R}^2)$. For instance, $M'(u)$ is first defined by shape differentiation for $u \in E_\rho(\Omega)$, and that requires to consider the functions $\phi'_j(u)$ and $\xi'_j(u)$, $j = 1, k$ verifying (34) and (44). On the other hand, these functions only need $\langle u, \nu\rangle$, the normal component of the variation, to be defined. Thus, for $\langle u, \nu\rangle \in H^s(\partial\Omega)$, $s \leqslant 1$, one still can define by density (unique) solutions of (34) and (44) associated to u and thus traces on $\partial\Omega$ of these elements. Finally, using Lemma 5.5, the function defined in the left-hand side of (43) is well defined and, by an obvious abuse of notation, we use $M'(u)$ to denote it. We now have defined $M'(u_{q_*})$ and we refer to it as the shape differential of M for the variation u_{q_*}.

Remark 5.5. For presentation ease, we use the arc-length σ for parameterizing all points q in a neighborhood of the fixed point $q_* \in \partial\Omega$.

Definition 5.3. Let Ω be a domain of $\mathbb{D}_3$ not verifying condition (B_k). A variation u (defined with $\langle u, \nu\rangle \in H^s(\partial\Omega)$, $s \in (0, 2)$) is said to be *admissible* if

$$\int_{\partial\Omega} \langle u, \nu\rangle \frac{\partial\phi_1}{\partial\nu} \frac{\partial\xi_k}{\partial\nu} d\sigma(q) = 0. \tag{60}$$

By applying Green's second formula and using (33) and (38), one sees that condition (60) is necessary (and sufficient) for the existence of solutions of the PDEs given in (53), (54) and then (55) after an appropriate choice of c_1 and c_2. Moreover, remark that if $\frac{\partial\xi_k}{\partial\nu} \equiv 0$ on $\partial\Omega$ (and thus $\frac{\partial\xi_1}{\partial\nu} \equiv 0$), then every variation is admissible.

Lemma 5.6. *For every $q_* \in \partial\Omega$, one can choose the smooth function η and the parameter α introduced in* (59) *such that u_{q_*} is an admissible variation.*

Proof of Lemma 5.6. We may assume that $\frac{\partial\xi_k}{\partial\nu}$ (and thus $\frac{\partial\xi_1}{\partial\nu}$) is not identically equal to 0 on $\partial\Omega$. Assume first that $\frac{\partial\xi_k}{\partial\nu}(q_*) \neq 0$. Eq. (60) can clearly be stated as an affine relation $L(\eta) = l$, where L is a linear form and $l \in \mathbb{R}$. Notice that L is not null. Indeed, $\frac{\partial\xi_k}{\partial\nu}(q)$ is not equal to zero in an open neighborhood of q_*. Then, by choosing α small enough, $\frac{\partial\xi_k}{\partial\nu}(q(\sigma))$ is not equal to zero for some σ in $(-L, -\alpha) \cup (\alpha, L)$. It is therefore always possible to select η so that u_{q_*}

is an admissible variation. It is immediate to extend the above construction to the case where $\frac{\partial \xi_k}{\partial \nu}(q_*) = 0$ and there exists a sequence of points $q \in \partial\Omega$ converging to q_* such that $\frac{\partial \xi_k}{\partial \nu}(q) \neq 0$.

It remains to treat the case where $\frac{\partial \xi_k}{\partial \nu} \equiv 0$ on an open neighborhood $\mathcal{N}$ of $q_* \in \partial\Omega$. It is then possible to choose $\alpha > 0$ small enough so that $q(\sigma) \in \mathcal{N}$ for $\sigma \in (-2\alpha, 2\alpha)$ and $\eta \equiv 0$ on $(2\alpha, L) \cup (-L, -\alpha)$. Then, the corresponding u_{q_*} is admissible. □

Definition 5.4. We say that a function g defined on $\partial\Omega$ is 2-regular if there exists two smooth (i.e., C^∞) functions $h, \tilde{h}$ defined on $\partial\Omega$ such that $g(\sigma) = \sigma^2 \ln(|\sigma|)h(\sigma) + \tilde{h}(\sigma)$ for σ in an open neighborhood of zero. We will use sometimes the symbol $\mathcal{R}_2$ to denote an arbitrary 2-regular function. In addition, we use the symbol $\mathcal{R}_1$ to denote an arbitrary C^1 function in an open neighborhood of zero. Note that a 2-regular function is necessarily of class C^1. Finally, we use the notation $O(\sigma)$ to denote an arbitrary C^1 function equal to zero at $\sigma = 0$ and with uniformly bounded derivative over some open neighborhood of zero.

In the next paragraph, we will prove that the irregular parts of $\frac{\partial \phi_1'}{\partial \nu}(u_{q_*})$, $\frac{\partial \phi_k'}{\partial \nu}(u_{q_*})$, $\frac{\partial \xi_{1,b}'}{\partial \nu}(u_{q_*})$, $\frac{\partial \xi_{1,d}'}{\partial \nu}(u_{q_*})$, $\frac{\partial \xi_{k,b}'}{\partial \nu}(u_{q_*})$ and $\frac{\partial \xi_{k,d}'}{\partial \nu}(u_{q_*})$ involved in $M'(u_{q_*}) = M_b'(u_{q_*}) + M_d'(u_{q_*})$ only occur at the point q_* and we intend to calculate them exactly.

5.4.1. Expression of $M_b'(u_{q_*})$

The main result of this section is the following theorem.

Theorem 5.7. *There exists an open neighborhood of zero $\mathcal{N}_1 \subset \mathcal{N}_0$ such that, if $\sigma \in \mathcal{N}_1$, one has*

$$M_b'(u_{q_*})(\sigma) = \frac{1}{\pi}\left\{\lambda_1 \frac{\partial \phi_1(0)}{\partial \nu}\frac{\partial \xi_k(0)}{\partial \nu} + \lambda_k \frac{\partial \phi_k(0)}{\partial \nu}\frac{\partial \xi_1(0)}{\partial \nu}\right\}\sigma \ln|\sigma| + \mathcal{R}_1. \tag{61}$$

For the rest of the paper, we set

$$a_1 := -\frac{1}{2\pi}, \qquad a_2 := \frac{1}{8\pi}. \tag{62}$$

Note that the constant $1/\pi$ appearing in the right-hand side of (61) is equal to $-4(a_1 + 2a_2)$.

The proof of this theorem is based on the following proposition.

Proposition 5.8. *We have*

$$\begin{aligned}
\frac{\partial \phi_1'(u_{q_*})}{\partial \nu}(\sigma) = -2\bigg\{ & a_1 \frac{\partial \phi_1}{\partial \nu}(0)\, p.v.\left(\frac{1}{\sigma}\right) + a_1 \frac{\partial}{\partial \tau}\left(\frac{\partial \phi_1}{\partial \nu}\right)(0) \ln|\sigma| \\
& + \left\{a_1 \frac{\partial^2}{\partial \tau^2}\left(\frac{\partial \phi_1}{\partial \nu}\right) + (a_1 + 2a_2)\lambda_1 \frac{\partial \phi_1}{\partial \nu}\right\}(0)\sigma \ln|\sigma|\bigg\} \\
& - a_1 \frac{\partial \phi_1}{\partial \nu}(0) L_1(\sigma) - a_1 \frac{\partial}{\partial \tau}\left(\frac{\partial \phi_1}{\partial \nu}\right)(0) L_2(\sigma) + \mathcal{R}_1,
\end{aligned} \tag{63}$$

$$\begin{aligned}\frac{\partial \phi_k'(u_{q_*})}{\partial \nu}(\sigma) = -2\bigg\{ & a_1 \frac{\partial \phi_k}{\partial \nu}(0)\ p.v.\left(\frac{1}{\sigma}\right) + a_1 \frac{\partial}{\partial \tau}\left(\frac{\partial \phi_k}{\partial \nu}\right)(0) \ln|\sigma| \\ & + \left\{ a_1 \frac{\partial^2}{\partial \tau^2}\left(\frac{\partial \phi_k}{\partial \nu}\right) + (a_1 + 2a_2)\lambda_k \frac{\partial \phi_k}{\partial \nu}\right\}(0)\sigma \ln|\sigma| \bigg\} \\ & - a_1 \frac{\partial \phi_k}{\partial \nu}(0) L_1(\sigma) - a_1 \frac{\partial}{\partial \tau}\left(\frac{\partial \phi_k}{\partial \nu}\right)(0) L_2(\sigma) + \mathcal{R}_1, \end{aligned} \tag{64}$$

$$\begin{aligned}\frac{\partial \xi_{1,b}'(u_{q_*})}{\partial \nu}(\sigma) = -2\bigg\{ & a_1 \frac{\partial \xi_1}{\partial \nu}(0)\ p.v.\left(\frac{1}{\sigma}\right) + a_1 \frac{\partial}{\partial \tau}\left(\frac{\partial \xi_1}{\partial \nu}\right)(0) \ln|\sigma| \\ & + \left\{ a_1 \frac{\partial^2}{\partial \tau^2}\left(\frac{\partial \xi_1}{\partial \nu}\right) + (a_1 + 2a_2)\lambda_k \frac{\partial \xi_1}{\partial \nu}\right\}(0)\sigma \ln|\sigma| \bigg\} \\ & - a_1 \frac{\partial \xi_1}{\partial \nu}(0) L_1(\sigma) - a_1 \frac{\partial}{\partial \tau}\left(\frac{\partial \xi_1}{\partial \nu}\right)(0) L_2(\sigma) + \mathcal{R}_1, \end{aligned} \tag{65}$$

$$\begin{aligned}\frac{\partial \xi_{k,b}'(u_{q_*})}{\partial \nu}(\sigma) = -2\bigg\{ & a_1 \frac{\partial \xi_k}{\partial \nu}(0)\ p.v.\left(\frac{1}{\sigma}\right) + a_1 \frac{\partial}{\partial \tau}\left(\frac{\partial \xi_k}{\partial \nu}\right)(0) \ln|\sigma| \\ & + \left\{ a_1 \frac{\partial^2}{\partial \tau^2}\left(\frac{\partial \xi_k}{\partial \nu}\right) + (a_1 + 2a_2)\lambda_1 \frac{\partial \xi_k}{\partial \nu}\right\}(0)\sigma \ln|\sigma| \bigg\} \\ & - a_1 \frac{\partial \xi_k}{\partial \nu}(0) L_1(\sigma) - a_1 \frac{\partial}{\partial \tau}\left(\frac{\partial \xi_k}{\partial \nu}\right)(0) L_2(\sigma) + \mathcal{R}_1, \end{aligned} \tag{66}$$

where $L_1(\sigma) := T_0(p.v.(\frac{1}{\sigma}))$, $L_2(\sigma) := T_0(\ln|\sigma|)$, with T_0 the linear operator defined in (111). *Recall that $\mathcal{R}_1$ is used to denote an arbitrary C^1 function of $\partial\Omega$.*

Proof of Proposition 5.8. Explicit computation is only provided for (63) since expressions for $\frac{\partial \phi_k'(u_{q_*})}{\partial \nu}$, $\frac{\partial \xi_{1,b}'(u_{q_*})}{\partial \nu}$ and $\frac{\partial \xi_{k,b}'(u_{q_*})}{\partial \nu}$ are derived in a similar way. From (107), we first easily get that the contribution of $\lambda_1'(u_{q_*})\phi_1$ to $\frac{\partial \phi_1'}{\partial \nu}(q_*)$ is a term of class C^2 and thus of type $\mathcal{R}_2$. We next apply Proposition B.8 with $g = \frac{\partial \phi_1}{\partial \nu}$. It yields

$$\begin{aligned} E_1\left(\mathcal{H}_0 \frac{\partial \phi_1}{\partial \nu}\right)(\sigma) = {} & \frac{\partial \phi_1(0)}{\partial \nu} a_1 \text{ p.v.}\left(\frac{1}{\sigma}\right) + \frac{\partial}{\partial \tau}\left(\frac{\partial \phi_1}{\partial \nu}\right)(0) a_1 \ln|\sigma| \\ & + \left\{ a_1 \frac{\partial^2}{\partial \tau^2}\left(\frac{\partial \phi_1}{\partial \nu}\right) + (a_1 + 2a_2)\lambda_1 \frac{\partial \phi_1}{\partial \nu}\right\}(0)\sigma \ln|\sigma| + \mathcal{R}_2. \end{aligned} \tag{67}$$

According to Theorem B.4, we get

$$\begin{aligned}\frac{\partial \phi_1'}{\partial \nu}(\sigma) = -2\bigg\{ & \frac{\partial \phi_1(0)}{\partial \nu} a_1 \text{ p.v.}\left(\frac{1}{\sigma}\right) + \frac{\partial}{\partial \tau}\left(\frac{\partial \phi_1}{\partial \nu}\right)(0) a_1 \ln|\sigma| \\ & + \left\{ a_1 \frac{\partial^2}{\partial \tau^2}\left(\frac{\partial \phi_1}{\partial \nu}\right) + (a_1 + 2a_2)\lambda_1 \frac{\partial \phi_1}{\partial \nu}\right\}(0)\sigma \ln|\sigma| \bigg\} \end{aligned}$$

$$
\begin{aligned}
&- \frac{\partial\phi_1(0)}{\partial\nu} a_1 L_1(\sigma) - \frac{\partial}{\partial\tau}\left(\frac{\partial\phi_1}{\partial\nu}\right)(0) a_1 L_2(\sigma) \\
&- \left(a_1 \frac{\partial^2}{\partial\tau^2}\left(\frac{\partial\phi_1}{\partial\nu}\right) + (a_1 + 2a_2)\lambda_1 \frac{\partial\phi_1}{\partial\nu}\right)(0) L_3(\sigma) + \mathcal{R}_1,
\end{aligned}
$$

where $L_3(\sigma) := T_0(\sigma \ln|\sigma|)$. Recalling that $\ln|\sigma|$ belongs to $H^{1/2-\varepsilon}(\partial\Omega)$ for every $\varepsilon > 0$ and the regularizing effect of the operator T_0, one immediately gets that $\sigma \ln|\sigma| \in H^{3/2-\varepsilon}(\partial\Omega)$ and $L_3(\sigma) \in H^{5/2-\varepsilon}(\partial\Omega)$ for every $\varepsilon > 0$. It implies that $L_3(\cdot)$ is a C^1 function of $\partial\Omega$. $\square$

Remark 5.6. For the rest of the paper, we will need information about the regularity of $L_j(\sigma)$, $j = 1, 2$. As done in the above argument, we have that $\text{p.v.}(\frac{1}{\sigma}) \in H^{-1/2-\varepsilon}(\partial\Omega)$ for every $\varepsilon > 0$ and, thanks to the regularizing effect of the operator T_0, we get that $L_1(\cdot) \in H^{1/2-\varepsilon}(\partial\Omega)$ for every $\varepsilon > 0$. Similarly, we get that $L_2(\cdot) \in H^{3/2-\varepsilon}(\partial\Omega)$ and $T_0(H^{3/2-\varepsilon}(\partial\Omega)) \subset \mathcal{R}_1$ for every $\varepsilon > 0$.

We are now able to prove Theorem 5.7.

Proof of Theorem 5.7. Let $\sigma \in \mathcal{N}_0$ and we eventually reduce the size of the neighborhood later on. Our first goal consists in computing explicitly the coefficient associated to $\text{p.v.}(\frac{1}{\sigma})$ in $M_b'(u_{q_*})$. Using Proposition 5.8 and Remark 5.6, we have

$$
\begin{aligned}
M_b'(u_{q_*})(\sigma) &= \frac{\partial\phi_k'(u_{q_*})}{\partial\nu}(\sigma)\frac{\partial\xi_1}{\partial\nu}(\sigma) + \frac{\partial\phi_k}{\partial\nu}(\sigma)\frac{\partial\xi_{1,b}'(u_{q_*})}{\partial\nu}(\sigma) \\
&\quad + \frac{\partial\phi_1'(u_{q_*})}{\partial\nu}(\sigma)\frac{\partial\xi_k}{\partial\nu}(\sigma) + \frac{\partial\phi_1}{\partial\nu}(\sigma)\frac{\partial\xi_{k,b}'(u_{q_*})}{\partial\nu}(\sigma) \\
&= -2\left(\frac{\partial\phi_k}{\partial\nu}(0) a_1 \,\text{p.v.}\left(\frac{1}{\sigma}\right) + \mathcal{P}_0(\sigma)\right)\left(\frac{\partial\xi_1}{\partial\nu}(0) + O(\sigma)\right) \\
&\quad - 2\left(\frac{\partial\xi_1}{\partial\nu}(0) a_1 \,\text{p.v.}\left(\frac{1}{\sigma}\right) + \mathcal{P}_0(\sigma)\right)\left(\frac{\partial\phi_k}{\partial\nu}(0) + O(\sigma)\right) \\
&\quad - 2\left(\frac{\partial\phi_1}{\partial\nu}(0) a_1 \,\text{p.v.}\left(\frac{1}{\sigma}\right) + \mathcal{P}_0(\sigma)\right)\left(\frac{\partial\xi_k}{\partial\nu}(0) + O(\sigma)\right) \\
&\quad - 2\left(\frac{\partial\xi_k}{\partial\nu}(0) a_1 \,\text{p.v.}\left(\frac{1}{\sigma}\right) + \mathcal{P}_0(\sigma)\right)\left(\frac{\partial\phi_1}{\partial\nu}(0) + O(\sigma)\right) \\
&= -4a_1\left(\frac{\partial\phi_k}{\partial\nu}(0)\frac{\partial\xi_1}{\partial\nu}(0) + \frac{\partial\phi_1}{\partial\nu}(0)\frac{\partial\xi_k}{\partial\nu}(0)\right)\text{p.v.}\left(\frac{1}{\sigma}\right) + \mathcal{P}_0(\sigma).
\end{aligned}
$$

where $\mathcal{P}_0(\sigma)$ denotes any function belonging to $H^{1/2-\varepsilon}(\partial\Omega)$ for every $\varepsilon > 0$ in some open neighborhood of zero. Then, we have

$$
M_b'(u_{q_*})(\sigma) = -4a_1 M(0)\,\text{p.v}\left(\frac{1}{\sigma}\right) + \mathcal{P}_0(\sigma). \tag{68}
$$

Since $M \equiv 0$ on $\partial\Omega$, we have in particular $M(0) = 0$. In consequence, there is not any term in $\text{p.v}(\frac{1}{\sigma})$ in $M_b'(u_{q_*})$.

The next step consists in identifying the least regular term of $\mathcal{P}_0$. We begin by writing that

$$\begin{aligned}
M_b'(u_{q_*})(\sigma) = &-\biggl(2a_1\frac{\partial}{\partial\tau}\biggl(\frac{\partial\phi_k}{\partial\nu}\biggr)(0)\ln|\sigma| + 2\biggl\{a_1\frac{\partial^2}{\partial\tau^2}\biggl(\frac{\partial\phi_k}{\partial\nu}\biggr) + (a_1+2a_2)\lambda_k\frac{\partial\phi_k}{\partial\nu}\biggr\}(0)\sigma\ln|\sigma| \\
&- a_1\frac{\partial\phi_k}{\partial\nu}(0)L_1(\sigma) - a_1\frac{\partial}{\partial\tau}\biggl(\frac{\partial\phi_k}{\partial\nu}\biggr)(0)L_2(\sigma) + \mathcal{R}_1(\sigma)\biggr) \\
&\times\biggl(\frac{\partial\xi_1}{\partial\nu}(0) + \frac{\partial}{\partial\tau}\biggl(\frac{\partial\xi_1}{\partial\nu}\biggr)(0)\sigma + O\bigl(\sigma^2\bigr)\biggr) \\
&-\biggl(2a_1\frac{\partial}{\partial\tau}\biggl(\frac{\partial\xi_1}{\partial\nu}\biggr)(0)\ln|\sigma| + 2\biggl\{a_1\frac{\partial^2}{\partial\tau^2}\biggl(\frac{\partial\xi_1}{\partial\nu}\biggr) + (a_1+2a_2)\lambda_k\frac{\partial\xi_1}{\partial\nu}\biggr\}(0)\sigma\ln|\sigma| \\
&- a_1\frac{\partial\xi_1}{\partial\nu}(0)L_1(\sigma) - a_1\frac{\partial}{\partial\tau}\biggl(\frac{\partial\xi_1}{\partial\nu}\biggr)(0)L_2(\sigma) + \mathcal{R}_1(\sigma)\biggr) \\
&\times\biggl(\frac{\partial\phi_k}{\partial\nu}(0) + \frac{\partial}{\partial\tau}\biggl(\frac{\partial\phi_k}{\partial\nu}\biggr)(0)\sigma + O\bigl(\sigma^2\bigr)\biggr) \\
&-\biggl(2a_1\frac{\partial}{\partial\tau}\biggl(\frac{\partial\phi_1}{\partial\nu}\biggr)(0)\ln|\sigma| + 2\biggl\{a_1\frac{\partial^2}{\partial\tau^2}\biggl(\frac{\partial\phi_1}{\partial\nu}\biggr) + (a_1+2a_2)\lambda_1\frac{\partial\phi_1}{\partial\nu}\biggr\}(0)\sigma\ln|\sigma| \\
&- a_1\frac{\partial\phi_1}{\partial\nu}(0)L_1(\sigma) - a_1\frac{\partial}{\partial\tau}\biggl(\frac{\partial\phi_1}{\partial\nu}\biggr)(0)L_2(\sigma) + \mathcal{R}_1(\sigma)\biggr) \\
&\times\biggl(\frac{\partial\xi_k}{\partial\nu}(0) + \frac{\partial}{\partial\tau}\biggl(\frac{\partial\xi_k}{\partial\nu}\biggr)(0)\sigma + O\bigl(\sigma^2\bigr)\biggr) \\
&-\biggl(2a_1\frac{\partial}{\partial\tau}\biggl(\frac{\partial\xi_k}{\partial\nu}\biggr)(0)\ln|\sigma| + 2\biggl\{a_1\frac{\partial^2}{\partial\tau^2}\biggl(\frac{\partial\xi_k}{\partial\nu}\biggr) + (a_1+2a_2)\lambda_1\frac{\partial\xi_k}{\partial\nu}\biggr\}(0)\sigma\ln|\sigma| \\
&- a_1\frac{\partial\xi_k}{\partial\nu}(0)L_1(\sigma) - a_1\frac{\partial}{\partial\tau}\biggl(\frac{\partial\xi_k}{\partial\nu}\biggr)(0)L_2(\sigma) + \mathcal{R}_1(\sigma)\biggr) \\
&\times\biggl(\frac{\partial\phi_1}{\partial\nu}(0) + \frac{\partial}{\partial\tau}\biggl(\frac{\partial\phi_1}{\partial\nu}\biggr)(0)\sigma + O\bigl(\sigma^2\bigr)\biggr).
\end{aligned}$$

Rearranging the terms and using Remark 5.6, we get

$$\begin{aligned}
M_b'(u_{q_*})(\sigma) = &-a_1\biggl(\frac{\partial M}{\partial\tau}(0)\bigl(2\ln|\sigma| + \sigma L_1(\sigma) + L_2(\sigma)\bigr) + M(0)L_1(\sigma) + 2\frac{\partial^2 M}{\partial\tau^2}(0)\sigma\ln|\sigma|\biggr) \\
&- (4a_1+8a_2)\biggl\{\lambda_1\frac{\partial\phi_1(0)}{\partial\nu}\frac{\partial\xi_k(0)}{\partial\nu} + \lambda_k\frac{\partial\phi_k(0)}{\partial\nu}\frac{\partial\xi_1(0)}{\partial\nu}\biggr\}\sigma\ln|\sigma| + \mathcal{R}_1(\sigma).
\end{aligned}$$

Since $M(0)\equiv 0$ on $\partial\Omega$, we have $\frac{\partial M}{\partial\tau}(0)\equiv 0$ and $\frac{\partial^2 M}{\partial\tau^2}(0) = 0$ on $\partial\Omega$. As a consequence, the above equation reduces Eq. (61). $\square$

5.4.2. Contribution of $M_d'(u_{q_*})$

We prove in this section the following theorem regarding the Taylor expansion of $M_d'(u_{q_*})$ in an open neighborhood of zero.

Theorem 5.9. *There exists an open neighborhood of zero $\mathcal{N}_2 \subset \mathcal{N}_0$ such that, if $\sigma \in \mathcal{N}_2$, one has*

$$M_d'(u_{q_*})(\sigma) = \frac{1}{\pi}\mu(q_*)\frac{\partial\phi_1}{\partial\nu}(0)\frac{\partial\phi_k}{\partial\nu}(0)\sigma\ln|\sigma| + \mathcal{R}_1. \tag{69}$$

The proof of this theorem is based on the following proposition.

Proposition 5.10. *We keep the notations above, then we have*

$$\frac{\partial\xi_{1,d}'(u_{q_*})}{\partial\nu}(\sigma) = -(2a_1 + 4a_2)\mu(q_*)\frac{\partial\phi_1}{\partial\nu}(0)\sigma\ln|\sigma| + \mathcal{R}_1, \tag{70}$$

$$\frac{\partial\xi_{k,d}'(u_{q_*})}{\partial\nu}(\sigma) = -(2a_1 + 4a_2)\mu(q_*)\frac{\partial\phi_k}{\partial\nu}(0)\sigma\ln|\sigma| + \mathcal{R}_1. \tag{71}$$

Proof of Theorem 5.9. We note that $2a_1 + 4a_2 = -\frac{1}{2\pi}$. Assuming Proposition 5.10. We have

$$\begin{aligned} M_d'(u_{q_*})(\sigma) &= \frac{\partial\phi_k}{\partial\nu}\frac{\partial\xi_{1,d}'(u_{q_*})}{\partial\nu} + \frac{\partial\phi_1}{\partial\nu}\frac{\partial\xi_{k,d}'(u_{q_*})}{\partial\nu} \\ &= \left(\frac{\partial\phi_k}{\partial\nu}(0) + O(\sigma)\right)\left(\frac{1}{2\pi}\mu(q_*)\frac{\partial\phi_1}{\partial\nu}(0)\sigma\ln|\sigma| + \mathcal{R}_1\right) \\ &\quad + \left(\frac{\partial\phi_1}{\partial\nu}(0) + O(\sigma)\right)\left(\frac{1}{2\pi}\mu(q_*)\frac{\partial\phi_k}{\partial\nu}(0)\sigma\ln|\sigma| + \mathcal{R}_1\right) \\ &= \frac{1}{\pi}\mu(q_*)\frac{\partial\phi_1}{\partial\nu}(0)\frac{\partial\phi_k}{\partial\nu}(0)\sigma\ln|\sigma| + \mathcal{R}_1. \qquad \square \end{aligned}$$

Recall now the system verified by $\xi_{1,d}'$.

$$\begin{cases} -(\Delta + \lambda_k)\xi_{1,d}'(u_{q_*}) = \lambda_k'(u_{q_*})\xi_1 + \mu(q + q_*)\phi_1'(u_{q_*}) & \text{in } \Omega, \\ \xi_{1,d}'(u_{q_*}) = 0 & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega \phi_k\xi_{1,d}'(u_{q_*}) = 0. \end{cases} \tag{72}$$

The function $\xi_{k,d}'$ verifies a similar system by exchanging the indices 1 and k and we will omit the corresponding argument.

Remark 5.7. By classical elliptic regularity theory presented in [32], we know that $\phi_1' \in H^{1-\epsilon}(\Omega)$, and then $\xi_{1,d}' \in H^{3-\epsilon}(\Omega)$. By taking the trace, we have $\frac{\partial\xi_{1,d}'}{\partial\nu} \in H^{3/2-\epsilon}(\partial\Omega)$. A straightforward computation shows that the last term $\sigma\ln|\sigma|$ in our expansion of $M_b'(u_{q^*})$ is in $H^{3/2-\epsilon}(\partial\Omega)$. Hence, it is necessary to compute exactly the first singular term of $\frac{\partial\xi_{1,d}'}{\partial\nu}$.

Remark 5.8. The term $M_d'(u_{q^*})$ cannot be treated by a direct functional analysis argument: if there were a family of functional spaces X^s with a well established theory of elliptic equations such as that for Sobolev spaces and if there would exist $s_1 \neq s_2$ such that $\mathcal{H}_0 \in X^{s_1}$ and

$\ln|\sigma| \in X^{s_2}$, then $\frac{\partial \xi'_{1,d}}{\partial \nu}$ and $\sigma \ln|\sigma|$ would not be in the same X^s. However, we cannot distinguish these two functions even in the family of Besov spaces. We can also note that $\mathcal{H}_0$ is a bounded variation function and $\ln|\sigma|$ is not, but elliptic theory in the space of bounded variation functions is not easy. For these reasons, it seems that an exact computation of the first term of $\frac{\partial \xi'_{1,d}}{\partial \nu}$ is necessary.

Let us first prove the following technical lemma, which expresses integrals over Ω by means of boundary integrals over $\partial\Omega$.

Lemma 5.11. *Let k, m be two distinct positive integers. Assume that a function h verifies $(\Delta + \lambda_k)h = 0$ in Ω. Then we have*

$$\frac{\partial}{\partial \nu_p} \int_{\Omega} h(q) G_m(p,q)\, dq = \frac{-1}{\lambda_m - \lambda_k}\left((E_m - E_k)(h|_{\partial\Omega}) + \left(K_k^* - K_m^*\right)\left(\frac{\partial h}{\partial \nu}\right)\right), \tag{73}$$

where $G_m(\cdot,\cdot)$ is the fundamental solution of the Helmholtz equation corresponding to λ_m and verifying the Sommerfeld condition and the operators E_k, E_m, K_k^*, K_m^* are defined in Section B.1.

Proof of Lemma 5.11. Green's second formula says that

$$\int_{\Omega} \big(g_1 \Delta(g_2) - g_2 \Delta(g_1)\big) = \int_{\partial\Omega} \left(g_1 \frac{\partial g_2}{\partial \nu} - g_2 \frac{\partial g_1}{\partial \nu}\right) d\sigma(q),$$

where g_1, g_2 are arbitrary functions such that the above integrals exist. Choose $g_1 = h$ and $g_2 = cG_k$ where c is a real number to be determined later. We have

$$c \int_{\Omega} h(q)\big[(\Delta + \lambda_k) G_m(p,q)\big]\, dq = c \int_{\partial\Omega} \left(h(q) \frac{\partial G_m}{\partial \nu_q}(p,q) - G_m(p,q) \frac{\partial h}{\partial \nu_q}\right) d\sigma(q). \tag{74}$$

Since $(\Delta + \lambda_m) G_m(p,q) = \delta_p$, we then get

$$\begin{aligned} &ch(p) + c(\lambda_k - \lambda_m) \int_{\Omega} h(q) G_m(p,q)\, dq \\ &\quad = c \int_{\partial\Omega} \left(h(q) \frac{\partial G_m}{\partial \nu_q}(p,q) - G_m(p,q) \frac{\partial h}{\partial \nu_q}\right) d\sigma(q). \end{aligned} \tag{75}$$

Setting $c := \frac{1}{\lambda_m - \lambda_k}$, we then get, for $p \in \Omega$,

$$\int_{\Omega} h(q) G_m(p,q)\, dq = \frac{1}{\lambda_m - \lambda_k}\left(-D_m(h)(p) + S_m\left(\frac{\partial h}{\partial \nu}\right)(p) + h(p)\right), \tag{76}$$

where S_m, D_m are respectively the single-layer and double-layer potentials associated to G_m (cf. Section B.1).

By applying the normal derivative operator to the two side terms of Eq. (76) and then taking into account the jump relations (103) and $(\frac{1}{2}I + K_k^*)\frac{\partial}{\partial \nu}h = E_k(h)$, we get that

$$\begin{aligned}
\frac{\partial}{\partial \nu}\int_\Omega h(q)G_m(p,q)\,dq &= \frac{1}{\lambda_m - \lambda_k}\frac{\partial}{\partial \nu}\left(-D_m(h)(p) + S_m\left(\frac{\partial h}{\partial \nu}\right)(p) + h(p)\right) \\
&= \frac{1}{\lambda_m - \lambda_k}\left(-E_m(h|_{\partial\Omega}) + \left(\frac{-1}{2} + K_m^*\right)\left(\frac{\partial h}{\partial \nu}\right) + \frac{\partial h}{\partial \nu}\right)(p) \\
&= \frac{1}{\lambda_m - \lambda_k}\left(-E_m(h|_{\partial\Omega}) + K_m^*\left(\frac{\partial h}{\partial \nu}\right) + \frac{1}{2}\frac{\partial h}{\partial \nu}\right)(p) \\
&= \frac{1}{\lambda_m - \lambda_k}\left(-E_m(h|_{\partial\Omega}) + K_m^*\left(\frac{\partial h}{\partial \nu}\right) + \left[E_k(h|_{\partial\Omega}) - K_k^*\frac{\partial h}{\partial \nu}\right]\right)(p) \\
&= \frac{1}{\lambda_m - \lambda_k}\left((E_k - E_m)(h|_{\partial\Omega}) + \left(K_m^* - K_k^*\right)\left(\frac{\partial h}{\partial \nu}\right)\right)(p). \qquad \Box
\end{aligned}$$

We are now able to provide an argument for Proposition 5.10.

Proof of Proposition 5.10. According to (107), we first easily get that the contribution of $\lambda_k'(u_{q_*})\xi_1$ to $\frac{\partial \xi_{1,d}'}{\partial \nu}(q_*)$ is a term of class C^2 and that

$$\left(\frac{1}{2}I + K_k^*\right)\frac{\partial \xi_{1,d}'}{\partial \nu}(p) = \frac{\partial}{\partial \nu_p}\int_\Omega \mu(q+q_*)\phi_1'(q)G_k(p,q)\,dq + \mathcal{R}_1. \tag{77}$$

We need the Taylor expansion of the right-hand side of (77) when a boundary point p belongs to an open neighborhood (in $\partial\Omega$) of q_* (i.e., $(0,0)$). For that purpose, we perform the following decomposition.

$$\begin{aligned}
\frac{\partial}{\partial \nu_p}\int_\Omega \mu(q+q_*)\phi_1'(q)G_k(p,q)\,dq &= \int_\Omega \mu(q+q_*)\phi_1'(q)\frac{\partial G_k}{\partial \nu_p}(p,q)\,dq \\
&= \mu(p+q_*)I_1(p) + I_2(p),
\end{aligned}$$

where

$$I_1(p) = \int_\Omega \phi_1'(q)\frac{\partial G_k}{\partial \nu_p}(p,q)\,dq = \frac{\partial}{\partial \nu_p}\int_\Omega \phi_1'(q)G_k(p,q)\,dq,$$

and

$$I_2 = \int_\Omega \left[\mu(q+q_*) - \mu(p+q_*)\right]\phi_1'(q)\frac{\partial G_k}{\partial \nu_p}(p,q)\,dq.$$

We first treat $I_1(p)$. Since ϕ_1' verifies $(\Delta+\lambda_1)\phi_1'=0$, we can apply Lemma 5.11 and we get

$$I_1(p)=\frac{-1}{\lambda_k-\lambda_1}\left((E_k-E_1)\left(\phi_1'\big|_{\partial\Omega}\right)+\left(K_1^*-K_k^*\right)\left(\frac{\partial\phi_1'}{\partial\nu}\right)\right). \tag{78}$$

Using the arc-length σ, recall that $p=O(\sigma)$. Thus, we have $p+q_*=q_*+O(\sigma)$ and we write $I_1(\sigma)$ for $I_1(p(\sigma))$. According to (116), we first deduce that

$$\begin{aligned}\frac{1}{\lambda_k-\lambda_1}(E_k-E_1)\left(\phi_1'\right)&=\frac{1}{\lambda_k-\lambda_1}(a_1+2a_2)(\lambda_k-\lambda_1)\frac{\partial\phi_1}{\partial\nu}(0)\sigma\ln|\sigma|+\mathcal{R}_2\\&=(a_1+2a_2)\frac{\partial\phi_1}{\partial\nu}(0)\sigma\ln|\sigma|+\mathcal{R}_2.\end{aligned} \tag{79}$$

As K_k^* and K_1^* have the same principal part, by Lemma B.5, we know that $(K_k^*-K_1^*)\frac{\partial\phi_1'}{\partial\nu}$ is a 2-regular term. Then,

$$I_1(\sigma)=-(a_1+2a_2)\frac{\partial\phi_1}{\partial\nu}(0)\sigma\ln|\sigma|+\mathcal{R}_2. \tag{80}$$

We now treat $I_2(p)$. Taking into account the Taylor expansion of μ at $p+q*$, we can rewrite $I_2(p)=d\mu(p+q_*)J_2(p)+R(p)$ where

$$J_2(p)=\int_\Omega(q-p)\frac{\partial G_k}{\partial\nu_p}(p,q)\phi_1'(q)\,dq,$$

and

$$R(p)=\int_\Omega O\left(\|q-p\|^2\right)\frac{\partial G_k}{\partial\nu_p}(p,q)\phi_1'(q)\,dq.$$

Since $R(\cdot)$ is a more regular term than $J_2(\cdot)$, it is enough to prove that J_2 is of class C^1.

Note that $J_2=\int_\Omega H(p,q)\phi'(q)\,dq$ with $H(\cdot,\cdot)$ the convolution kernel given by $H(p,q):=(q-p)\frac{\partial G_k}{\partial\nu_p}(p,q)$, $p\neq q$. The kernel H is no longer singular (it is actually uniformly bounded on its domain of definition) and straightforward computations yield that H defines a pseudodifferential operator of class $-3/2$. Recall that $\mathcal{H}_0\in H^{1/2-\varepsilon}(\partial\Omega)$ for every $\varepsilon>0$, we deduce that $\phi_1'\in H^{1-\varepsilon}(\Omega)$ for every $\varepsilon>0$, then $I_2\in H^{5/2-\varepsilon}(\partial\Omega)$ for every $\varepsilon>0$. Then $\sigma\mapsto J_2(p(\sigma))$ admits a continuous derivative in an open neighborhood of zero. We conclude that the contribution of J_2 to $\frac{\partial\xi_{1,d}'}{\partial\nu}(\sigma)$ yields an $\mathcal{R}_1$ term.

By Theorem B.4, we finally get

$$\frac{\partial\xi_{1,d}'}{\partial\nu}(\sigma)=-(2a_1+4a_2)\mu(q_*)\frac{\partial\phi_1}{\partial\nu}(0)\sigma\ln|\sigma|+\mathcal{R}_1.\qquad\square \tag{81}$$

5.5. Proof of Proposition 5.4

Collecting the results of Theorems 5.7 and 5.9 in (43), we get that, for σ in some open neighborhood of zero, one has

$$\begin{aligned} M'(u_{q_*})(\sigma) &= \frac{1}{\pi}\left\{\lambda_1 \frac{\partial\phi_1(0)}{\partial\nu}\frac{\partial\xi_k(0)}{\partial\nu} + \lambda_k \frac{\partial\phi_k(0)}{\partial\nu}\frac{\partial\xi_1(0)}{\partial\nu} + \mu(q_*)\frac{\partial\phi_1(0)}{\partial\nu}\frac{\partial\phi_k(0)}{\partial\nu}\right\}\sigma\ln|\sigma| + \mathcal{R}_1 \\ &= -\mathcal{H}_0\frac{\partial M}{\partial\nu}(0). \end{aligned} \tag{82}$$

The left-hand side of (82) is continuous at $\sigma = 0$, which implies that $\frac{\partial M}{\partial\nu}(0) = 0$. Then the left-hand side of (82) must be of class C^1 at $\sigma = 0$, implying that the coefficient of $\sigma\ln|\sigma|$ must also be equal to zero. We finally get that

$$\lambda_1 \frac{\partial\phi_1(0)}{\partial\nu}\frac{\partial\xi_k(0)}{\partial\nu} + \lambda_k \frac{\partial\phi_k(0)}{\partial\nu}\frac{\partial\xi_1(0)}{\partial\nu} + \mu(q_*)\frac{\partial\phi_1(0)}{\partial\nu}\frac{\partial\phi_k(0)}{\partial\nu} = 0, \tag{83}$$

and, since q_* is an arbitrary point of $\partial\Omega$, we get

$$\lambda_1 \frac{\partial\phi_1}{\partial\nu}(q)\frac{\partial\xi_k}{\partial\nu}(q) + \lambda_k \frac{\partial\phi_k}{\partial\nu}(q)\frac{\partial\xi_1}{\partial\nu}(q) + \mu(q)\frac{\partial\phi_1}{\partial\nu}(q)\frac{\partial\phi_k}{\partial\nu}(q) = 0 \quad \text{on } \partial\Omega. \tag{84}$$

Consider now equations (38) and (84) as a linear system with $\frac{\partial\xi_1}{\partial\nu}(q)$ and $\frac{\partial\xi_k}{\partial\nu}(q)$ as unknowns. After an elementary algebraic manipulation, we have, for every $q \in \partial\Omega$,

$$\frac{\partial\phi_k}{\partial\nu}(q)\left\{\frac{\partial\xi_1}{\partial\nu}(q) - \frac{1}{\lambda_1 - \lambda_k}\mu(q)\frac{\partial\phi_1}{\partial\nu}(q)\right\} = 0, \tag{85}$$

$$\frac{\partial\phi_1}{\partial\nu}(q)\left\{\frac{\partial\xi_k}{\partial\nu} - \frac{1}{\lambda_k - \lambda_1}\mu(q)\frac{\partial\phi_k}{\partial\nu}(q)\right\} = 0. \tag{86}$$

As ϕ_1 and ϕ_k are eigenfunctions of $-\Delta_\Omega^D$, by Holmgren uniqueness theorem (see [43, Proposition 4.3, p. 433]), their normal derivatives cannot be equal to zero on a subset of $\partial\Omega$ with nonnull measure. Then, by a simple density argument, we get

$$\frac{\partial\xi_1}{\partial\nu}(q) - \frac{1}{\lambda_1 - \lambda_k}\mu(q)\frac{\partial\phi_1}{\partial\nu}(q) = 0 \quad \text{on } \partial\Omega, \tag{87}$$

$$\frac{\partial\xi_k}{\partial\nu}(q) - \frac{1}{\lambda_k - \lambda_1}\mu(q)\frac{\partial\phi_k}{\partial\nu}(q) = 0 \quad \text{on } \partial\Omega. \tag{88}$$

What we have proved so far is that, if property (B_k), $k > 1$, is not generic then a certain property (C_k) is not as well, where the latter property is defined exactly as in Definition 5.2 except that the function M defined in (30) is replaced by the function $S : \partial\Omega \to \mathbb{R}^2$ defined by

$$S(q) := \frac{\partial\xi_1}{\partial\nu}(q) - \frac{1}{\lambda_1 - \lambda_k}\mu(q)\frac{\partial\phi_1}{\partial\nu}(q) \quad \text{for } q \in \partial\Omega. \tag{89}$$

As in Proposition 5.4, it now amounts to prove that the function S defined in (89) cannot be identically equal to zero on any $E_\rho(\Omega)$ with $\rho > 0$. We can follow the same strategy developed in Section 5.2 and use the computations made in Section 5.4.

Reasoning by contradiction, we assume that $S \equiv 0$ on $\partial\Omega$. Taking the shape differentiation of that equation and using a variation u_{q_*} for an arbitrary $q_* \in \partial\Omega$, we get

$$\frac{\partial \xi_1'(u_{q_*})}{\partial \nu}(q) - \frac{1}{\lambda_1-\lambda_k}\mu(q)\frac{\partial \phi_1'(u_{q_*})}{\partial \nu}(q) = -\mathcal{H}_0\frac{\partial}{\partial \nu}\left\{\frac{\partial \xi_1}{\partial \nu}(q) - \frac{1}{\lambda_1-\lambda_k}\mu(q)\frac{\partial \phi_1}{\partial \nu}(q)\right\}. \tag{90}$$

Using Propositions 5.8 and 5.10, we have at $\sigma = 0$

$$\begin{aligned}
&-2\left(\frac{\partial \xi_1}{\partial \nu}(0) - \frac{1}{\lambda_1-\lambda_k}\frac{\partial \phi_1}{\partial \nu}(0)\mu(q_*)\right)a_1 \text{ p.v.}\left(\frac{1}{\sigma}\right)\\
&\quad + 2\left(\frac{\partial}{\partial \tau}\left(\frac{\partial \xi_1}{\partial \nu}\right)(0) - \frac{1}{\lambda_1-\lambda_k}\frac{\partial}{\partial \tau}\left(\frac{\partial \phi_1}{\partial \nu}\right)(0)\mu(q_*)\right)a_1 \ln|\sigma| + O\big(\sigma\ln|\sigma|\big)\\
&= -\mathcal{H}_0\frac{\partial}{\partial \nu}\left\{\frac{\partial \xi_1}{\partial \nu} - \frac{1}{\lambda_1-\lambda_k}\frac{\partial \phi_1}{\partial \nu}\mu(q)\right\}(0).
\end{aligned}$$

By (87), we simplify the previous equation and get

$$\begin{aligned}
&2\left(\frac{\partial}{\partial \tau}\left(\frac{\partial \xi_1}{\partial \nu}\right)(0) - \frac{1}{\lambda_1-\lambda_k}\frac{\partial}{\partial \tau}\left(\frac{\partial \phi_1}{\partial \nu}\right)(0)\mu(q_*)\right)a_1 \ln|\sigma| + O\big(\sigma\ln|\sigma|\big)\\
&\quad = \mathcal{H}_0\frac{\partial}{\partial \nu}\left\{\frac{\partial \xi_1}{\partial \nu} - \frac{1}{\lambda_1-\lambda_k}\frac{\partial \phi_1}{\partial \nu}\mu\big(q(\sigma)\big)\right\}
\end{aligned}$$

Since the right-hand side remains bounded in neighborhood of $\sigma = 0$, it is necessary that

$$\frac{\partial}{\partial \tau}\left(\frac{\partial \xi_1}{\partial \nu}\right)(0) - \frac{1}{\lambda_1-\lambda_k}\frac{\partial}{\partial \tau}\left(\frac{\partial \phi_1}{\partial \nu}\right)(0)\mu(q_*) = 0. \tag{91}$$

On the other hand, by taking the tangent derivative of (87) at $q = q_*$, we have

$$\frac{\partial S}{\partial \tau}\big(q(\tau)\big) = \frac{\partial}{\partial \tau}\left(\frac{\partial \xi_1}{\partial \nu}\right)(0) - \frac{1}{\lambda_1-\lambda_k}\left(\frac{\partial}{\partial \tau}\left(\frac{\partial \phi_1}{\partial \nu}\right)(0)\mu(q_*) + \frac{\partial \phi_1}{\partial \nu}(0)d\mu(q_*)\cdot\tau_0\right) = 0, \tag{92}$$

where τ_0 is the unit tangent vector on $\partial\Omega$ at the point q_*. From (91) and (92), we end up with

$$\frac{\partial \phi_1}{\partial \nu}(0)\, d\mu(q_*)\cdot\tau_0 = 0. \tag{93}$$

As the previous reasoning is valid almost everywhere on $\partial\Omega$, we have

$$\frac{\partial \phi_1}{\partial \nu}(q)\, d\mu(q)\cdot\tau_q = 0, \quad \text{for } q \in \partial\Omega. \tag{94}$$

By condition (36) and by continuity of the map $q \mapsto d\mu(q)\cdot\tau_q$ for $q \in \partial\Omega$, we get that $\frac{\partial\phi_1}{\partial\nu}$ is equal to zero on an open neighborhood of $\bar{q}$ on $\partial\Omega$ (defined in (36)). This is not possible by

Holmgren uniqueness theorem. We finally derived a contradiction and Proposition 5.4 is now proved.

6. Conclusion, conjectures, perspectives

We recapitulate all the controllability results known for (3) in the following array.

	Spectral controllability in time T of (3)	Exact controllability in time T of (3)
1D $\Omega = (0, 1)$	*yes* under (H1) $\forall T > 0$ *no* under *no* (H1)	*yes* under (H2) (i.e., generically with respect to μ) in $H^3_{(0)}((0,1),\mathbb{C})$ with $L^2((0,T),\mathbb{R})$-controls $\forall T > 0$ *no* under *no* (H1)
2D	*yes* under (H3) (i.e., generically with respect to (Ω, μ)) with $T > T_{\min}(\Omega)$ *no* under (H3) and (H4) with $T < T_{\min}(\Omega)$ *no* under *no* (H3)	*yes* under (H3) and (H4) in abstract spaces with $T > T_{\min}(\Omega)$ *no* under *no* (H3)
3D	*no* under (H4)	*no* under (H4)

In this array, we have used the notation

$$T_{\min}(\Omega) := 2\pi d(\Omega)$$

where $d(\Omega) > 0$ is such that

$$\sharp\{k \in \mathbb{N}^*;\ \lambda_k - \lambda_1 \in [0, t]\} \sim d(\Omega)t, \quad \text{when } t \to +\infty.$$

and the assumptions

(H1) $\langle \mu\varphi_1, \varphi_k\rangle_{L^2(\Omega)} \neq 0$, for every $k \in \mathbb{N}^*$,
(H2) there exists $c_1, c_2 > 0$ such that,

$$\frac{c_1}{k^3} \leqslant \left|\langle \mu\varphi_1, \varphi_k\rangle\right| \leqslant \frac{c_2}{k^3}, \quad \forall k \in \mathbb{N}^*,$$

(H3) any eigenvalue λ of $-\Delta^D_\Omega$ has multiplicity $m \leqslant n$ ($n = 2, 3$ is the space dimension: $\Omega \subset \mathbb{R}^n$) and the vectors $\langle \mu\phi_1, \phi_{k_1}\rangle, \ldots, \langle \mu\phi_1, \phi_{k_m}\rangle$ are linearly independent in $\mathbb{R}^n$, where $k_1 < \cdots < k_m$ and $\phi_{k_1}, \ldots, \phi_{k_m}$ are the eigenvectors associated to λ.
(H4) there exists $\tilde{\mu} \in C^0(\overline{\Omega}, \mathbb{R})$ such that $\mu(q) = \tilde{\mu}(q)e_1$.

The assumption (H4) is not necessary for the nonspectral controllability of (3) in small time in 2D (see Remark 3.3). We conjecture that, in 2D, the system (3) is not spectral controllable in small time under (H3). This is an open problem.

Similarly, the assumption (H4) is not necessary for the nonspectral controllability of (3) in any time $T > 0$ in 3D. We conjecture that, in 3D, and with any time $T > 0$, the system (3) is not spectral controllable in time T under (H3). This is an open problem.

A strategy to prove these conjectures could be the adaptation of Haraux and Jaffard's result (Theorem 3.2) to vector exponential families.

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Appendix A. Shape differentiation

The material presented here is borrowed from [42] and [35].

A.1. Main definitions

Let Ω be a domain in $\mathbb{D}_3$. For a positive integer l, we consider perturbations u in the space $W^{l,\infty}(\Omega,\mathbb{R}^2)$ with norm

$$\|u\|_{l,\infty} := \sup \mathrm{ess}\big\{\big|D^\alpha u(x)\big|;\ 0 \leqslant \alpha \leqslant l,\ x \in \Omega\big\}.$$

Then, the domain $\Omega + u$ is defined by

$$\Omega + u := (\mathrm{Id}_+ u)(\Omega) = \big\{x + u(x),\ x \in \Omega\big\}.$$

Lemma A.1. *(Cf. [42].) Let $l \in \mathbb{N}^*$ and $u \in W^{l,\infty}(\Omega,\mathbb{R}^2)$ be such that $\|u\|_{l,\infty} \leqslant 1/2$. Then, the map $\mathrm{Id}_+ u$ is invertible. Furthermore, there exits $w \in W^{l,\infty}(\Omega + u,\mathbb{R}^2)$ such that $(\mathrm{Id}_+ u)^{-1} = \mathrm{Id}_+ w$ and $\|w\|_{l,\infty} \leqslant C_l \|u\|_{l,\infty}$ where C_l is a constant independent on u.*

Remark A.1. According to this result, if Ω is of class C^j, we can choose $l = j + 1$ so that the new domain $\Omega + u$ is also of class C^j. In particular, if we need domains of class C^3, we can choose $W^{4,\infty}(\Omega,\mathbb{R}^2)$ as the perturbation space.

We now consider a function

$$v : u \in W^{l,\infty}\big(\Omega,\mathbb{R}^2\big) \to\ v(u) \in W^{m,r}(\Omega + u)$$

where $1 \leqslant r < \infty$ and $m \leqslant l$ are integers. In practice, $v(u)$ is solution of a suitable problem, which depends on the perturbation function u. We are interested in the study of the regularity of the function $v(u)$ with respect to the perturbation function u.

Definition A.1 *(First order local variation).* Let $k \geqslant m \geqslant 1$ and $1 \leqslant r < \infty$. We say that the function $v(u)$ has a first order local variation at $u = 0$ on $W^{m,\,r}(\Omega + u)$ for all $u \in W^{l,\infty}(\Omega,\mathbb{R}^2)$ if there exists a linear map $v'(u)$ from $u \in W^{l,\infty}(\Omega,\mathbb{R}^2)$ to $v'(u) \in W^{m-1,r}_{\mathrm{loc}}(\Omega)$ such that, for every open set $\omega \Subset \Omega$,

$$v(u) = v(0) + v'(u) + \theta(u) \quad \text{in } \omega,$$

when $\|u\|_{l,\infty}$ is small enough and

$$\frac{\|\theta(u)\|_{m-1,r}}{\|u\|_{l,\infty}} \to 0 \quad \text{as } \|u\|_{l,\infty} \to 0.$$

Remark A.2. The first order local variation can also be defined as

$$v'(u) = \lim_{t\to 0} \frac{v(tu)|_\omega - v(0)|_\omega}{t} \quad \text{in } W^{m-1,r}(\omega),$$

where $\omega \Subset \Omega$.

The following theorem provides sufficient conditions for the existence of the first order local variation.

Theorem A.2. *(Cf. [42].) Let Ω be a $C^{0,1}$ domain. Consider the map $u \to v(u) \in W^{m,r}(\Omega + u)$ defined on a neighborhood of $u = 0$ in $W^{k,\infty}(\Omega, \mathbb{R}^2)$, with $k \geqslant m \geqslant 1$ and $1 \leqslant r < \infty$. Assume that there exists a linear continuous map $u \in W^{k,\infty}(\Omega) \to \dot v(u) \in W^{m,r}(\Omega)$ such that*

$$v(u) \circ (\mathrm{Id}_{+}u) = v(0) + \dot v(u) + \theta(u) \quad \textit{in } W^{m,r}(\Omega),$$

for all $u \in W^{k,\infty}(\Omega, \mathbb{R}^2)$ small enough, where

$$\frac{\|\theta(u)\|_{m-1,r}}{\|u\|_{k,\infty}} \to 0 \quad \textit{as } \|u\|_{k,\infty} \to 0.$$

Furthermore, we assume that for every $u \in W^{k,\infty}(\Omega, \mathbb{R}^2)$ small enough,

$$v(u) = 0 \quad \textit{on } \partial\Omega + u.$$

Then, for each $\omega \Subset \Omega$, the function $u \to v(u)|_\omega \in W^{m-1,r}(\omega)$ defined on a neighborhood of $u = 0$ in $W^{k,\infty}(\Omega, \mathbb{R}^2)$ is differentiable at $u = 0$.

Moreover, the map $u \to v(u)|_\omega$ has a first order local variation and this variation at $u = 0$ in the direction u_1 denoted by $v'(u_1)$ verifies $v'(u_1) \in W^{m-1,r}(\Omega)$ and

$$v'(u_1) = -\langle u_1, \nu\rangle \frac{\partial v(0)}{\partial \nu} \quad \textit{on } \partial\Omega. \tag{95}$$

A.2. Regularity of the eigenvalues and eigenfunctions

By applying [36, Theorem 3] in the same way as in [35], we get the following result.

Theorem A.3. *Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain of class C^1. Let λ be an eigenvalue of multiplicity h of $-\Delta^D_\Omega$, with associated orthonormal eigenfunctions $y_1, \dots, y_h$. Then, there exist h real-valued continuous functions, $u \mapsto \lambda_i^{\Omega+u}$, and h continuous functions with values in $H^2 \cap H^1_0(\Omega, \mathbb{R})$, $u \mapsto y_i(u)$, for $i = 1, \dots, h$, defined in a neighborhood U of $u = 0$ in $W^{4,\infty}(\Omega, \mathbb{R}^3)$ such that the following properties hold,*

- *$\lambda_i^{\Omega} = \lambda$ for $i = 1, \ldots, h$,*
- *for every $u \in U$, $\varphi_i^{\Omega+u} := y_i(u) \circ (I+u)^{-1}$ is an eigenfunction of $-\Delta_{\Omega+u}^D$ associated to the eigenvalue $\lambda_i^{\Omega+u}$,*
- *for every $u \in U$, the family $(\varphi_1^{\Omega+u} \ldots, \varphi_h^{\Omega+u})$ is orthonormal in $L^2(\Omega+u, \mathbb{R})$,*
- *for each open interval $I \subset \mathbb{R}$, such that the intersection of I with the set of eigenvalues of $-\Delta_\Omega^D$ contains only λ, there exists a neighborhood $U_I \subset U$ such that, for every $u \in U_I$, there exist exactly h eigenvalues (counting the multiplicity), $\lambda_i^{\Omega+u}$, $1 \leqslant i \leqslant h$, of $-\Delta_{\Omega+u}^D$ contained in I,*
- *for each $u \in W^{2,\infty}(\Omega, \mathbb{C})$ and $1 \leqslant i \leqslant h$, the map*

$$\mathbb{R} \to \mathbb{R} \times H^2 \cap H_0^1(\Omega, \mathbb{R}),$$
$$t \mapsto \big(\lambda_i^{\Omega+tu}, y_i(tu)\big)$$

is analytic in a neighborhood of $t = 0$.

A.3. Local variations of the eigenvalues and eigenfunctions

Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain of class C^1. Let λ be an eigenvalue of multiplicity h of $-\Delta_\Omega^D$, with associated orthonormal eigenfunctions $y_1, \ldots, y_h$. Let $\varphi_i(u) \in H^2 \cap H_0^1(\Omega+u, \mathbb{R})$, $i = 1, \ldots, h$ be the eigenfunctions of $-\Delta_{\Omega+u}^D$ associated to the eigenvalues $\lambda_i(u)$, $i = 1, \ldots, h$, where $\lambda_i(0) = \lambda$ for $i = 1, \ldots, h$.

According to the result of the previous section, the functions $t \mapsto \lambda_i(tu)$ and $t \mapsto \varphi_i(tu)$ are analytic in a neighborhood of 0. Let us denote by

$$\lambda_i'(u_0)\left(\text{resp. } \frac{d\varphi_i}{du}\bigg]_{u_0}\right)$$

the value of the directional derivative of λ_i (resp. φ_i) at $u = 0$ in the direction u_0,

$$\lambda_i'(u_0) := \lim_{t\to 0} \frac{\lambda_i(tu_0) - \lambda_i(0)}{t}.$$

For $i = 1, \ldots, h$, $\frac{d\varphi_i}{du}]_{u_0} \in H^2(\Omega, \mathbb{R})$ and, for every open subset $\omega \Subset \Omega$,

$$\frac{d\varphi_i}{du}\bigg]_{u_0} := \lim_{t\to 0} \frac{\varphi_i(tu_0)|_\omega - \varphi_i(0)|_\omega}{t} \quad \text{in } H^2\big(\omega, \mathbb{R}^3\big).$$

We have, for every $t \in \mathbb{R}$,

$$\begin{cases} -\Delta\varphi_i(tu_0) = \lambda_i(tu_0)\varphi_i(tu_0) \quad \text{in } \Omega + tu_0, \\ \varphi_i(tu_0) = 0 \quad \text{on } \partial(\Omega + tu_0), \\ \displaystyle\int_{\Omega+tu_0} \big|\varphi_i(tu_0)(q)\big|^2\, dq = 1. \end{cases}$$

Thus, using classical results on shape differentiation (see [42]), we get

$$\begin{cases} -(\Delta+\lambda_i)\left.\dfrac{d\varphi_i}{du}\right]_{u_0} = \lambda_i'(u_0)\varphi_i \quad \text{in } \Omega, \\ \left.\dfrac{d\varphi_i}{du}\right]_{u_0} = -u_0.\nabla\varphi_i \quad \text{on } \partial\Omega, \\ \displaystyle\int_\Omega \varphi_i(q)\left.\dfrac{d\varphi_i}{du}\right]_{u_0}(q)\,dq = 0. \end{cases} \tag{96}$$

Remark A.3. We note that all results stated above can be easily extended for C^3 domains and variations $u \in W^{4,\infty}(\Omega, \mathbb{R}^2)$.

Appendix B. The Dirichlet-to-Neumann map for the Helmholtz equation

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a connected boundary $\partial\Omega$ of class C^3 and outward unit normal ν. For $k > 0$, we consider the following problem

$$\begin{cases} (\Delta + k^2)u = F, & \text{in } \Omega, \\ u = f, & \text{on } \partial\Omega. \end{cases} \tag{97}$$

The goal of this section is to study the Dirichlet-to-Neumann map associated to (97) when $-k^2$ is an eigenvalue of the interior Dirichlet problem. In Section B.1, we recall some useful background results (see, for instance, [4,13,34,44]). In Section B.2, we study precisely the Dirichlet-to-Neumann map associated to (97).

B.1. Preliminary results on Helmholtz equation

A standard approach for studying the Helmholtz equations consists in the representation of the solution using the single and double layer potentials respectively defined by

$$S_k(f)(p) := \int_{\partial\Omega} G_k(p,q) f(q)\,d\sigma(q), \quad \forall p \in \mathbb{R}^2 \backslash \partial\Omega, \tag{98}$$

and

$$D_k(f)(p) := \int_{\partial\Omega} \frac{\partial G_k(p,q)}{\partial \nu_q} f(q)\,d\sigma(q), \quad \forall p \in \mathbb{R}^2 \backslash \partial\Omega, \tag{99}$$

where $G_k(.,.)$ is the fundamental solution of the Helmholtz equation that satisfies the Sommerfeld condition and $f \in L^2(\partial\Omega)$. Here the notation $\frac{\partial}{\partial \nu_q}$ stands for the outward unit normal to $\partial\Omega$ at the point q. Then the solution of (97) is given by the third Green formula,

$$u = -S_k\left(\frac{\partial u}{\partial \nu}\right) + D_k(f) + F * G_k, \tag{100}$$

where

$$F * G_k(p) := \int_\Omega F(q) G_k(p,q)\,dq, \quad \forall p \in \Omega.$$

For the reader's convenience, we recall the following useful standard result which highlights the difference between the fundamental solution G_0 of the Laplace equation and G_k (see [2] and [33]).

B.1.1. Fundamental solution

Proposition B.1. *Let $k > 0$. The fundamental solution for the Helmholtz equation is*

$$G_k(p,q) = -\frac{i}{4} H_0^1\big(k|p-q|\big) \tag{101}$$

where H_0^1 denotes the Hankel function of the first kind of order 0. If $G_0(p,q) := \frac{1}{2\pi}\ln|p-q|$ is the fundamental solution of the Laplace equation, then we have

$$G_k(p,q) = G_0(p,q) + g_k(p,q), \tag{102}$$

where $g_k = g_k^{(1)} + g_k^{(2)}$ with

$$g_k^{(1)}(p,q) := -\frac{1}{2\pi}\ln\left(\frac{k}{2}\right) + \frac{1}{2\pi}\ln\left(\frac{k|p-q|}{2}\right)\sum_{j=1}^{\infty}\frac{(-1)^j}{(j!)^2}\left(\frac{k|p-q|}{2}\right)^{2j},$$

and

$$g_k^{(2)}(p,q) := -\frac{i}{4}J_0\big(k|p-q|\big) + \sum_{j=1}^{\infty}\frac{(-1)^j}{(j!)^2}\psi(j+1)\left(\frac{k|p-q|}{2}\right)^{2j},$$

with ψ, the digamma function and J_0, the Bessel function of first kind.

The interested reader can find details in [1] and [12].

B.1.2. Jump relations

Now, let us state jump relations satisfied by the layer potentials and their normal derivative. We recall the standard notations

$$f|_\pm(p) = \lim_{t\to 0^+} f(x \pm t\nu_p), \quad p \in \partial\Omega,$$

and

$$\frac{\partial}{\nu_p} f|_\pm(p) = \lim_{t\to 0^+}\big\langle \nabla f(p \pm t\nu_p), \nu_p\big\rangle, \quad p \in \partial\Omega.$$

We quote from [4, Lemma 11.1, p. 186] the following result.

Theorem B.2. *Let Ω be a C^3 domain in $\mathbb{R}^2$ and let $f \in L^2(\partial\Omega)$. We have*

$$\begin{aligned}\big(S_k(f)\big)\big|_+(p) = \big(S_k(f)\big)\big|_-(p) = S_k(f)(p), \quad \text{a.e. } p \in \partial\Omega,\\ \left(\frac{\partial}{\partial \nu_p} S_k(f)\right)\bigg|_\pm(p) = \left(\pm\frac{1}{2}I + K_k^*\right) f(p), \quad \text{a.e. } p \in \partial\Omega,\\ D_k(f)|_\pm(p) = \left(\mp\frac{1}{2}I + K_k\right) f(p), \quad \text{a.e. } p \in \partial\Omega,\end{aligned} \tag{103}$$

where K_k is the operator defined by

$$K_k\phi(p) := \text{p.v.} \int\limits_{\partial\Omega} \frac{\partial G_k(p,q)}{\partial \nu_q} \phi(q)\, d\sigma(q), \quad p \in \partial\Omega, \tag{104}$$

and where K_k^* is its $L^2(\partial\Omega)$-adjoint.

An other operator will be of interest and will play a major role in our computations. It is the normal derivative of $D_k(f)$,

$$E_k(f)(p) := \frac{\partial}{\partial \nu_p}\left(\int\limits_{\partial\Omega} \frac{\partial G_k}{\partial \nu_q}(p,q) f(q)\, d\sigma(q)\right), \quad p \in \partial\Omega. \tag{105}$$

Remark B.1. There is not a jump relation for the normal derivative of the double-layer potential across the boundary $\partial\Omega$.

B.1.3. Mapping properties in Sobolev spaces

The following results are also needed (see [44, Chapter 7] and [13, Chapter 3]).

Theorem B.3. *Let Ω be a C^3 domain and $s \in \mathbb{R}$. Then,*

(i) *the operator S_k is bounded from $H^s(\partial\Omega)$ into $H^{s+1}(\partial\Omega)$,*
(ii) *the operators K_k and its adjoint K_k^* are bounded from $H^s(\partial\Omega)$ into $H^{s+1}(\partial\Omega)$,*
(iii) *the operators $\frac{I}{2} \pm K_k^*$ and $\frac{I}{2} \pm K_k$ are bounded from: $H^s(\partial\Omega)$ into $H^s(\partial\Omega)$,*
(iv) *the operator $K_k^* - K_0^*$ is continuous from $H^s(\partial\Omega)$ into $H^{s+3}(\partial\Omega)$.*
(v) *the operator E_k is bounded from $H^s(\partial\Omega)$ into $H^{s-1}(\partial\Omega)$.*

Proof. The results concerning the single and double layer potential are developed in (cf. [34, Chapter 4, paragraph 2])) where are studied the boundedness properties of singular integral operators whose kernels are the restriction to $\partial\Omega$ of kernels defined in $\mathbb{R}^2$.

In $\mathbb{R}^2$, the layer potential kernel associated to Helmholtz equation is $K(x) = H_0^{(1)}(k|x|)$ where $H_0^{(1)}$ is the Hankel function of order 0. A Taylor expansion shows that the kernel is pseudo-homogeneous of class -1. Thanks to [34, Theorem 4.3.1], we conclude that S_k is bounded from $H^s(\partial\Omega)$ into $H^{s+1}(\partial\Omega)$,

Concerning the double layer potential, its regularity property is due to the fact that its kernel is pseudo-homogeneous of class -1. From [34, Theorem 4.3.1], K_k and K_k^* are bounded from

$H^s(\partial\Omega)$ into $H^{s+1}(\partial\Omega)$ for every real s. We point out that one can find the detailed computations in [34, Example 4.5, Section 4.3.3].

A Taylor expansion shows that the kernel of the operator $K_k^* - K_0^*$ has the same property as $E(x, y)$, the kernel of the single layer potential corresponding to the biharmonic equation (cf. [17]). We recall that

$$\frac{\partial E(x,y)}{\partial \nu_y} = \frac{1}{8\pi}\langle \nu_y, x-y\rangle \big(2\ln|x-y|+1\big).$$

The factor $\langle \nu_y, x-y\rangle$ is regular on $\partial\Omega \times \partial\Omega$ and furthermore for small $|x-y|$ it satisfies

$$\langle \nu_y, x-y\rangle = O\big(|x-y|^2\big). \tag{106}$$

Thus, for an element (x, y) living near the diagonal $\partial\Omega \times \partial\Omega$, we have

$$\frac{\partial E(x,y)}{\partial \nu_y} = O\big(|x-y|^2 \ln|x-y|\big).$$

It follows that $E(x, y)$ and the kernel of $K_k^* - K_0^*$ have the same smoothing effects. Furthermore, from [34, Example 4.3, p. 216], we get that the kernel of $K_k^* - K_0^*$ is pseudo-homogeneous of class -3. Thanks to [34, Theorem 4.3.1], it comes that $K_k^* - K_0^*$ is continuous from $H^s(\partial\Omega)$ into $H^{s+3}(\partial\Omega)$, for every real s.

To finish, we see E_k as a pseudodifferential operator on $\partial\Omega$ whose leading symbol is of the form $p(\xi) = -\frac{1}{2}|\xi|$. Consequently, the operator E_k is continuous from $H^s(\partial\Omega)$ into $H^{s-1}(\partial\Omega)$. □

B.2. Dirichlet-to-Neumann map

The goal of this section is the study of the singularities of the normal derivative of the solution of (97). From (100), (103) and (105), we deduce

$$\left(\frac{1}{2}I + K_k^*\right)\frac{\partial u}{\partial \nu} = E_k(f) + \frac{\partial}{\partial \nu}(F * G_k). \tag{107}$$

In Section B.2.1, we study the inverse of the operator $(\frac{1}{2}I + K_k^*)$, thanks to the reduced resolvent theory. In Section B.2.2, we study the normal derivative of the double-layer potential $E_k(f)$.

B.2.1. Singular perturbation problem and reduced resolvent

Notice that, when $-k^2$ is an eigenvalue of the interior Dirichlet problem for the Laplacian, the integral equation (107) is not invertible. The associated operator $(\frac{1}{2}I + K_k^*)$ is in fact invertible except for these critical values.

In this subsection, we show how to solve (107) in an efficient manner. More precisely, we consider a general right-hand side v, which is assumed to belong to the range of $\frac{1}{2}I + K_k^*$ and whose Taylor expansion in an open neighborhood of zero takes the following form,

$$v(\sigma) = \alpha_1 \text{ p.v.}\left(\frac{1}{\sigma}\right) + \alpha_2 \ln|\sigma| + \alpha_3\sigma\ln|\sigma| + \mathcal{R}_2, \tag{108}$$

where α_1, α_2 and α_3 are arbitrary real numbers and where σ denotes the oriented counterclockwise arc-length of the boundary $\partial\Omega$ and $\mathcal{R}_2$ is an error term defined in Definition 5.4.

The main idea is to break up the explicit formula of $\frac{\partial u}{\partial \nu}$ into two parts. The first part reflects the singular behavior of $\frac{\partial u}{\partial \nu}$ and it will not depend on the eigenvalue $-k^2$ of the Laplacian. The second part is a regular remainder of the type $\mathcal{R}_2$. Precisely, the goal of this subsection is the proof of the following result.

Theorem B.4. *Assume that* $\frac{\partial u}{\partial \nu}$ *satisfies the equation*

$$\left(\frac{1}{2}I + K_k^*\right)\frac{\partial u}{\partial \nu} = v, \tag{109}$$

where v *is given by* (108). *Then, we have*

$$\frac{\partial u}{\partial \nu} = 2v + T_0 v + \mathcal{R}_2, \tag{110}$$

where the linear operator T_0 *given by*

$$T_0 := -2\left(\frac{1}{2}I + K_0^*\right)^{-1} K_0^*, \tag{111}$$

defines a bounded operator from $H^s(\partial\Omega)$ *into* $H^{s+1}(\partial\Omega)$, *for every* $s \in \mathbb{R}$.

For the proof of the result, precise information on $(K_k^* - K_0^*)$ p.v.$(\frac{1}{\sigma})$ is needed. Although we know the higher smoothing effect of $K_k^* - K_0^*$ the operator, we have to show the following result.

Lemma B.5. *Let* $k > 0$. *The distribution* $(K_k^* - K_0^*)$ *p.v.*$(\frac{1}{\sigma})$ *is of the type* $\mathcal{R}_2$.

Note that the above distribution makes sense thanks to Remark 5.5.

Proof. We are led to study the Taylor expansion of

$$I(\sigma_0) = \int_{-L}^{L} (\sigma - \sigma_0)^2 \ln|\sigma - \sigma_0| \text{ p.v.}\left(\frac{1}{\sigma}\right) d\sigma, \tag{112}$$

for σ_0 in an open neighborhood of zero. We may assume $\sigma_0 > 0$ and we fix $\alpha > 0$ small enough. We have to evaluate the following limit

$$\begin{aligned}
&\lim_{\epsilon\to 0}\left(\int_{-\alpha}^{-\epsilon} (\sigma - \sigma_0)^2 \ln|\sigma - \sigma_0| \frac{1}{\sigma}\, d\sigma + \int_{\epsilon}^{\alpha} (\sigma - \sigma_0)^2 \ln|\sigma - \sigma_0| \frac{1}{\sigma}\, d\sigma\right) \\
&= \lim_{\epsilon\to 0} \int_{\epsilon}^{\alpha} \frac{d\sigma}{\sigma}\left[(\sigma - \sigma_0)^2 \ln|\sigma - \sigma_0| - (\sigma + \sigma_0)^2 \ln|\sigma + \sigma_0|\right] \\
&= I_1(\sigma_0) + I_2(\sigma_0) + I_3(\sigma_0),
\end{aligned}$$

where we set

$$I_1(\sigma_0) := \lim_{\epsilon\to 0}\int_\epsilon^\alpha \sigma \ln\frac{|\sigma-\sigma_0|}{\sigma_0+\sigma}\,d\sigma,$$

$$I_2(\sigma_0) := -2\sigma_0 \lim_{\epsilon\to 0}\int_\epsilon^\alpha \big(\ln|\sigma-\sigma_0| + \ln|\sigma+\sigma_0|\big)\,d\sigma,$$

$$I_3(\sigma_0) := \sigma_0^2 \lim_{\epsilon\to 0}\int_\epsilon^\alpha \ln\frac{|\sigma-\sigma_0|}{\sigma_0+\sigma}\frac{d\sigma}{\sigma}.$$

We first estimate $I_1(\sigma_0)$. The function in the integral is integrable at $\sigma = 0$, then $I_1(\sigma_0) = \int_0^\alpha \sigma \ln\frac{|\sigma-\sigma_0|}{\sigma_0+\sigma}\,d\sigma$. We first make the change of variable $t = \sigma/\sigma_0$. We get $I_1(\sigma_0) = \sigma_0^2(C_0 + J_1(\alpha/\sigma_0))$, where $C_0 = \int_0^1 t\ln\frac{|t-1|}{t+1}\,dt$ and $J_1(X) = \int_1^X t\ln\frac{|t-1|}{t+1}\,dt$ for $X \geqslant 1$. By integrating by part J_1, we obtain

$$I_1(\sigma_0) = C_0\sigma_0^2 + \frac{\alpha^2-\sigma_0^2}{2}\ln\left(\frac{\alpha-\sigma_0}{\alpha+\sigma_0}\right) - \sigma_0(\alpha-\sigma_0).$$

Then I_1 is of class C^2 in a neighborhood of zero.

We next consider $I_2(\sigma_0)$. We have

$$\begin{aligned} I_2(\sigma_0) &= -2\sigma_0\left\{\int_{-\sigma_0}^{\alpha-\sigma_0} \ln|s|\,ds + \int_{\sigma_0}^{\alpha+\sigma_0} \ln|s|\,ds\right\} \\ &= -2\sigma_0\big\{(\alpha-\sigma_0)\ln|\alpha-\sigma_0| + (\alpha+\sigma_0)\ln|\alpha+\sigma_0| - 2\alpha\big\}, \end{aligned}$$

which show that I_2 is real analytic in an open neighborhood of zero.

Finally, we estimate $I_3(\sigma_0)$. We have

$$\begin{aligned} I_3(\sigma_0) &= \sigma_0^2 \lim_{\epsilon\to 0}\int_\epsilon^\alpha \ln\frac{|\sigma-\sigma_0|}{\sigma_0+\sigma}\frac{d\sigma}{\sigma} = \sigma_0^2 \lim_{\epsilon\to 0}\int_{\epsilon/\sigma_0}^{\alpha/\sigma_0} \ln\frac{|1-t|}{1+t}\frac{dt}{t} \\ &= \sigma_0^2 C_1 + \sigma_0^2 H_1(\alpha/\sigma_0), \end{aligned}$$

where $C_1 = \int_0^1 \ln\frac{1-t}{1+t}\frac{dt}{t}$ and $H_1(X) = \int_1^X \ln\frac{t-1}{t+1}\frac{dt}{t}$ for $X \geqslant 1$. Making the change of variable $v = \frac{t-1}{t+1}$ in H_1, we have $H_1(\alpha/\sigma_0) = 2\int_0^\beta \frac{\ln v}{1-v^2}\,dv$, where $\beta = \frac{\alpha-\sigma_0}{\alpha+\sigma_0}$. We note that $\beta < 1$. Then,

$$\begin{aligned} \int_0^\beta \frac{\ln v}{1-v^2}\,dv &= \int_0^\beta \ln v \sum_{n\geqslant 0} v^{2n}\,dv = \sum_{n\geqslant 0}\int_0^\beta (\ln v) v^{2n}\,dv \\ &= \sum_{n\geqslant 0}\frac{\beta^{2n+1}\ln\beta}{2n+1} - \sum_{n\geqslant 0}\frac{\beta^{2n+1}}{(2n+1)^2} = S_1 + S_2. \end{aligned}$$

For S_1, we have

$$\begin{aligned}
S_1 &= \frac{1}{2}\ln(\beta)\left(\sum_{n\geqslant 1}\frac{\beta^n}{n} - \sum_{n\geqslant 1}\frac{(-\beta)^n}{n}\right)\\
&= \frac{1}{2}\ln(\beta)\big(-\ln(1-\beta)+\ln(1+\beta)\big)\\
&= \frac{1}{2}\ln\frac{1+\beta}{1-\beta}\ln\beta\\
&= \frac{1}{2}\ln\left(1-\frac{2\sigma_0}{\alpha+\sigma_0}\right)\ln\frac{\alpha}{\sigma_0}.
\end{aligned}$$

For S_2, we begin by computing $\frac{dS_2}{d\sigma_0}$.

$$\begin{aligned}
\frac{dS_2}{d\sigma_0} &= -\sum_{n\geqslant 0}\frac{\beta^{2n}}{2n+1}\frac{d\beta}{d\sigma_0} = \frac{1}{\beta}\sum_{n\geqslant 0}\frac{\beta^{2n+1}}{2n+1}\frac{2\alpha}{(\alpha+\sigma_0)^2}\\
&= \frac{\alpha}{\alpha^2-\sigma_0^2}\ln\frac{\alpha}{\sigma_0} = \frac{1}{\alpha(1-\frac{\sigma_0^2}{\alpha^2})}\ln\frac{\alpha}{\sigma_0}.
\end{aligned}$$

Recall that $I_3(\sigma_0) = \sigma_0^2 C_1 + \sigma_0^2(S_1+S_2)$, the computations above show that I_3 is a 2-regular term. □

We are now ready to prove Theorem B.4.

Proof of Theorem B.4. We subdivide the proof in several steps.

Step 1: We begin to recall some results on the reduced resolvent theory (cf. [27, Chapter I, paragraph 5]). Since $\lambda = 0$ is an eigenvalue of $(\frac{1}{2}I + K_k^*)$, the resolvent

$$R(\lambda) = \left(\left(\frac{1}{2}-\lambda\right)I + K_k^*\right)^{-1}$$

has a singularity at $\lambda = 0$. Since the dimension of the eigenspace associated to $\lambda = 0$ is equal to one, the resolvent is expanded as a Laurent series

$$\left(\left(\frac{1}{2}-\lambda\right)I + K_k^*\right)^{-1} = \frac{A_{-1,k}}{\lambda} + \sum_{n=0}^{\infty}\lambda^n A_{0,k}^{n+1}$$

in a neighbourhood of $\lambda = 0$. The notations $A_{-1,k}$ and $A_{0,k}$ stand for

$$A_{-1,k} := \frac{1}{2i\pi}\int_\Gamma \left(\left(\frac{1}{2}-\lambda\right)I + K_k^*\right)^{-1} d\lambda,$$

and

$$A_{0,k} := \frac{1}{2i\pi} \int_\Gamma \frac{1}{\lambda}\left(\left(\frac{1}{2}-\lambda\right)I + K_k^*\right)^{-1} d\lambda,$$

where Γ is a small positively oriented circle enclosing 0 in $\mathbb{C}$. According to [27], the operator $P_0 := -A_{-1,k}$ is a projector on the null space associated to $\lambda = 0$ and moreover

$$A_{0,k}P_0 = P_0 A_{0,k} = 0,$$
$$\left(\frac{1}{2}I + K_k^*\right)A_{0,k} = A_{0,k}\left(\frac{1}{2}I + K_k^*\right) = I - P_0.$$

The last equalities show that $A_{0,k}$ is the "inverse" of $(\frac{1}{2}I + K_k^*)$ restrained to the complementary subspace to the null space associated to $\lambda = 0$.

Step 2: Using the reduced resolvent method, one gets

$$\frac{\partial u}{\partial \nu} = A_{0,k}v + \mathcal{U}(v), \tag{113}$$

where $\mathcal{U}(\sqsubseteq)$ is an arbitrary element belonging to $\mathrm{Ker}(\frac{1}{2}I + K_k^*)$.

Recall that $\mathrm{Ker}(\frac{1}{2}I + K_k^*)$ coincides with the span of the traces of all normal derivatives on $\partial\Omega$ of Dirichlet eigenfunctions of the Laplacian with eigenvalue $-k^2$ (see [16, p. 684]). We then deduce that $\mathcal{U}(\sqsubseteq)$ is of type $\mathcal{R}_2$.

We can now rewrite Eq. (113) as follows

$$\begin{aligned}
\frac{\partial u}{\partial \nu} &= \left(\frac{1}{2}I + K_0^*\right)^{-1} v + \left[A_{0,k} - \left(\frac{1}{2}I + K_0^*\right)^{-1}\right]v + \mathcal{U}(v)\\
&= \left(\frac{1}{2}I + K_0^*\right)^{-1} v + \frac{1}{2i\pi}\int_\Gamma \frac{d\lambda}{\lambda}\left[\left(\left(\frac{1}{2}-\lambda\right)I + K_k^*\right)^{-1} - \left(\frac{1}{2}I + K_0^*\right)^{-1}\right]v + \mathcal{U}(v)\\
&= \left(\frac{1}{2}I + K_0^*\right)^{-1} v\\
&\quad + \frac{1}{2i\pi}\int_\Gamma \frac{d\lambda}{\lambda}\left[\left(\left(\frac{1}{2}-\lambda\right)I + K_k^*\right)^{-1}\left(K_0^* - K_k^* + \lambda I\right)\left(\frac{1}{2}I + K_0^*\right)^{-1}\right]v + \mathcal{U}(v)\\
&= \left(\frac{1}{2}I + K_0^*\right)^{-1} v + A_{0,k}\left[\left(K_0^* - K_k^*\right)\left(\frac{1}{2}I + K_0^*\right)^{-1}\right]v + A_{-1,k}\left[\left(\frac{1}{2}I + K_0^*\right)^{-1} v\right]\\
&\quad + \mathcal{U}(v).
\end{aligned} \tag{114}$$

Writing

$$\left(\frac{1}{2}I + K_0^*\right)^{-1} = 2I - 2\left(\frac{1}{2}I + K_0^*\right)^{-1} K_0^*,$$

it follows that

$$\frac{\partial u}{\partial \nu} = 2v - 2\left(\frac{1}{2}I + K_0^*\right)^{-1} K_0^* v + \mathcal{V}(v) + \mathcal{W}(v),$$

where

$$\mathcal{V}(v) := A_{0,k}\left[\left(K_0^* - K_k^*\right)\left(\frac{1}{2}I + K_0^*\right)^{-1}\right]v,$$

and

$$\mathcal{W}(v) := A_{-1,k}\left[\left(\frac{1}{2}I + K_0^*\right)^{-1}\right]v + \mathcal{U}(v).$$

Since $-A_{-1,k}$ is a projector on the null eigenspace associated to the zero eigenvalue, the remainder $\mathcal{W}(v)$ belongs to $\mathcal{R}_2$. The smoothing effects of $K_k^* - K_0^*$ described in Lemma B.5 and Theorem B.3(iv) show that $\mathcal{V}(v)$ belongs also to $\mathcal{R}_2$. Concerning the term $T_0 v$, its regularity is deduced from the fact that

$$\frac{1}{2}I + K_0^* : H^s(\partial\Omega) \to H^s(\partial\Omega)$$

is an isomorphism and that K_0^* is a bounded operator from $H^s(\partial\Omega) \to H^{s+1}(\partial\Omega)$ for every real s. $\square$

B.2.2. Normal derivative of the double-layer potential

In [16], the normal derivative of a double-layer potential is investigated in dimension three. For our purpose, we adapt their computations in dimension two.

Lemma B.6. *Let $k \in \mathbb{C}$ with $\mathrm{Im}\, k \geqslant 0$ and $f \in \mathcal{D}'(\partial\Omega)$. We have*

$$\bigl\langle E_k(f), \psi\bigr\rangle = -\left\langle G_k * \frac{\partial f}{\partial \tau}, \frac{\partial \psi}{\partial \tau}\right\rangle + k^2 \int_{\partial\Omega} \psi(p) \int_{\partial\Omega} f(q) G_k(p,q)\langle \nu_q, \nu_p\rangle \, d\sigma(q)\, d\sigma(p),$$
$$\forall \psi \in \mathcal{D}(\partial\Omega),$$

where $\langle .,. \rangle$ refers to the $\mathcal{D}'(\partial\Omega)/\mathcal{D}(\partial\Omega)$-duality, and $*$ is the convolution product on $\partial\Omega$.

Remark B.2. For details about the convolution product defined on $\partial\Omega$, we can refer to [41, Chapitre IV, pp. 166–168].

Lemma B.7. *Let $f := \mathcal{H}_0 g$ where $\mathcal{H}_0$ is the Heaviside function with a jump at zero and $g : \partial\Omega \to \mathbb{R}$ is smooth. We have*

$$E_k(f)\bigl(p(\sigma)\bigr) = g(0)\frac{\partial G_k}{\partial \tau}\bigl(p(\sigma)\bigr) + \frac{\partial g}{\partial \tau}(0) G_k + G_k * \mathcal{H}_0\left(\frac{\partial^2 g}{\partial \tau^2} + k^2 g\right) + O\bigl(\sigma^2\bigr), \tag{115}$$

in the space of distributions $\mathcal{D}'(\partial\Omega)$.

Proof of Lemma B.7. We apply Lemma B.6 to $f=\mathcal{H}_0 g$. On the one hand, we have

$$-\left\langle G_k*\frac{\partial f}{\partial\tau},\frac{\partial\psi}{\partial\tau}\right\rangle=-\left\langle G_k*\left(\delta_0 g(0)+\mathcal{H}_0\frac{\partial g}{\partial\tau}\right),\frac{\partial\psi}{\partial\tau}\right\rangle$$
$$=g(0)\left\langle\frac{\partial G_k}{\partial\tau},\psi\right\rangle+\frac{\partial g}{\partial\tau}(0)\langle G_k,\psi\rangle+\left\langle G_k*\left(\mathcal{H}_0\frac{\partial^2 g}{\partial\tau^2}\right),\psi\right\rangle.$$

On the other hand, using $\langle\nu_{p(0)},\nu_{q(\sigma)}\rangle=1+O(\sigma^3)$ in a neighborhood of $\sigma=0$, we get

$$k^2\int_{\partial\Omega}\psi(p)\int_{\partial\Omega}f(q)G_k(p,q)\langle\nu_q,\nu_p\rangle d\sigma(q)\,d\sigma(p)$$
$$=k^2\int_{\partial\Omega}\psi(p)\int_{\partial\Omega}f(q)G_k(p,q)\big[1+O(\sigma^2)\big]\,d\sigma(q)\,d\sigma(p)$$
$$=\langle k^2G_k*f,\psi\rangle+\langle O(\sigma^2),\psi\rangle.\qquad\square$$

Then, we have the following result.

Proposition B.8. *Let $f=\mathcal{H}_0 g$, with $\mathcal{H}_0$ the Heaviside function with jump at zero and $g:\partial\Omega\to\mathbb{R}$ a smooth function. We have*

$$E_k(f)\big(p(\sigma)\big)=g(0)a_1\,p.v.\left(\frac{1}{\sigma}\right)+\frac{\partial g}{\partial\tau}(0)a_1\ln|\sigma|+\left\{a_1\frac{\partial^2 g}{\partial\tau^2}+(a_1+2a_2)k^2g\right\}(0)\sigma\ln|\sigma|$$
$$+\mathcal{R}_2,\tag{116}$$

where a_1 and a_2 are defined in (62).

Proof. According to Proposition B.1, we get

$$G_k(p,0)=a_1\ln|p|+c_k+a_2k^2|p|^2\ln|p|+O\big(|p|^2\big)\quad\text{when } p\to 0,\tag{117}$$

where c_k is a constant depending on k^2. Thus, using (57) and (58) we get

$$G_k\big(p(\sigma),0\big)=a_1\ln|\sigma|+c_k+a_2k^2\sigma^2\ln|\sigma|+O\big(\sigma^2\big).\tag{118}$$

Similar computations show that

$$\frac{\partial G_k}{\partial\tau}\big(p(\sigma),0\big)=a_1\,\text{p.v.}\left(\frac{1}{\sigma}\right)+2a_2k^2\sigma\ln|\sigma|+O(\sigma).\tag{119}$$

Consider $\tilde g=\frac{\partial^2 g}{\partial\tau^2}+k^2g$ and calculate now $G_k*\mathcal{H}_0\tilde g$. Since G_k is a compactly supported distribution, then $G_k*\mathcal{H}_0$ is a primitive of G_k (see for example [41, Chapitre IV, p. 168]). Thanks to the fact that $\tilde g$ is a smooth function, we conclude that

$$G_k * \mathcal{H}_0 \tilde{g} = \tilde{g}(0) a_1 \sigma \ln|\sigma| + \alpha_k + O(\sigma), \tag{120}$$

where α_k is a constant of integration. This ends the proof of the proposition. □

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