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MAGNETIZATION SWITCHING ON SMALL FERROMAGNETIC ELLIPSOIDAL SAMPLES

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Abstract. The study of small magnetic particles has become a very important topic, in particular for the development of technological devices such as those used for magnetic recording. In this field, switching the magnetization inside the magnetic sample is of particular relevance. We here investigate mathematically this problem by considering the full partial differential model of Landau-Lifschitz equations triggered by a uniform (in space) external magnetic field.

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1. Introduction

Ferromagnetic materials nowadays are used in numerous technological devices (magnetic recording, cellular phones, etc). Among these applications, magnetic storage is probably one of the most important areas. Devices such as hard-disks, or magnetic RAM are composed of several small ferromagnetic particles capable of being magnetized in two opposite directions, allowing for the storage of one bit of information.

Being able to switch the magnetization in a quick and sure way into this sample is therefore of prime interest. Not surprisingly, the switching of the magnetization in small elongated particles has received a lot of attention (see for instance [2,16] or [3] and references therein) after the pioneering work of Kikuchi [14] where an analytical solution is given in the case of a spherical particle uniformly magnetized.

However, if physicists have worked a lot on such problems by giving strategies to switch the magnetization with the help of an external magnetic field inside nanoscale ferromagnetic particles, the dynamics of the magnetization is usually modelized with a monodomain particle for which the Landau-Lifschitz equation takes the particular form of an ordinary differential equation. This is probably due to the fact that for particles in which the magnetization is not constant, the Landau-Lifschitz equation becomes a non-linear and non local partial differential equation. This equation remains largely badly understood since in all generality, strong solutions are only known to exist locally in time [5] and whenever weak solutions are considered [1,18] although they are defined for all time, such solutions are likely to be nonunique. In this article, we address the question of studying mathematically the possibility of switching the magnetization inside an elongated particle with

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external magnetic fields that are uniform in space (but may be variable in time). As we shall see, although we will restrict to small ellipsoidal ferromagnetic particles, we will consider the full PDE problem, and both weak and strong solutions.

The magnetization m inside a ferromagnetic body, located in a space domain Ω, is a three dimensional vector field, defined on Ω and constrained to be of constant magnitude through the sample. After a suitable renormalization, we consider this magnitude to be equal to 1. The evolution of the magnetization inside a ferromagnetic body is modelized by the Landau-Lifschitz equation,

$$\frac{\partial m}{\partial t} = \alpha[H(m) - \langle H(m), m\rangle m] - m \wedge H(m), \text{ in } \Omega. \tag{1.1}$$

Here, $H(m)$ is the total magnetic field induced by several physical phenomena (exchange, stray-field, anisotropy, exterior field), $\alpha > 0$ is a damping coefficient which depends on the material (we refer the reader to [4] or [12] for a more complete description of the physical model). In this equation, $\langle ., .\rangle$ denotes the Euclidean scalar product on $\mathbb{R}^3$ and $\wedge$ is the vectorial product. Equivalently, at least for smooth solutions, the equation (1.1) may be written under the so-called Gilbert form

$$\frac{\partial m}{\partial t} - \alpha \left(m \wedge \frac{\partial m}{\partial t} \right) = -(1 + \alpha^2) \left(m \wedge H(m) \right), \tag{1.2}$$

and under the form,

$$\alpha \frac{\partial m}{\partial t} + \left(m \wedge \frac{\partial m}{\partial t} \right) = (1 + \alpha^2)[H(m) - \langle H(m), m\rangle m]. \tag{1.3}$$

For a ferromagnetic body without anisotropy, the magnetic field $H(m)$ can be expressed, in order to emphasize the dependence on the (non-constant in time) external magnetic field, as

$$H(m) = -\frac{\partial \mathcal{E}}{\partial m} + H_{\text{ext}},$$

where $\mathcal{E}(m)$ is the micromagnetic energy associated to a given magnetization m,

$$\mathcal{E}(m) := \frac{A}{2} \int_\Omega |\nabla m|^2 - \frac{1}{2} \int_\Omega \langle H_d(m), m\rangle. \tag{1.4}$$

This leads to

$$H(m) := A\Delta m + H_d(m) + H_{\text{ext}}, \tag{1.5}$$

where H_{ext} is the uniform in space external magnetic field applied to the sample, A is the so-called exchange constant [4], and $H_d(m)$ is the stray field generated by the magnetization m itself *via* the following dimensionless Maxwell equations

$$\begin{cases} H_d(m) = \nabla\phi, \text{ in } \mathbb{R}^3, \\ \Delta\phi = -\text{div}(\overline{m}), \text{ in } \mathbb{R}^3, \\ H_d(m) \text{ vanishes at infinity,} \end{cases} \tag{1.6}$$

where

$$\overline{m} = \begin{cases} m \text{ in } \Omega, \\ 0 \text{ in } \mathbb{R}^3 \setminus \Omega. \end{cases} \tag{1.7}$$

It is well known that $H_d(m)$ is the $L^2(\mathbb{R}^3)$-orthogonal projection of $-\overline{m}$ on gradients from which we deduce

$$\|H_d(m)\|_{L^2(\mathbb{R}^3)} \leqslant \|\overline{m}\|_{L^2(\mathbb{R}^3)}.$$

We will frequently use the following consequence

$$\forall m \in L^2(\Omega),\ \|H_d(m)\|_{L^2(\Omega)} \leqslant \|m\|_{L^2(\Omega)}. \tag{1.8}$$

The natural boundary conditions are of Neumann type, thus, we will work on the following Cauchy problem

$$\begin{cases} \dfrac{\partial m}{\partial t} = \alpha[H(m) - \langle H(m), m\rangle m] - m \wedge H(m),\ x \in \Omega,\ t \in (0,T) \\ \dfrac{\partial m}{\partial \nu}(t,x) = 0,\ x \in \partial\Omega,\ t \in (0,T), \\ m(0,x) = m_0(x),\ x \in \Omega. \end{cases} \tag{1.9}$$

It is a non linear control system in which

- the state is the magnetization m, with $m(t) : \Omega \to S^2$, for every t;
- the control is the external magnetic field $H_{\rm ext} : t \in \mathbb{R}_+ \mapsto \mathbb{R}^3$.

This means that we seek the possibility of using a time dependent magnetic field $H_{\rm ext}$ in order to control the magnetization m. In this article, we are interested in the existence and the properties of a such control $H_{\rm ext}$ that steers m from $m(0) = u$ to $m(T) = -u$ where u and $-u$ are global minimizers of the micromagnetic energy $\mathcal{E}$. To further simplify the presentation, we will furthermore assume that the exchange constant A in (1.5) is equal to 1. Let us also quote the paper by Carbou *et al.* [6] which also treats a control problem in micromagnetics, but in the completely different context of moving a wall in a nano-wire. We will also give a couple of results in the problem similar to (1.9) but posed in 2D. By this, we mean that the domain is bidimensional, but the magnetization still takes values into S^2. However, the stray field satisfies (1.6) and (1.7) but in $\mathbb{R}^2$. This models an infinite ferromagnetic cylindrical rod along the axis of which the solution is invariant. As a consequence of (1.6) and (1.7) the component of the stray field parallel to the axis of the cylinder vanishes.

This article is organized as follows.

In Section 2, we consider a ferromagnetic body having an ellipsoidal shape. Then, the stray field of uniform magnetizations is uniform, thus, a subclass of solutions of (1.9) are uniform magnetizations that solve an ordinary differential equation (ODE) presented in Section 2.1. The goal of Section 2 is the study of the switching for those uniform magnetizations. We may assume that $\pm e_1$ are global minimizers of the micromagnetic energy. In Section 2.2, we prove the existence of external magnetic fields $H_{\rm ext}$ that produce the switching from $m(0) = +e_1$ to $m(T) = -e_1$ for every $T > 0$. Then, we justify the existence of optimal magnetic fields realizing this switching, and we show that the associated solutions m are not 2-dimensional, excepted when Ω is a sphere. In Section 2.3, we study the cost of the optimal control as $T \to +\infty$. We prove that this cost converges to zero if and only if there exists two orthogonal global minimizers of the micromagnetic energy.

The Section 3 is dedicated to weak solutions for the partial differential equation (PDE) (1.9). In Section 3.1, we prove the existence of weak solutions of (1.9). In Section 3.2, we study their convergence to uniform magnetizations when the size of the domain Ω goes to zero. This already shows that the external field found in Section 2 allows an approximate switching on any sufficiently small domain in the very general sense of weak solutions. To go further, we need more regularity and strong solutions. This imposes restrictions on either the shape of the domain or the regularity and smallness of the initial condition. Namely, Section 4 is dedicated to smooth solutions of (1.9). In Section 4.1, we present preliminary results useful for the proof done in the next section. In Section 4.2, we prove the existence and uniqueness of local (in time) smooth solutions when Ω is a bounded domain of $\mathbb{R}^2$ or $\mathbb{R}^3$. Then, in Section 4.3, we prove that such local solutions indeed provide global solutions when Ω is a 2D bounded domain and when the initial condition is in a H^1-neighborhood of constant magnetizations. In Section 4.4, we prove the existence of global smooth solutions when Ω is a small 3D ellipsoid domain and when the initial condition is in a H^2-neighborhood of constant magnetizations. Contrarily to the preceding results where we follow the strategy developed by [5], the latter result involves ideas completely different.

In Section 5, we work with small 2D or 3D ellipsoid domains Ω. We propose explicit external fields that exponentially stabilize the uniform stationary solutions.

In Section 6, we propose a way to realize the approximate switching of PDE solutions on small 2D or 3D ellipsoidal domains.

In all this article we will use the following notations. The family (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$, if $x \in \mathbb{R}^3$, we write its components $x^{(1)}$, $x^{(2)}$ and $x^{(3)}$. The same letter C denotes different constants that can change from one line to another. Whenever possible, we have explicited the parameters on which those constants depend. When Ω is an open bounded subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ and $T > 0$, Q_T denotes $(0,T) \times \Omega$. Eventually, for every map $f : \Omega \to \mathbb{R}^3$, we denote by $f_\sharp$ its space average

$$f_\sharp := \frac{1}{|\Omega|} \int_\Omega f(x) \mathrm{d}x.$$

2. Magnetization switching on ellipsoidal domains: ODE study

2.1. A simplified Landau-Lifschitz equation

It is well known that, when Ω is a 3D ellipsoidal domain over which the magnetization is constant, the stray field is also constant on Ω and therefore satisfies

$$H_d(m) = -Dm \text{ on } \Omega$$

where D is a symmetric positive matrix. Up to an orthonormal change of coordinates, we may take

$$D := \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix},$$

where $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \alpha_3 \leqslant 1$ depend on the size of the three axis of the ellipsoid (the last inequality comes from (1.8)).

In this case, the Landau-Lifschitz equations becomes the ordinary differential system

$$\begin{cases} \dfrac{\mathrm{d}m}{\mathrm{d}t} = \alpha[H_0(m) - \langle H_0(m), m\rangle m] - m \wedge H_0(m), \\ m(0) = m_0, \\ m : \mathbb{R}_+ \to S^2, \end{cases} \tag{2.1}$$

where

$$H_0(m) = -Dm + H_{\text{ext}}. \tag{2.2}$$

The existence of solutions is a classical matter.

Proposition 2.1. *For every $H_{\text{ext}} \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^3)$, for every $m_0 \in S^2$, there exist $T > 0$ and a unique function $m \in C^0([0,T), \mathbb{R}^3)$, such that,*

$$m(t) = m_0 + \int_0^t \{\alpha[H_0(m) - \langle H_0(m), m\rangle m] - m \wedge H_0(m)\} \,\mathrm{d}\tau, \tag{2.3}$$

for every $t \in [0,T)$.

Moreover, if $H_{\text{ext}} \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^3)$, then, $m \in H^1((0,T), \mathbb{R}^3)$, the first equation of (2.1) holds in $L^2((0,T), \mathbb{R}^3)$ and $|m| \equiv 1$.

If $H_{\text{ext}} \in C^0(\mathbb{R}_+, \mathbb{R}^3)$, then $m \in C^1([0,+\infty), S^2)$, and the first equation of (2.1) holds for every $t \in [0,+\infty)$.

2.2. Optimal control

Viewing H_{ext} as a control parameter, (2.1) turns out to be a flat system *i.e.* for every reference path $m_{\text{ref}} \in H^1((0,T), S^2)$ (for some $T > 0$), there exists an external field $H_{\text{ext}}[m_{\text{ref}}(t)] \in L^2((0,T), \mathbb{R}^3)$ such that the unique solution m of (2.1) with initial condition $m_0 = m_{\rm ref}(0)$, and external field $H_{\rm ext}(m_{\rm ref})$ coincides with $m_{\rm ref}$

$$m(t) = m_{\rm ref}(t), \quad \forall t \in (0,T).$$

Indeed, (2.1) rewrites as

$$\alpha \frac{{\rm d}m}{{\rm d}t} + m \wedge \frac{{\rm d}m}{{\rm d}t} = (1+\alpha^2)[-Dm + H_{\rm ext} + \langle Dm - H_{\rm ext}, m\rangle m], \tag{2.4}$$

thus, for every path $m \in H^1((0,T),S^2)$, the magnetic field

$$H_{\rm ext}(m) := \frac{1}{1+\alpha^2}\left\{\alpha \frac{{\rm d}m}{{\rm d}t} + m \wedge \frac{{\rm d}m}{{\rm d}t}\right\} + Dm - \langle Dm, m\rangle m \tag{2.5}$$

belongs to $L^2((0,T),\mathbb{R}^3)$ and, since $\langle H_{\rm ext}(m), m\rangle = 0$, allows to follow m. The other solutions are such that $\tilde{H}_{\rm ext} - \langle \tilde{H}_{\rm ext}, m\rangle m = H_{\rm ext}(m)$, and therefore $H_{\rm ext}(m)$ has the minimal $L^2((0,T),\mathbb{R}^3)$-norm, among all possible solutions.

For $T>0$, we introduce the set

$$V_T := \{m \in H^1((0,T),S^2); m(0) = e_1, m(T) = -e_1\},$$

and the functional $J_T : V_T \to \mathbb{R}_+$,

$$J_T(m) := \int_0^T |H_{\rm ext}(m)(t)|^2 {\rm d}t.$$

Proposition 2.2. *Let $T>0$. There exists a solution $m^{{\rm opt},T} \in V_T$ of the minimization problem*

$$J_T(m^{{\rm opt},T}) = \min\{J_T(\xi), \xi \in V_T\}. \tag{2.6}$$

This solution is not unique and satisfies

$$\begin{cases} \dfrac{2}{1+\alpha^2}\left\{-\dfrac{{\rm d}^2 m}{{\rm d}t^2} + \dfrac{\rm d}{{\rm d}t}[m \wedge Dm] + \dfrac{{\rm d}m}{{\rm d}t} \wedge Dm + D\left(m \wedge \dfrac{{\rm d}m}{{\rm d}t}\right)\right\} \\ \qquad\qquad\qquad + 2D^2 m - 4\langle Dm, m\rangle Dm = \lambda m, \\ m(0) = e_1, m(T) = -e_1, \end{cases} \tag{2.7}$$

where $\lambda : [0,T] \to \mathbb{R}$.

If $\alpha_1 = \alpha_2 = \alpha_3$, then any 2 dimensional path

$$m_\theta(t) := \cos\left(\frac{\pi t}{T}\right) e_1 + \sin\left(\frac{\pi t}{T}\right)[\cos(\theta) e_2 + \sin(\theta) e_3], \quad \theta \in (0,2\pi), \tag{2.8}$$

is optimal, otherwise, no optimal solution can be only 2 dimensional.

Proof of Proposition 2.2. Let $T>0$. We have, for every $m \in V_T$,

$$J_T(m) = \int_0^T \frac{1}{1+\alpha^2}\left|\frac{{\rm d}m}{{\rm d}t}\right|^2 + \frac{2}{1+\alpha^2}\langle m \wedge \frac{{\rm d}m}{{\rm d}t}, Dm\rangle + |Dm - \langle Dm, m\rangle m|^2. \tag{2.9}$$

The existence of an optimal path is a consequence of the direct method of calculus of variations.

For $m = (m^{(1)}, m^{(2)}, m^{(3)})^t \in V_T$, we define

$$\widetilde{m} = (m^{(1)}, -m^{(2)}, -m^{(3)})^t. \tag{2.10}$$

An easy calculation gives $J_T(m) = J_T(\widetilde{m})$, showing that the solution to (2.6) cannot be unique.

The equation (2.7) is the Euler-Lagrange equation of the optimization problem (2.6).

Moreover, any function $m \in V_T$ solution of (2.7) satisfies at time 0

$$\frac{2}{1+\alpha^2}\frac{\mathrm{d}^2 m}{\mathrm{d}t^2}(0) = \frac{2}{1+\alpha^2}(2\alpha_1 - \alpha_2 - \alpha_3)\frac{\mathrm{d}m}{\mathrm{d}t}(0) \wedge e_1 - (\lambda + 2\alpha_1^2)e_1. \tag{2.11}$$

Let m be an optimal path from e_1 to $-e_1$ on $[0,T]$. It is clear from (2.7) that $m \in \mathcal{C}^1([0,T], S^2)$ and we have

$$\forall t \in [0,T],\ \frac{\mathrm{d}m}{\mathrm{d}t}(t) \perp m(t).$$

Now, since m is not constant, the Cauchy-Lipschitz theorem ensures that

$$\frac{\mathrm{d}m}{\mathrm{d}t}(0) \neq 0$$

and this vector is orthogonal to $m(0) = e_1$ because $|m| \equiv 1$. Hence,

$$\frac{\mathrm{d}^2 m}{\mathrm{d}t^2}(0) \wedge e_1 = -(2\alpha_1 - \alpha_2 - \alpha_3)\frac{\mathrm{d}m}{\mathrm{d}t}(0).$$

Therefore, in the case $2\alpha_1 - \alpha_2 - \alpha_3 \neq 0$, the optimal path can not be 2-dimensional.

Now, let us assume that $\alpha_1 = \alpha_2 = \alpha_3$. The Euler equation (2.7) reduces to

$$-\frac{2}{1+\alpha^2}\frac{\mathrm{d}^2 m}{\mathrm{d}t^2} = (\lambda + 2\alpha_1^2)m, \tag{2.12}$$

which is the equation of geodesics on the sphere. Those geodesics are bi-dimensional. □

Remark 2.1. When $\alpha_1 < \alpha_2 = \alpha_3$, for every $m = (m^{(1)}, m^{(2)}, m^{(3)})^t \in H^1((0,T), \mathbb{R}^3)$, the function $\tilde{m} := (m^{(1)}, -m^{(3)}, m^{(2)})$ satisfies $J_T(m) = J_T(\tilde{m})$. Thus, the optimal path is not unique, even up to the symmetry defined by (2.10).

However, when $\alpha_1 < \alpha_2 < \alpha_3$, the uniqueness of the optimal path, up to the symmetry defined in (2.10), is an open problem.

As a sake of example, we provide hereafter the solution obtained by a numerical method based on a shooting strategy, when one solves the boundary value problem (2.7) with $D = \mathrm{diag}(0.02, 0.5, 1)$. The three components of the magnetization are shown in Figure 1 where it is clear that the solution is not bidimensional only.

Remark 2.2. Of course, minimizing J_T over V_T leads to discontinuous external fields, which, in practice might be undesirable. Other functionals can be chosen ensuring more regular H_{ext}, for instance, minimizing

$$\widetilde{J}_T(m) := \int_0^T \left|\frac{\mathrm{d}}{\mathrm{d}t}H_{\mathrm{ext}}(m)\right|^2 \mathrm{d}t$$

over

$$\widetilde{V}_T := \left\{ m \in H^2((0,T), S^2); m(0) = e_1, m(T) = -e_1, \frac{\mathrm{d}m}{\mathrm{d}t}(0) = \frac{\mathrm{d}m}{\mathrm{d}t}(T) = 0 \right\}$$

leads to optimal external fields in $H_0^1((0,T), \mathbb{R}^3)$ ($H_{\mathrm{ext}}(0) = H_{\mathrm{ext}}(T) = 0$ corresponds to a switch off for the magnetic source).

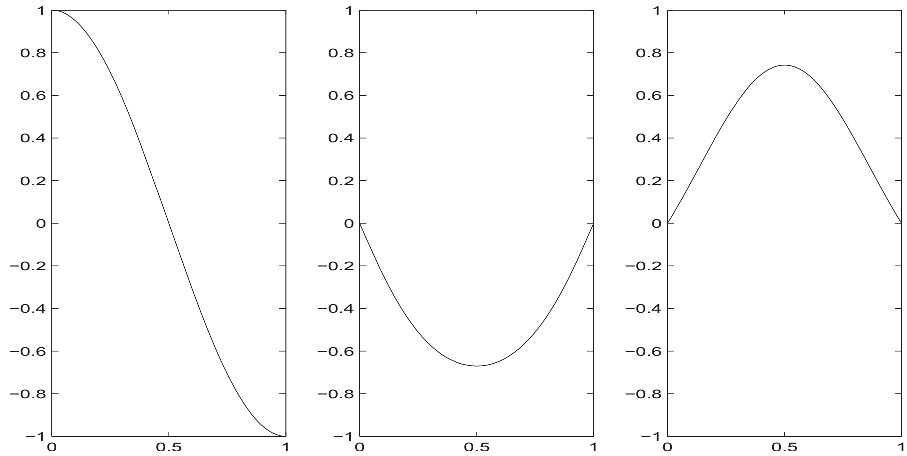

FIGURE 1. The three components of the optimal magnetization with respect to time.

2.3. Behavior of the optimal cost as $T \to +\infty$

The asymptotic behavior, when $T \to +\infty$, of the optimal cost

$$f(T) := J_T(m^{\mathrm{opt},T})$$

is given by the following proposition.

Proposition 2.3. *If $\alpha_1 = \alpha_2 \leqslant \alpha_3$, then $f(T) \to 0$, when $T \to +\infty$. Otherwise,*

$$f(T) \geqslant 4\left(\frac{1}{1+\alpha^2} - \frac{1}{(1+\alpha^2)^{3/2}}\right)(\alpha_2 - \alpha_1), \tag{2.13}$$

thus $f(T)$ does not converge to zero when $T \to +\infty$.

Proof of Proposition 2.3. When $\alpha_1 = \alpha_2 \leqslant \alpha_3$, computing explicitly the energy of the path (2.8), for $\theta = 0$, we get

$$f(T) \leqslant J_T(m_0) = \frac{1}{1+\alpha^2}\frac{\pi^2}{T}$$

proving that $f(T) \to 0$ when $T \to +\infty$.

Now, let us assume that $\alpha_1 < \alpha_2 \leqslant \alpha_3$. Let $T > 0$ and $m \in V_T$. We have

$$\begin{aligned}\int_0^T \frac{2}{1+\alpha^2}\langle m \wedge \frac{\mathrm{d}m}{\mathrm{d}t}, Dm\rangle &= \int_0^T \frac{2}{1+\alpha^2}\langle m \wedge \frac{\mathrm{d}m}{\mathrm{d}t}, Dm - \langle Dm, m\rangle m\rangle \\ &\leqslant \int_0^T \frac{1}{(1+\alpha^2)^{3/2}}\left|\frac{\mathrm{d}m}{\mathrm{d}t}\right|^2 + \frac{1}{(1+\alpha^2)^{1/2}}|Dm \wedge m|^2. \end{aligned}\tag{2.14}$$

We introduce the notations

$$C(\alpha) := \frac{1}{1+\alpha^2} - \frac{1}{(1+\alpha^2)^{3/2}},$$

$$T_1 := \sup\{t > 0; m^{(1)}(\tau) > 0, \quad \forall \tau \in (0, t)\},$$

then $m(T_1) \in \mathrm{Span}(e_2, e_3)$. We deduce from (2.9) and the last inequality that

$$\begin{aligned} J_T(m) &\geqslant C(\alpha)\int_0^T \left|\frac{\mathrm{d}m}{\mathrm{d}t}\right|^2 + |Dm\wedge m|^2 \\ &\geqslant 2C(\alpha)\int_0^T \left|\left\langle\frac{\mathrm{d}m}{\mathrm{d}t}, Dm - \langle Dm, m\rangle m\right\rangle\right| \\ &= 2C(\alpha)\left(\int_0^{T_1}\left|\left\langle\frac{\mathrm{d}m}{\mathrm{d}t}, Dm\right\rangle\right| + \int_{T_1}^T\left|\left\langle\frac{\mathrm{d}m}{\mathrm{d}t}, Dm\right\rangle\right|\right) \\ &\geqslant 2C(\alpha)\left(\left|\int_0^{T_1}\left\langle\frac{\mathrm{d}m}{\mathrm{d}t}, Dm\right\rangle\right| + \left|\int_{T_1}^T\left\langle\frac{\mathrm{d}m}{\mathrm{d}t}, Dm\right\rangle\right|\right) \\ &= 4C(\alpha)\left[\langle Dm(T_1), m(T_1)\rangle - \alpha_1\right] \\ &\geqslant 4C(\alpha)(\alpha_2-\alpha_1). \end{aligned}$$

This holds for every $m \in V_T$, which justifies (2.13). □

3. PDE weak solutions

3.1. Existence of 3D global weak solutions

Weak solutions for Landau-Lifschitz equations have been proven to exist in [1,15,18] although either without a possibly variable in time external field or without the stray-field. We here follow the strategy of [1], and show necessary adaptations to our case.

Definition 3.1. Let $m_0 \in H^1(\Omega, S^2)$ and $H_{\mathrm{ext}} \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^3)$. A function m is a weak solution of (1.9) if

- for every $T > 0$, $m \in H^1(Q_T, S^2)$;
- for every $T > 0$, for every $\Phi \in H^1(Q_T, \mathbb{R}^3)$, there holds

$$\int_{Q_T}\left\langle\frac{\partial m}{\partial t}, \Phi\right\rangle - \alpha\left\langle m\wedge\frac{\partial m}{\partial t}, \Phi\right\rangle \mathrm{d}x\mathrm{d}t = \\ -(1+\alpha^2)\int_{Q_T} -\sum_{j=1}^3\left\langle m\wedge\frac{\partial m}{\partial x_j}, \frac{\partial\Phi}{\partial x_j}\right\rangle + \langle m\wedge(H_d(m)+H_{\mathrm{ext}}), \Phi\rangle\,\mathrm{d}x\mathrm{d}t; \tag{3.1}$$

- $m(0, x) = m_0(x)$ in the trace sense;
- for almost every $T > 0$,

$$\mathcal{E}(m(T)) + \frac{\alpha}{1+\alpha^2}\int_0^T\left\|\frac{\partial m}{\partial t}(t)\right\|^2_{L^2(\Omega)}\mathrm{d}t \leqslant \mathcal{E}(m_0) + \int_0^T\int_\Omega\left\langle H_{\mathrm{ext}}, \frac{\partial m}{\partial t}\right\rangle, \tag{3.2}$$

where $\mathcal{E}(m)$ is the micromagnetic energy defined by

$$\mathcal{E}(m) := \frac{1}{2}\int_\Omega|\nabla m|^2 + \frac{1}{2}\int_{\mathbb{R}^3}|H_d(m)|^2.$$

Let us now state the main result of this section.

Theorem 3.1. *Let $m_0 \in H^1(\Omega, S^2)$ and $H_{\mathrm{ext}} \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^3)$. There exists a weak solution of (1.9).*

Remark 3.1. Notice that such weak solutions may not be unique (see [1], Thm. 1.6).

The proof of Theorem 3.1 readily follows the arguments of [1], Theorem 1.5. We write a complete proof in order to precise the necessary adaptations, due to the presence of $H_d(m)$ and H_{ext} in (1.9).

Proof of Theorem 3.1. First, following [1], we construct, through a Galerkin method, weak solutions to the penalized system

$$(P^k)\begin{cases} \alpha\dfrac{\partial m^k}{\partial t} + m^k \wedge \dfrac{\partial m^k}{\partial t} = (1+\alpha^2)\Big[\Delta m^k + H_d(m^k) + H_{\text{ext}} - k(|m^k|^2-1)m^k\Big] \text{ in } \Omega, \\ m^k(0) = m_0, \\ \dfrac{\partial m^k}{\partial \nu}(t,x) = 0,\ x \in \partial\Omega, \end{cases}$$

for $k \in \mathbb{R}_+^*$, where the constraint $|m| = 1$ is relaxed.

Let $(\varphi_j)_{j\in\mathbb{N}}$ be an orthonormal basis of $L^2(\Omega,\mathbb{R})$ consisting of eigenvectors of the Laplace operator with Neumann boundary conditions,

$$\begin{cases} -\Delta\varphi_j = \lambda_j\varphi_j, \text{ in } \Omega, \\ \dfrac{\partial\varphi_j}{\partial\nu} = 0, \text{ on } \partial\Omega, \end{cases} \tag{3.3}$$

where $(\lambda_j)_{j\in\mathbb{N}}$ is a non decreasing sequence. Let $n \in \mathbb{N}^*$. We are looking for approximate solutions

$$m_n(t,x) = \sum_{j=0}^{n-1} y_j(t)\varphi_j(x)$$

where

- for every $T > 0$, $y_j \in H^1((0,T),\mathbb{R}^3)$;
- for almost every $t \in (0,+\infty)$, for every $l \in \{0,\dots,n-1\}$,

$$\begin{cases} \displaystyle\int_\Omega \left\{ \frac{1}{\alpha^2+1}\left(\alpha\frac{\partial m_n}{\partial t} + m_n \wedge \frac{\partial m_n}{\partial t}\right) \right. \\ \qquad \left. -\Big[\Delta m_n + H_d(m_n) + H_{\text{ext}} - k(|m_n|^2-1)m_n\Big]\right\}(t,x)\varphi_l(x)\mathrm{d}x = 0, \\ \displaystyle\int_\Omega \Big[m_n(0,x) - m_0(x)\Big]\varphi_l(x)\mathrm{d}x = 0. \end{cases} \tag{3.4}$$

These relations produce a differential system that can be written as

$$\begin{cases} \dfrac{\partial Y}{\partial t} - A(Y)\dfrac{\partial Y}{\partial t} = F(Y) + B(t), \text{ for almost every } t \in (0,+\infty), \\ Y(0) = Y_0, \end{cases} \tag{3.5}$$

where $Y(t) := ((y_0(t),\dots,y_{n-1}(t))^\tau \in \mathbb{R}^{3n})$, $A(Y)$ is a $3n\times 3n$ skew-symmetric matrix (thus $I - A(Y)$ is always invertible), $F : \mathbb{R}^{3n} \to \mathbb{R}^{3n}$ is a locally Lipschitz nonlinear map and $B(t) := (B_0(t),\dots,B_{n-1}(t))^\tau \in \mathbb{R}^{3n}$ has components

$$B_j := \frac{(1+\alpha^2)}{\alpha}H_{\text{ext}}(t)\delta_{j,0}.$$

A fixed point argument gives the existence of $\tilde{T}_n > 0$ and the existence and uniqueness of $Y \in C^0((0,\tilde{T}_n),\mathbb{R}^{3n})$ such that, for every $t \in (0,\tilde{T}_n)$,

$$Y(t) = Y_0 + \int_0^t (I - A(Y(s)))^{-1}[F(Y(s)) + B(s)]\mathrm{d}s,$$

moreover, $Y \in H^1((0,\tilde{T}_n),\mathbb{R}^{3n})$ and (3.5) is satisfied. □

Remark 3.2. If $H_{\text{ext}} \in C^0(\mathbb{R}_+, \mathbb{R}^3)$ then $Y \in C^1((0, T_n), \mathbb{R}^{3n})$, the first equality of (3.5) holds for every $t \in (0, T_n)$ and the proof may be finished exactly as in [1]. When $H_{\text{ext}} \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^3)$, the first equality of (3.5) may only hold a.e., this changes a few details at the end of the proof.

Let $T_n \in (0, +\infty]$ and $Y \in C^0((0, T_n), \mathbb{R}^{3n})$ be a maximal solution. In order to prove that $T_n = +\infty$, we proceed by contradiction, assuming that $T_n < +\infty$ and $Y(t)$ is not bounded when $t \to T_n$. Noticing that

$$\|Y(t)\|^2 = \int_\Omega |m_n(t,x)|^2 \mathrm{d}x,$$

we provide estimates on m_n showing that $m_n \in L^\infty((0, T_n), L^2(\Omega))$, which gives the contradiction.

Multiplying the first equation of (3.4) by $\frac{\mathrm{d}y_l}{\mathrm{d}t}(t)$ (which is finite for almost every $t \in (0, T_n)$) and summing for $l = 0, ..., n-1$, we find, for almost every $t \in (0, T_n)$,

$$\begin{aligned} \frac{\alpha}{\alpha^2+1} \int_\Omega \left|\frac{\partial m_n}{\partial t}\right|^2 = -\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\int_\Omega \left(|\nabla m_n|^2 + \frac{k}{2}(|m_n|^2-1)^2\right) + \frac{1}{2}\int_{\mathbb{R}^3} |H_d(m_n)|^2\right] \\ + \left\langle H_{\text{ext}}, \int_\Omega \frac{\partial m_n}{\partial t}\right\rangle. \end{aligned} \tag{3.6}$$

Integrating this relation between 0 and t leads to

$$\mathcal{E}_k(m_n(t)) + \frac{\alpha}{\alpha^2+1}\int_0^t \int_\Omega \left|\frac{\partial m_n}{\partial t}\right|^2 \leqslant \mathcal{E}_k(m_{n0}) + \int_0^t \left\langle H_{\text{ext}}, \int_\Omega \frac{\partial m_n}{\partial t}\right\rangle \tag{3.7}$$

for almost every $t \in (0, T_n)$, where $m_{n0}(x) := m_n(0, x)$, and

$$\mathcal{E}_k(m) := \frac{1}{2}\int_\Omega \left(|\nabla m|^2 + \frac{k}{2}(|m|^2-1)^2\right) + \frac{1}{2}\int_{\mathbb{R}^3} |H_d(m)|^2.$$

Using the inequality

$$\left\langle H_{\text{ext}}, \frac{\partial m_n}{\partial t}\right\rangle \leqslant \frac{\alpha^2+1}{2\alpha}|H_{\text{ext}}|^2 + \frac{\alpha}{2(\alpha^2+1)}\left|\frac{\partial m_n}{\partial t}\right|^2,$$

we deduce that

$$\mathcal{E}_k(m_n(t)) + \frac{\alpha}{2(\alpha^2+1)}\int_0^t \int_\Omega \left|\frac{\partial m_n}{\partial t}\right|^2 \leqslant \mathcal{E}_k(m_{n0}) + |\Omega|\frac{\alpha^2+1}{2\alpha}\int_0^t |H_{\text{ext}}|^2. \tag{3.8}$$

Moreover, since for every $t \in (0, T_n)$,

$$\int_\Omega |m_n(t,x)|^2 \mathrm{d}x \leqslant \left(|\Omega| \int_\Omega (|m_n(t,x)|^2 - 1)^2\right)^{1/2} + |\Omega|, \tag{3.9}$$

then (3.8) provides a bound of the right-hand side, and we get that $m_n \in L^\infty((0, T_n), L^2(\Omega))$, which gives the contradiction. Finally, the maximal solution $Y(t)$ is defined for every $t \in (0, +\infty)$, and for every $T > 0$, $m_n \in H^1(Q_T)$.

Let $T>0$. Since $H^1(\Omega)$ is embedded into $L^4(\Omega)$, the right hand side of (3.8), is bounded uniformly with respect to $n\in\mathbb{N}^*$ and $t\in[0,T]$ by

$$\mathcal{R}_k := C_k\|m_0\|_{H^1(\Omega)} + |\Omega|\frac{\alpha^2+1}{2\alpha}\int_0^T |H_{\text{ext}}|^2$$

where $C_k = C(\Omega,k)$.

The inequality (3.8) shows that $(|m_n|^2-1)$, $\dfrac{\partial m_n}{\partial t}$ and ∇m_n are bounded in $L^2(Q_T,\mathbb{R}^3)$, indeed,

$$\begin{aligned}&\int_0^T \mathcal{E}_k(m_n(t))\,\mathrm{d}t \leqslant T\mathcal{R}_k,\\ &\frac{\alpha}{2(\alpha^2+1)}\int_0^T\int_\Omega\left|\frac{\partial m_n}{\partial t}(t,x)\right|^2 \mathrm{d}x\mathrm{d}t \leqslant \mathcal{R}_k.\end{aligned} \tag{3.10}$$

Thus m_n is also bounded in $H^1(Q_T)$ (the uniform L^2 bound is given by (3.9)) and there exists $m^k\in H^1(Q_T)$ such that, up to the extraction of a subsequence,

$$\begin{aligned}&m_n \to m^k \text{ weakly in } H^1(Q_T),\\ &m_n \to m^k \text{ strongly in } L^2(Q_T),\\ &|m_n|^2-1 \to |m^k|^2-1 \text{ weakly in } L^2(Q_T).\end{aligned}$$

Passing to the limit $(n\to+\infty)$ in (3.7) gives (since $\mathcal{E}_k(m_0)=\mathcal{E}_0(m_0)$)

$$\mathcal{E}_k(m^k(t)) + \frac{\alpha}{\alpha^2+1}\int_0^t\int_\Omega\left|\frac{\partial m^k}{\partial t}\right|^2 \leqslant \mathcal{E}_0(m_0) + \int_0^t\left\langle H_{\text{ext}}, \int_\Omega \frac{\partial m^k}{\partial t}\right\rangle \tag{3.11}$$

for almost every $t\in[0,T]$.

Now, let $N\in\mathbb{N}^*$ and $\varphi=\sum_{j=0}^{N-1}\alpha_j(t)\varphi_j$, where $\forall j\in\{0,...,N-1\}$, $\alpha_j\in\mathcal{C}^\infty([0,T],\mathbb{R}^3)$. For every $n\in\mathbb{N}^*$ with $n\geqslant N$, we have

$$\begin{aligned}\int_{Q_T}\left\langle \alpha\frac{\partial m_n}{\partial t} + m_n\wedge\frac{\partial m_n}{\partial t},\varphi\right\rangle \;=\;& -(\alpha^2+1)\int_{Q_T}\langle\nabla m_n,\nabla\varphi\rangle\\ &+(\alpha^2+1)\int_{Q_T}\langle H_d(m_n)+H_{\text{ext}}-k(|m_n|^2-1)m_n,\varphi\rangle,\end{aligned}$$

which gives, passing to the limit $n\to+\infty$

$$\begin{aligned}\int_{Q_T}\left\langle \alpha\frac{\partial m^k}{\partial t} + m^k\wedge\frac{\partial m^k}{\partial t},\varphi\right\rangle \;=\;& -(\alpha^2+1)\int_{Q_T}\langle\nabla m^k,\nabla\varphi\rangle\\ &+(\alpha^2+1)\int_{Q_T}\langle H_d(m^k)+H_{\text{ext}}-k(|m^k|^2-1)m^k,\varphi\rangle.\end{aligned} \tag{3.12}$$

Indeed, thanks to (1.8), we have

$$\begin{aligned}\|H_d(m_n-m^k)\|_{L^2(Q_T)} &\leqslant \|m_n-m^k\|_{L^2(Q_T)}\\ &\to 0 \text{ when } n\to+\infty.\end{aligned}$$

By density, (3.12) also holds for every $\varphi\in H^1(Q_T,\mathbb{R}^3)$.

From (3.11), we get that ∇m^k, $H_d(m^k)$, $\sqrt{k}(|m^k|^2-1)$, $\dfrac{\partial m^k}{\partial t}$ are bounded in $L^2(Q_T)$, and therefore, there exists $m \in H^1(Q_T)$ such that, up to the extraction of a suitable subsequence,

$$\begin{array}{l} m^k \to m \text{ weakly in } H^1(Q_T), \\ m^k \to m \text{ strongly in } L^2(Q_T), \\ |m^k|^2 - 1 \to 0 \text{ strongly in } L^2(Q_T). \end{array} \tag{3.13}$$

We hence first obtain, $|m| = 1$ a.e. on Q_T, and in order to pass to the limit $k \to +\infty$ in (3.12), we take $\Phi \in \mathcal{C}^\infty(Q_T)$, and test (3.12) with $\varphi(t,x) := m^k \wedge \Phi \in H^1(Q_T)$. We get

$$\begin{gathered} \int_{Q_T} \alpha \left\langle \frac{\partial m^k}{\partial t} \wedge m^k, \Phi \right\rangle - \left\langle m^k, \frac{\partial m^k}{\partial t} \right\rangle \langle m^k, \Phi\rangle + |m^k|^2 \left\langle \frac{\partial m^k}{\partial t}, \Phi \right\rangle = \\ (\alpha^2+1) \int_{Q_T} -\langle \nabla m^k \wedge m^k, \nabla \Phi\rangle + \langle (H_d(m^k) + H_{\text{ext}}) \wedge m^k, \Phi\rangle. \end{gathered} \tag{3.14}$$

All the terms pass to the limit easily but two, namely, $\displaystyle\int_{Q_T} \left\langle m^k, \frac{\partial m^k}{\partial t} \right\rangle \langle m^k, \Phi\rangle$ and $\displaystyle\int_{Q_T} |m^k|^2 \left\langle \frac{\partial m^k}{\partial t}, \Phi \right\rangle$. We decompose this latter term as

$$\begin{aligned} \int_{Q_T} |m^k|^2 \left\langle \frac{\partial m^k}{\partial t}, \Phi \right\rangle &= \int_{Q_T} (|m^k|^2 - 1) \left\langle \frac{\partial m^k}{\partial t}, \Phi \right\rangle + \int_{Q_T} \left\langle \frac{\partial m^k}{\partial t}, \Phi \right\rangle \\ &\to \int_{Q_T} \left\langle \frac{\partial m}{\partial t}, \Phi \right\rangle \end{aligned}$$

when k tends to infinity, from (3.13).

For the other term, we perform an integration by parts

$$\begin{aligned} -\int_{Q_T} \left\langle m^k, \frac{\partial m^k}{\partial t} \right\rangle \langle m^k, \Phi\rangle &= -\int_{Q_T} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[|m^k|^2 - 1]\langle m^k, \Phi\rangle \\ &= \frac{1}{2}\int_{Q_T} [|m^k|^2-1]\frac{\mathrm{d}}{\mathrm{d}t}[\langle m^k, \Phi\rangle] \\ &\to 0 \text{ as } k \to +\infty. \end{aligned}$$

Remark 3.3. This step is different from [1]: we do not need here that $|m^k| \leqslant 1$ a.e. which was proved with a Stampacchia argument in [1].

Eventually, in order to show that m satisfies the energy inequality (3.2), we just pass to the weak $H^1(Q_T)$ limit in (3.11).

3.2. Convergence of weak solutions to ODE solutions when the size of the domain goes to zero

Let Ω be a bounded open subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ such that $|\Omega| = 1$. In this section, we consider the weak solutions of the Landau-Lifschitz PDE on the domain $\Omega_\lambda := \sqrt{\lambda}\Omega$, when $\lambda \to 0$, $\lambda > 0$. A change of space and time variables shows that it is equivalent to study the weak solutions of the following Landau-Lifschitz PDE on the fixed domain Ω,

$$\begin{cases} \dfrac{\partial m}{\partial t} = \alpha[H_\lambda(m) - \langle H_\lambda(m), m\rangle m] - m \wedge H_\lambda(m), \quad x \in \Omega, \quad t \in (0,T) \\ \dfrac{\partial m}{\partial \nu}(t,x) = 0, \quad x \in \partial\Omega, \quad t \in (0,T), \\ m(0,x) = m_0(x), \quad x \in \Omega, \end{cases} \tag{3.15}$$

with an effective magnetic field

$$H_\lambda(m) := \frac{\Delta m}{\lambda} + H_d(m) + H_{\text{ext}} \tag{3.16}$$

associated to the micromagnetic energy

$$\mathcal{E}_\lambda(m) := \int_\Omega \frac{1}{2\lambda}|\nabla m|^2 + \frac{1}{2}\int_{\mathbb{R}^3} |H_d(m)|^2. \tag{3.17}$$

When the domain is small ($\lambda << 1$), non constant in space magnetizations are penalized. Therefore it is expected that solutions of (3.15), (3.16) should tend to the solutions of the ODE (2.1) with D defined by

$$D\widetilde{m} := -\frac{1}{|\Omega|}\int_\Omega H_d(\widetilde{m}\chi_\Omega), \quad \forall \widetilde{m} \in S^2,$$

where χ_Ω is the characteristic function of Ω. This is precisely the purpose of this section. We also quote the paper by DeSimone [10] in which the same kind of result is shown but for static problems, using Γ-convergence theory.

The convergence result proved in this section shows that the external magnetic field found in Section 2 allows an approximate switching for the PDE solutions, on any sufficiently small domain, in the very general sense of weak solutions.

Proposition 3.1. *Let Ω be a bounded open subset of $\mathbb{R}^2$ or $\mathbb{R}^3$, $\alpha > 0$, $H_{\text{ext}} \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^3)$, $\overline{m} \in S^2$. Let $(m_{\lambda 0})_{\lambda > 0}$ be a sequence of $H^2(\Omega, S^2)$ such that $\dfrac{\partial m_{\lambda 0}}{\partial \nu} \equiv 0$ on $\partial\Omega$ for every $\lambda > 0$,*

$$m_{\lambda 0\sharp} \to \overline{m} \text{ and } \int_\Omega |\nabla m_{\lambda 0}|^2 = o\left(\sqrt{\lambda}\right) \text{ when } \lambda \to 0. \tag{3.18}$$

Let m_λ be a weak solution of (3.15) such that $m_\lambda(0) = m_{\lambda 0}$ and m_{ref} be the solution of (2.1) with initial data $m_0 = \overline{m}$. Then, for every $T > 0$,

$$\|m_\lambda - m_{\text{ref}}\|_{C^0([0,T],H^1(\Omega))} \to 0 \text{ when } \lambda \to 0.$$

Remark 3.4. The assumption (3.18) is not restrictive as the minimizers of $\mathcal{E}_\lambda$ do satisfy it.

Proof of Proposition 3.1. The proof follows 3 steps.

First step: Bounds on ∇m_λ and $\dfrac{\partial m_\lambda}{\partial t}$. We know (from the definition of 'weak solution') that

$$\mathcal{E}_\lambda(m_\lambda(t)) + \frac{\alpha}{2(\alpha^2+1)}\int_0^t \left\|\frac{\partial m_\lambda}{\partial t}\right\|^2_{L^2} \leqslant \mathcal{E}_\lambda(m_\lambda(0)) + |\Omega|\frac{(\alpha^2+1)}{2\alpha}\int_0^t |H_{\text{ext}}|^2$$

with $\mathcal{E}_\lambda$ defined by (3.17). Thus, we have the following bounds

$$\|\nabla m_\lambda\|^2_{C^0([0,T],L^2)} \leqslant 2\lambda\left(\mathcal{E}_\lambda(m_\lambda(0)) + |\Omega|\frac{\alpha^2+1}{2\alpha}\int_0^T |H_{\text{ext}}|^2\right), \tag{3.19}$$

and

$$\int_0^T \left\|\frac{\partial m_\lambda}{\partial t}\right\|_{L^2}^2 \leqslant \frac{2(\alpha^2+1)}{\alpha}\left(\mathcal{E}_\lambda(m_\lambda(0)) + |\Omega|\frac{\alpha^2+1}{2\alpha}\int_0^T |H_{\rm ext}|^2\right). \tag{3.20}$$

Second step: Study of $m_\sharp$. Taking in (3.1) a test function Φ constant in space (but not in time) leads to an equation for $m_{\lambda\sharp}$

$$\frac{\mathrm{d}m_{\lambda\sharp}}{\mathrm{d}t} - \alpha m_{\lambda\sharp}\wedge\frac{\mathrm{d}m_{\lambda\sharp}}{\mathrm{d}t} = -(1+\alpha^2)m_{\lambda\sharp}\wedge H_0(m_{\lambda\sharp}) + f_\lambda(t), \tag{3.21}$$

where from the definition (2.2) of H_0

$$f_\lambda(t) := \frac{1}{|\Omega|}\int_\Omega \alpha(m_\lambda - m_{\lambda\sharp})\wedge\frac{\partial m_\lambda}{\partial t} - \frac{(1+\alpha^2)}{|\Omega|}\int_\Omega m_\lambda\wedge(H_d(m_\lambda)+Dm_{\lambda\sharp}).$$

Now, since $Dm_{\lambda\sharp\chi_\Omega} = -\frac{1}{|\Omega|}\int_\Omega H_d(m_{\lambda\sharp})$, we may write the latter term as

$$\int_\Omega m_\lambda\wedge(H_d(m_\lambda)+Dm_{\lambda\sharp}) \;=\; \int_\Omega m_\lambda\wedge H_d(m_\lambda - m_{\lambda\sharp}\chi_\Omega) + (m_\lambda - m_{\lambda\sharp})\wedge(H_d(m_{\lambda\sharp}\chi_\Omega)+Dm_{\lambda\sharp})$$

from which we can bound

$$|f_\lambda(t)| \le \frac{\alpha}{|\Omega|}\|m_\lambda - m_{\lambda\sharp}\|_{L^2}\left\|\frac{\partial m_\lambda}{\partial t}\right\|_{L^2} + \frac{(1+\alpha^2)}{\sqrt{|\Omega|}}\left(\|H_d(m_\lambda - m_{\lambda\sharp}\chi_\Omega)\|_{L^2} + 2\|m_\lambda - m_{\lambda\sharp}\|_{L^2}\right).$$

This implies, thanks to (3.19), (3.20) and Poincaré inequality that there exists $C_2 = C_2(\Omega,\alpha,T) > 0$ such that

$$\int_0^T |f_\lambda(t)|\mathrm{d}t \leqslant C_2[B(\lambda) + \sqrt{B(\lambda)}]\sqrt{\lambda}, \tag{3.22}$$

where

$$B(\lambda) := \mathcal{E}_\lambda(m_\lambda(0)) + |\Omega|\frac{\alpha^2+1}{2\alpha}\int_0^T |H_{\rm ext}|^2.$$

We notice that from (3.21) we can deduce

$$\left\langle m_{\lambda\sharp}, \frac{\mathrm{d}m_{\lambda\sharp}}{\mathrm{d}t}\right\rangle = \langle m_{\lambda\sharp}, f_\lambda(t)\rangle$$

from which we get

$$\begin{aligned}
\frac{\mathrm{d}m_{\lambda\sharp}}{\mathrm{d}t} &= F(m_{\lambda\sharp}) + \frac{1}{1+\alpha^2|m_{\lambda\sharp}|^2}\left(\alpha^2\langle m_{\lambda\sharp}, f_\lambda(t)\rangle m_{\lambda\sharp} + f_\lambda(t) + \alpha m_{\lambda\sharp}\wedge f_\lambda(t)\right),\\
\frac{\mathrm{d}m_{\rm ref}}{\mathrm{d}t} &= F(m_{\rm ref}),
\end{aligned}$$

where

$$\begin{aligned}
F:\mathbb{R}^3 &\to \mathbb{R}^3\\
m &\mapsto -\frac{1+\alpha^2}{1+\alpha^2|m|^2}\left(m\wedge H_0(m) + \alpha m\wedge(m\wedge H_0(m))\right)
\end{aligned}$$

satisfies

$$\forall m_1, m_2 \in B_{\mathbb{R}^3}(0,1),\ |F(m_1) - F(m_2)| \leqslant C_3(1+|H_{\rm ext}(t)|)|m_1 - m_2|,$$

for some constant $C_3 = C_3(\alpha, D)$. Using Gronwall lemma and (3.22), we get

$$|m_{\lambda\sharp} - m_{\mathrm{ref}}|_{C^0([0,T],\mathbb{R}^3)} \leqslant \left(|m_{\lambda 0\sharp} - \overline{m}| + C_2\sqrt{\lambda}[B + \sqrt{B}]\right) \mathrm{e}^{C_3(T + \int_0^T |H_{\mathrm{ext}}|)}.$$

Third step: Conclusion. Noticing that

$$\|m_\lambda - m_{\mathrm{ref}}\chi_\Omega\|_{L^2}^2 = 2|\Omega|(1 - \langle m_{\lambda\sharp}, m_{\mathrm{ref}}\rangle) = 2|\Omega|\langle m_{\mathrm{ref}}, m_{\mathrm{ref}} - m_{\lambda\sharp}\rangle,$$

we get

$$\|m_\lambda - m_{\mathrm{ref}}\,\chi_\Omega\|_{C^0([0,T],H^1)}^2 \leqslant 2\lambda B + 2|\Omega| \left(|m_{\lambda 0\sharp} - \overline{m}| + C_2\sqrt{\lambda}[B + \sqrt{B}]\right) \mathrm{e}^{C_3(T + \int_0^T |H_{\mathrm{ext}}|)}$$

and the conclusion comes from (3.18). □

It is well known (see [1] for examples) that, for $\lambda > 0$ fixed, there may not be uniqueness for the weak solutions of (3.15), (3.16). However, all these weak solutions converge in $C^0([0,T], H^1(\Omega))$ to the same function m_{ref}, when $\lambda \to 0$.

Although restricted to approximate controllability, the preceding result is very general in the sense that it applies to a wide class of solutions to Landau-Lifschitz equations. To go further and obtain stronger results (like stabilization and convergence to minimizers), we need stronger solutions. This is precisely the aim of the following sections.

4. Global PDE smooth solutions

In this section, we investigate the existence and uniqueness of global smooth solutions of the Landau-Lifschitz equation (1.9) on the domain $\Omega_\lambda := \sqrt{\lambda}\Omega$, with $\lambda > 0$. As already explained, it is equivalent to study (3.15) and (3.16). First, we show existence and uniqueness of local (in time) smooth solutions.

Theorem 4.1. *Let Ω be a bounded regular open subset of $\mathbb{R}^2$ or $\mathbb{R}^3$, $\alpha > 0$, $\lambda > 0$, $H_{\mathrm{ext}} \in C^0(\mathbb{R}_+, \mathbb{R}^3)$ and $m_0 \in H^2(\Omega, S^2)$ be such that $\dfrac{\partial m_0}{\partial \nu} = 0$ on $\partial\Omega$. There exist a time $T^* = T^*(\Omega, \alpha, \lambda, \|m_0\|_{H^2(\Omega)}, \|H_{\mathrm{ext}}\|_{L^\infty})$ and a unique*

$$m \in C^0([0,T], H^2(\Omega, S^2)) \cap H^1((0,T), H^1(\Omega, S^2)) \cap L^2((0,T), H^3(\Omega, S^2)),$$

for all $T \in (0, T^*)$, satisfying (3.15) and (3.16). Moreover, such regular solutions depend continuously on m_0 for the topology $C^0([0,T], H^2(\Omega, S^2))$.

In the 2D case, this result can be improved since global existence holds for small λ (*i.e.* small domains $\Omega_\lambda = \sqrt{\lambda}\Omega$), with initial conditions m_0 in a H^1-neighborhood of constants, and for all (bounded and regular) domains Ω.

Theorem 4.2. *Let Ω be a bounded open subset of $\mathbb{R}^2$, $\alpha > 0$, $\lambda > 0$, $H_{\mathrm{ext}} \in L^\infty(\mathbb{R}_+, \mathbb{R}^3)$ with $\dot{H}_{\mathrm{ext}} \in L^1(\mathbb{R}_+, \mathbb{R}^3)$ and $C^*(\Omega)$ be the smallest constant C such that (4.13) holds. For every $m_0 \in H^2(\Omega, S^2)$ such that $\dfrac{\partial m_0}{\partial \nu} = 0$ on $\partial\Omega$ and*

$$\int_\Omega |\nabla m_0|^2 + \lambda \left(\int_\Omega |H_d(m_0)|^2 + 4\|H_{\mathrm{ext}}\|_{L^\infty(\mathbb{R}_+,\mathbb{R}^3)} + 2\|\dot{H}_{\mathrm{ext}}\|_{L^1(\mathbb{R}_+,\mathbb{R}^3)} \right) < \frac{1}{C^*(\Omega)}, \tag{4.1}$$

the smooth solution of (3.15) exists on $\mathbb{R}_+$.

In the 3D case, the result of Theorem 4.1 can also be improved, when Ω is an ellipsoidal domain. We get in that case global existence of smooth solutions for small λ (*i.e.* small ellipsoids $\Omega_\lambda = \sqrt{\lambda}\Omega$), and for initial conditions m_0 in a H^2-neighborhood of constants.

Theorem 4.3. *Let Ω be a 3D ellipsoid domain. There exists $C^{**}(\Omega) > 0$ such that, for every $\alpha > 0$, for every $\lambda \in (0,1)$, for every $H_{\rm ext} \in C^0 \cap L^\infty(\mathbb{R}_+, \mathbb{R}^3)$, for every $m_0 \in H^2(\Omega, S^2)$ with $\dfrac{\partial m_0}{\partial \nu} = 0$ on $\partial\Omega$ that satisfy*

$$C^{**}(\alpha+1)(1+\|H_{\rm ext}\|_{L^\infty(\mathbb{R}_+)}) \leqslant \frac{\alpha}{\lambda}, \tag{4.2}$$

$$C^{**}(\alpha+1)[\|\Delta m_0\|_{L^2} + \|\Delta m_0\|_{L^2}^2] < \alpha, \tag{4.3}$$

the smooth solution of (3.15) exists on $\mathbb{R}_+$ and verifies, for every $T > 0$

$$\|\Delta m(T)\|_{L^2}^2 + \frac{\alpha - N(T)}{\lambda} \int_0^T \|\nabla \Delta m(t)\|_{L^2}^2 \,\mathrm{d}t \leqslant \|\Delta m_0\|_{L^2}^2, \tag{4.4}$$

where

$$N(T) := \sup\left\{C^{**}(\alpha+1)[\|\Delta m(t)\|_{L^2} + \|\Delta m(t)\|_{L^2}^2]; t \in [0,T]\right\}. \tag{4.5}$$

In particular, we have

$$\|\Delta m(t)\|_{L^2} \leqslant \|\Delta m_0\|_{L^2}, \quad \forall t > 0. \tag{4.6}$$

Remark 4.1. As a corollary of Theorem 4.2, we have the existence of 2D global smooth solutions for (3.15) when λ is small enough, namely

$$\lambda\left(1 + 4\|H_{\rm ext}\|_{L^\infty(\mathbb{R}_+,\mathbb{R}^3)} + 2\|\dot{H}_{\rm ext}\|_{L^1(\mathbb{R}_+,\mathbb{R}^3)}\right) < \frac{1}{2C^*(\Omega)},$$

and for every initial data $m_0 \in H^2(\Omega, \mathbb{R}^3)$ which satisfies $\dfrac{\partial m_0}{\partial \nu} = 0$ on $\partial\Omega$ and

$$\int_\Omega |\nabla m_0|^2 < \frac{1}{2C^*(\Omega)}\cdot$$

This defines an H^1-neighborhood of uniform magnetizations of fixed size. Similarly, a corollary of Theorem 4.3 is the existence of 3D global smooth solutions (3.15) when λ is small enough, namely

$$\lambda \leqslant \frac{\alpha}{C^{**}(\alpha+1)(1+\|H_{\rm ext}\|_{L^\infty(\mathbb{R}_+)})},$$

and for every $m_0 \in H^2(\Omega, \mathbb{R}^3)$ which satisfies $\dfrac{\partial m_0}{\partial \nu} = 0$ on $\partial\Omega$ and

$$\|\Delta m_0\|_{L^2} < \min\left\{1, \frac{\alpha}{2C^{**}(\alpha+1)}\right\}.$$

This defines an H^2-neighborhood of uniform magnetizations of fixed size.

Remark 4.2. Theorem 4.2 provides the existence of global smooth solutions of (3.15) when λ is small enough and for initial conditions in a H^1-neighborhood of any minimizer m_λ^* of the micromagnetic energy $\mathcal{E}_\lambda$, defined by (3.17), when $\lambda > 0$ is small enough. Indeed,

$$\|\nabla m_\lambda^*\|_{L^2(\Omega)}^2 \leqslant 2\lambda\mathcal{E}_\lambda(m_\lambda^*) \leqslant 2\lambda\mathcal{E}_\lambda(e_1\chi_\Omega) = 2\lambda\|H_d(e_1\chi_\Omega)\|_{L^2(\Omega)}^2 \leqslant 2\lambda|\Omega|,$$

thus (4.1) holds when

$$\|\nabla(m_0 - m_\lambda^*)\|_{L^2(\Omega)}^2 + 2\lambda[1 + 2\|H_{\rm ext}\|_{L^\infty(\mathbb{R}_+,\mathbb{R}^3)} + \|\dot{H}_{\rm ext}\|_{L^1(\mathbb{R}_+,\mathbb{R}^3)}] < \frac{1}{C^*(\Omega)}\cdot$$

Therefore, Theorem 4.2 provides the existence of global smooth solutions of (3.15) when λ is small enough, namely

$$2\lambda[1+2\|H_{\rm ext}\|_{L^\infty(\mathbb{R}_+,\mathbb{R}^3)}+\|\dot{H}_{\rm ext}\|_{L^1(\mathbb{R}_+,\mathbb{R}^3)}]<\frac{1}{2C^*(\Omega)},$$

and for initial conditions in a H^1-neighborhood of m^*_λ, namely

$$\|\nabla(m_0-m^*_\lambda)\|^2_{L^2(\Omega)}\leqslant\frac{1}{2C^*(\Omega)}.$$

Theorem 4.2 also provides the existence of global smooth solutions of (3.15) for initial conditions in an H^2-neighborhood of any minimizer of the micromagnetic energy $\mathcal{E}_\lambda$, defined by (3.17), when $\lambda>0$ is small enough. Indeed, on a 3D ellipsoidal domain, for λ small enough, the minimizers of the micromagnetic energy $\mathcal{E}_\lambda$ are constant in space (see Prop. 5.1 below).

Remark 4.3. In 3D, the existence of global solutions is only proven on ellipsoidal domains on which the stray field has a particular structure. Indeed, on such domains, one has

$$H_d(m)=H_d(m_\sharp\chi_\Omega)+H_d(m-m_\sharp\chi_\Omega)$$

where $H_d(m_\sharp\chi_\Omega)$ is constant over Ω. In particular, we have the following inequality that will be crucial in the proof

$$\|\nabla H_d(m)\|_{L^2}\leqslant C\|\nabla m\|_{L^2},$$

where $C=C(\Omega)>0$, which is a consequence of Proposition 4.2 below and Poincaré inequality.

These results are more general than [5], Theorems 1.1–1.4:

- in Theorem 4.1, we take into account an external field $H_{\rm ext}$ which is not considered in [5], Theorems 1.1 and 1.2;
- in Theorem 4.2, we take into account the stray field $H_d(m)$ and the external field $H_{\rm ext}$ which are not considered in [5], Theorems 1.3 and 1.4;
- Theorem 4.3 deals with global solutions in a 3D case, this situation is not investigated in [5].

The proofs of Theorems 4.1 and 4.2 follow the ones of [5]. We detail them in order to justify the necessary adaptations due to the presence of H_d and $H_{\rm ext}$. The proof of Theorem 4.3, instead, involves new arguments.

This section is organized as follows. First, in Section 4.1, we recall Sobolev inequalities and some classical properties of the operator H_d, that will be useful. Then, we prove Theorems 4.1, 4.2 and 4.3 in Sections 4.2, 4.3 and 4.4 respectively.

4.1. Preliminaries

Proposition 4.1. *Let Ω be a bounded regular open subset of $\mathbb{R}^2$ or $\mathbb{R}^3$. There exists $C=C(\Omega)>0$ such that, for every $u\in H^2(\Omega,S^2)$ with $\frac{\partial u}{\partial\nu}=0$ on $\partial\Omega$,*

$$\|u\|_{H^2(\Omega)}\leqslant C\left(\|u\|^2_{L^2(\Omega)}+\|\Delta u\|^2_{L^2(\Omega)}\right)^{1/2},\tag{4.7}$$

$$\|u\|_{L^\infty(\Omega)}\leqslant C\left(\|u\|^2_{L^2(\Omega)}+\|\Delta u\|^2_{L^2(\Omega)}\right)^{1/2},\tag{4.8}$$

$$\|\nabla u\|_{L^p}\leqslant C\|\Delta u\|_{L^2},\quad\forall p\in[1,6],\tag{4.9}$$

$$\|D^2u\|_{L^2}\leqslant C\|\Delta u\|_{L^2}\tag{4.10}$$

and for every $u \in H^3(\Omega, S^2)$ *with* $\dfrac{\partial \Delta u}{\partial \nu} = \dfrac{\partial u}{\partial \nu} = 0$ *on* $\partial\Omega$,

$$\|\Delta u\|_{L^2} \leqslant C\|\nabla \Delta u\|_{L^2}, \tag{4.11}$$

$$\|D^2 u\|_{L^3} \leqslant C\|\Delta u\|_{L^2}^{1/2}\|\nabla \Delta u\|_{L^2}^{1/2}. \tag{4.12}$$

Let Ω *be a bounded regular open subset of* $\mathbb{R}^2$. *There exists* $C = C(\Omega) > 0$ *such that, for every* $u \in H^2(\Omega, S^2)$ *with* $\dfrac{\partial u}{\partial \nu} = 0$ *on* $\partial\Omega$,

$$\|\nabla u\|_{L^4(\Omega)} \leqslant C\|\nabla u\|_{L^2(\Omega)}^{1/2}\|\Delta u\|_{L^2(\Omega)}^{1/2}, \tag{4.13}$$

$$\|\nabla u\|_{L^6(\Omega)} \leqslant C\|\nabla u\|_{L^2(\Omega)}^{1/3}\|\Delta u\|_{L^2(\Omega)}^{2/3} \tag{4.14}$$

and for every $u \in H^3(\Omega, S^2)$ *with* $\dfrac{\partial \Delta u}{\partial \nu} = \dfrac{\partial u}{\partial \nu} = 0$ *on* $\partial\Omega$,

$$\|\nabla u\|_{L^\infty(\Omega)} \leqslant C\|\nabla u\|_{L^2(\Omega)}^{1/2}\|\nabla \Delta u\|_{L^2(\Omega)}^{1/2}. \tag{4.15}$$

Proof of Proposition 4.1. The inequality (4.7) results from the regularity of the operator $A = -\Delta + I$ with domain

$$D(A) := \left\{u \in H^2(\Omega); \frac{\partial u}{\partial \nu} \equiv 0 \text{ on } \partial\Omega\right\},$$

and (4.8) follows from (4.7) and the classical embedding $H^2(\Omega) \to L^\infty(\Omega)$. The inequality (4.9) with $p = 2$ is a consequence of the spectral decomposition

$$\|\nabla u\|_{L^2}^2 = \sum_{n=0}^{\infty} \lambda_n |\langle u, \varphi_n\rangle|^2 \leqslant C \sum_{n=0}^{\infty} \lambda_n^2 |\langle u, \varphi_n\rangle|^2 = C\|\Delta u\|_{L^2}^2,$$

where $(\varphi_n)_{n\in\mathbb{N}}$ is defined by (3.3). Thanks to the embedding $H^1(\Omega) \to L^p(\Omega)$, with $1 \leq p \leq 6$, the Poincaré inequality and (4.7), we get

$$\begin{aligned} \|\nabla u\|_{L^p} &= \|\nabla(u - u_\sharp)\|_{L^p} \\ &\leqslant C\|u - u_\sharp\|_{H^2} \\ &\leqslant C(\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2)^{1/2}, \end{aligned}$$

which leads to (4.9). The inequality (4.10) follows from the regularity of Laplace operator on regular domains, while (4.11) results from a spectral decomposition, in the same way as (4.9). Thanks to the embedding $H^{1/2}(\Omega) \to L^3(\Omega)$, we have

$$\|D^2 u\|_{L^3} \leqslant C\Big((\|u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2)^{1/2} + (\|u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2)^{1/4}\|\nabla \Delta u\|_{L^2}^{1/2}\Big).$$

Applying the previous inequality with u replaced by $u - u_\sharp$, using Poincaré inequality and (4.9) we get

$$\begin{aligned} \|D^2 u\|_{L^3} &\leqslant C\Big((\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2)^{1/2} + (\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2)^{1/4}\|\nabla \Delta u\|_{L^2}^{1/2}\Big) \\ &\leqslant C\Big(\|\Delta u\|_{L^2} + \|\Delta u\|_{L^2}^{1/2}\|\nabla \Delta u\|_{L^2}^{1/2}\Big), \end{aligned}$$

which gives (4.12) thanks to (4.11). Inequalities (4.13), (4.14), (4.15) are consequences of Galiardo Nirenberg inequalities. □

We will also need the following proposition.

Proposition 4.2 ([5], Lem. 2.3)**.** *If $m \in H^1(\Omega)$, the restriction of $H_d(m)$ to Ω belongs to $H^1(\Omega)$, and there exists $C = C(\Omega) > 0$ such that,*

$$\|H_d(m)\|_{H^1(\Omega)} \leqslant C\|m\|_{H^1(\Omega)}.$$

4.2. Local smooth solutions: proof of Theorem 4.1

This section is dedicated to the proof of Theorem 4.1, following the strategy of [5], Theorem 1.1. This proof is in six steps.

First step: Approximate problem

We denote by $\mathcal{V}_n$ the finite dimensional space built on the n first eigen-functions of $-\Delta$ on Ω with Neumann conditions, and by P_n the orthogonal projection from $L^2(\Omega, \mathbb{R})$ on $\mathcal{V}_n$. We seek a solution $m_n \in C^1([0, T_n), \mathcal{V}_n)$, with $T_n > 0$ of

$$\left\{\begin{array}{rl} \dfrac{\partial m_n}{\partial t} = & \alpha\Big\{\dfrac{1}{\lambda}\Delta m_n + H_{\mathrm{ext}} + \dfrac{1}{\lambda}P_n[|\nabla m_n|^2 m_n] - P_n[\langle H_{\mathrm{ext}}, m_n\rangle m_n] \\ & -P_n[m_n \wedge (m_n \wedge H_d(m_n))]\Big\} - P_n\left[m_n \wedge \left(\dfrac{1}{\lambda}\Delta m_n + H_d(m_n) + H_{\mathrm{ext}}\right)\right], \\ m_n(0) = P_n(m_0). & \end{array}\right. \tag{4.16}$$

Thanks to Cauchy-Lipschitz theorem, there exists a unique maximal solution of (4.16) defined on $[0, T_n)$ where $T_n \in (0, +\infty]$.

Second step: L^2 estimate

Taking the inner product in $L^2(\Omega)$ of (4.16) by m_n, and using (4.7) and (4.8), we get $C = C(\Omega, \alpha, \lambda) > 0$ such that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|m_n\|^2_{L^2}\right] + \frac{\alpha}{\lambda}\|\nabla m_n\|^2_{L^2} &= \int_\Omega \frac{\alpha}{\lambda}|\nabla m_n|^2|m_n|^2 + \alpha\langle H_{\mathrm{ext}}, m_n\rangle(1 - |m_n|^2) \\ &\leqslant C(1 + |H_{\mathrm{ext}}|)[1 + (\|m_n\|^2_{L^2} + \|\Delta m_n\|^2_{L^2})^2]. \end{aligned} \tag{4.17}$$

Thus, for every $T \in (0, T_n)$, for every $t \in [0, T]$, we have

$$\|m_n(t)\|^2_{L^2} + \frac{2\alpha}{\lambda}\int_0^t \|\nabla m_n\|^2_{L^2} \leqslant \|m_0\|^2_{L^2} + C[1 + \|H_{\mathrm{ext}}\|_{L^\infty(0,T)}]\int_0^t [1 + (\|m_n\|^2_{L^2} + \|\Delta m_n\|^2_{L^2})^2]. \tag{4.18}$$

Third step: H^2 estimate

Multiplying (4.16) by $\Delta^2 m_n$ and using integrations by parts, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\Delta m_n\|^2_{L^2(\Omega)}\right] + \frac{\alpha}{\lambda}\|\nabla\Delta m_n\|^2_{L^2(\Omega)} = I_1 + I_2 + I_3 + I_4, \tag{4.19}$$

where

$$\begin{aligned} I_1 &:= -\frac{\alpha}{\lambda}\int_\Omega \langle\nabla(|\nabla m_n|^2 m_n), \nabla\Delta m_n\rangle, \\ I_2 &:= \frac{1}{\lambda}\int_\Omega \langle\nabla m_n \wedge \Delta m_n, \nabla\Delta m_n\rangle, \\ I_3 &:= \int_\Omega \alpha\langle\nabla\left(m_n \wedge (m_n \wedge H_d(m_n))\right), \nabla\Delta m_n\rangle + \langle\nabla(m_n \wedge H_d(m_n)), \nabla\Delta m_n\rangle, \\ I_4 &:= \int_\Omega \alpha\langle H_{\mathrm{ext}}, \nabla m_n\rangle\langle m_n, \nabla\Delta m_n\rangle + \alpha\langle H_{\mathrm{ext}}, m_n\rangle\langle\nabla m_n, \nabla\Delta m_n\rangle + \langle\nabla m_n \wedge H_{\mathrm{ext}}, \nabla\Delta m_n\rangle. \end{aligned}$$

Working as in [5], third step of the proof of Theorem 1.1, we get $C = C(\Omega, \alpha, \lambda) > 0$ such that we can estimate the first three terms as

$$|I_1 + I_2 + I_3| \leqslant C[1 + (\|m_n\|_{L^2}^2 + \|\Delta m_n\|_{L^2}^2)^{5/4}]\|\nabla\Delta m_n\|_{L^2}^{3/2}.$$

For the fourth term, we have, thanks to Cauchy-Schwarz inequality,

$$\begin{aligned}|I_4| &= \int_\Omega (2\alpha|m_n| + 1)|H_{\text{ext}}||\nabla m_n||\nabla\Delta m_n| \\ &\leqslant (2\alpha\|m_n\|_{L^\infty} + 1)|H_{\text{ext}}|\|\nabla m_n\|_{L^2}\|\nabla\Delta m_n\|_{L^2}.\end{aligned}$$

Using (4.7) and (4.8), we get $C = C(\Omega, \alpha)$ such that

$$|I_4| \leqslant C|H_{\text{ext}}|(1 + \|m_n\|_{L^2}^2 + \|\Delta m_n\|_{L^2}^2)\|\nabla\Delta m_n\|_{L^2}.$$

Thus, there exists $C = C(\Omega, \alpha, \lambda) > 0$ such that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\Delta m_n\|_{L^2}^2\right] + \frac{\alpha}{\lambda}\|\nabla\Delta m_n\|_{L^2}^2 &\leqslant C\left(1 + (\|m_n\|_{L^2}^2 + \|\Delta m_n\|_{L^2}^2)^{5/4}\right)\|\nabla\Delta m_n\|_{L^2}^{3/2} \\ &\quad + C|H_{\text{ext}}|\left(1 + \|m_n\|_{L^2}^2 + \|\Delta m_n\|_{L^2}^2\right)\|\nabla\Delta m_n\|_{L^2} \\ &\leqslant \frac{\alpha}{4\lambda}\|\nabla\Delta m\|_{L^2}^2 + C\left(\frac{4\lambda}{\alpha}\right)^3\left[1 + (\|m_n\|_{L^2}^2 + \|\Delta m_n\|_{L^2}^2)^5\right] \\ &\quad + \frac{\alpha}{4\lambda}\|\nabla\Delta m\|_{L^2}^2 + C\frac{4\lambda}{\alpha}|H_{\text{ext}}|^2\left[1 + (\|m_n\|_{L^2}^2 + \|\Delta m_n\|_{L^2}^2)^2\right].\end{aligned}$$

Simplifying the terms containing $\|\nabla\Delta m_n\|_{L^2}$, and summing with (4.18), we get, for every $T \in (0, T_n)$, for every $t \in [0, T]$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\|m_n\|_{L^2(\Omega)}^2 + \|\Delta m_n\|_{L^2(\Omega)}^2\right] + \frac{\alpha}{\lambda}\left[\|\nabla m_n\|_{L^2(\Omega)}^2 + \|\nabla\Delta m_n\|_{L^2(\Omega)}^2\right] \leqslant \\ &\qquad C(1 + \|H_{\text{ext}}\|_{L^\infty(0,T)}^2)\left(1 + (\|m_n\|_{L^2(\Omega)}^2 + \|\Delta m_n\|_{L^2(\Omega)}^2)^5\right),\end{aligned} \tag{4.20}$$

where $C = C(\Omega, \alpha, \lambda)$. Thus, there exists $T^* > 0$ such that:

- for every n, the solution of the approximate problem (4.16) is defined (at least) on $(0, T^*)$ (see [5], Lem. 2.4);
- for every $T \in (0, T^*)$, $(m_n)_{n\in\mathbb{N}^*}$ is bounded in

$$L^2((0,T), H^3(\Omega)) \cap L^\infty((0,T), H^2(\Omega)) \cap H^1((0,T), H^1(\Omega, \mathbb{R}^3)).$$

Fourth step: Convergence

As in [5], these bounds allow to extract converging subsequences and pass to the limit $n \to +\infty$. In particular the terms with H_{ext} do not pose any difficulty. The proof of the conservation of the punctual norm of m is identical to [5].

Fifth step: L^2-stability

Let m_1, m_2 be two solutions of (3.15), $T^* := \min\{T_1^*, T_2^*\}$, $v := m_1 - m_2$. We prove that, for every $T \in (0, T^*)$, there exists $C > 0$ such that

$$\sup_{t\in[0,T]}\|v(t)\|_{L^2(\Omega)} \leqslant C\|v(0)\|_{L^2(\Omega)}. \tag{4.21}$$

One has

$$\begin{aligned}\frac{\partial v}{\partial t} \;=\;& \alpha\Big[\frac{1}{\lambda}\Delta v+\frac{1}{\lambda}|\nabla m_1|^2 v+\frac{1}{\lambda}\left(|\nabla m_1|^2-|\nabla m_2|^2\right)m_2\\ &- v\wedge(m_1\wedge H_d(m_1))-\ m_2\wedge(v\wedge H_d(m_1))-m_2\wedge(m_2\wedge H_d(v))\\ &-\langle H_{\rm ext},v\rangle m_1-\langle H_{\rm ext},m_2\rangle v\Big]\\ &+v\wedge\left(\frac{1}{\lambda}\Delta m_1+H_d(m_1)\right)+m_2\wedge\left(\frac{1}{\lambda}\Delta v+H_d(v)\right)+v\wedge H_{\rm ext}.\end{aligned}$$

Multiplying this equation by v, integrating over Ω, using the fact that $m_1,m_2\in L^\infty((0,T),H^2)\cap L^2((0,T),H^3)$, we get $f\in L^1(0,T)$ such that, for every $t\in[0,T]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{L^2}+\frac{\alpha}{\lambda}\|\nabla v\|^2_{L^2}\leqslant f(t)\|v\|^2_{L^2}.$$

We conclude (4.21) thanks to Gronwall Lemma.

Sixth step: H^2-stability

With the same notations as in the previous step, we prove that, for every $T\in(0,T^*)$, there exists $C=C(\Omega,\alpha,\lambda)>0$ such that

$$\sup_{t\in[0,T]}\|v(t)\|_{H^2(\Omega)}\leqslant C\|v(0)\|_{H^2(\Omega)}. \tag{4.22}$$

We go back to the Galerkin approximations. Taking the inner product of the equation on m_n with $\Delta^2 m_n$, integrating by parts on Ω, integrating in time between 0 and t and taking the limit when n tends to infinity gives the following inequality (thanks to the lower semi continuity of the norms under the weak topology)

$$\frac{1}{2}\|\Delta v(t)\|^2_{L^2(\Omega)}+\frac{\alpha}{\lambda}\int_0^t\|\nabla\Delta v\|^2_{L^2(\Omega)}\leqslant\frac{1}{2}\|\Delta v_0\|^2_{L^2(\Omega}+\int_0^t\sum_{j=1}^9 J_j(\tau)\mathrm{d}\tau, \tag{4.23}$$

where $J_1,\dots,J_9$ respectively stand for the following expressions

$$\begin{aligned}J_1 &:= \left|\int_\Omega\nabla(v\wedge\Delta m_1)\nabla\Delta v\right|,\\ J_2 &:= \left|\int_\Omega\nabla(v\wedge H_d(m_1))\nabla\Delta v\right|,\\ J_3 &:= \left|\int_\Omega\nabla(m_2\wedge\Delta v)\nabla\Delta v\right|,\\ J_4 &:= \left|\int_\Omega\nabla(m_2\wedge H_d(v))\nabla\Delta v\right|,\\ J_5 &:= \left|\int_\Omega\nabla(|\nabla m_1|^2 v)\nabla\Delta v\right|,\\ J_6 &:= \left|\int_\Omega\nabla((|\nabla m_1|^2-|\nabla m_2|^2)m_2)\nabla\Delta v\right|,\end{aligned}$$

$$
\begin{aligned}
J_7 &:= \left| \int_\Omega \nabla(v \wedge (m_1 \wedge H_d(m_1)))\nabla\Delta v \right|, \\
J_8 &:= \left| \int_\Omega \nabla(m_2 \wedge (v \wedge H_d(m_1) + m_2 \wedge H_d(v)))\nabla\Delta v \right|, \\
J_9 &:= \left| \int_\Omega \alpha\langle \nabla[\langle H_{\text{ext}}, v\rangle m_1], \nabla\Delta v\rangle - \alpha\langle \nabla[\langle H_{\text{ext}}, m_2\rangle v], \nabla\Delta v\rangle - \langle \nabla v \wedge H_{\text{ext}}, \nabla\Delta v\rangle \right|.
\end{aligned}
$$

Working as in [5], pp. 12–13, there exists $f_1 \in L^1(0,T)$ such that

$$
\left| \sum_{j=1}^{8} J_j \right| \leqslant \frac{\alpha}{4\lambda}\|\nabla\Delta v\|_{L^2}^2 + f_1(t)\left(\|v(t)\|_{L^2}^2 + \|\Delta v(t)\|_{L^2}^2\right). \tag{4.24}
$$

For the last term J_9, we get from Cauchy-Schwarz inequality that

$$
\begin{aligned}
|J_9| &\leqslant \int_\Omega \Big[\alpha(|m_1| + |m_2|) + 1\Big]|\nabla v||\nabla\Delta v||H_{\text{ext}}| + \alpha\Big[|\nabla m_1| + |\nabla m_2|\Big]|v||\nabla\Delta v||H_{\text{ext}}| \\
&\leqslant \Big[\alpha(\|m_1\|_{L^\infty} + \|m_2\|_{L^\infty}) + 1\Big]\|\nabla v\|_{L^2}\|\nabla\Delta v\|_{L^2}|H_{\text{ext}}| \\
&\quad + \alpha\Big[\|\nabla m_1\|_{L^\infty} + \|\nabla m_2\|_{L^\infty}\Big]\|v\|_{L^2}\|\nabla\Delta v\|_{L^2}|H_{\text{ext}}|.
\end{aligned}
$$

Using the embeddings $H^3(\Omega) \to W^{1,\infty}(\Omega)$, $H^2(\Omega) \to H^1(\Omega)$ and (4.7), there exists $C = C(\Omega, \alpha) > 0$ such that

$$
|J_9| \leqslant C\Big[1 + \|m_1\|_{H^3} + \|m_2\|_{H^3}\Big]|H_{\text{ext}}|(\|v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2)^{1/2}\|\nabla\Delta v\|_{L^2}.
$$

Since $H_{\text{ext}} \in L^\infty(0,T)$ and $m_1, m_2 \in L^2((0,T), H^3)$, this proves the existence of $f_2 \in L^1(0,T)$ such that

$$
|J_9| \leqslant \frac{\alpha}{4\lambda}\|\nabla\Delta v\|_{L^2}^2 + f_2(t)\left(\|v\|_{L^2}^2 + \|\nabla\Delta v\|_{L^2}^2\right). \tag{4.25}
$$

Therefore, using (4.23), (4.24), (4.25) we get $f \in L^1(0,T)$ such that

$$
\|\Delta v(t)\|_{L^2}^2 + \frac{\alpha}{\lambda}\int_0^t \|\nabla\Delta v\|_{L^2}^2 \leqslant \|\Delta v_0\|_{L^2}^2 + \int_0^t f(\tau)\Big[\|v(\tau)\|_{L^2}^2 + \|\Delta v(\tau)\|_{L^2}^2\Big]\mathrm{d}\tau.
$$

We conclude (4.22) by applying Gronwall Lemma. □

4.3. **2D global solutions: proof of Theorem 4.2**

In this section, we give the proof of Theorem 4.2. We follow the strategy of [5], Theorem 1.2, in three steps. Namely, let m be a local solution of (3.15). We prove that, under the assumption (4.1), m is bounded in $L^\infty((0,T), H^2(\Omega))$ for every $T > 0$, which gives the conclusion.

First step: Estimate on ∇m

Multiplying the first equation in (3.15) by $\frac{\partial m}{\partial t}$, we get

$$
\int_\Omega \left|\frac{\partial m}{\partial t}\right|^2 = -\alpha\frac{\mathrm{d}}{\mathrm{d}t}[\mathcal{E}_\lambda(m)] + \int_\Omega \alpha\left\langle H_{\text{ext}}, \frac{\partial m}{\partial t}\right\rangle + \left\langle m \wedge H_\lambda(m), \frac{\partial m}{\partial t}\right\rangle.
$$

Moreover, the first equation in (3.15) provides

$$\left|\frac{\partial m}{\partial t}\right|^2 = (1+\alpha^2)|m\wedge H(m)|^2,$$

and thus,

$$\mathcal{E}_\lambda(m(t)) + \frac{1}{\alpha}\left(1-\frac{1}{\sqrt{1+\alpha^2}}\right)\int_0^t \left\|\frac{\partial m}{\partial t}\right\|^2_{L^2(\Omega)} \leqslant \mathcal{E}_\lambda(m_0) + \int_0^t \left\langle H_{\rm ext}, \frac{{\rm d}m_\sharp}{{\rm d}t}\right\rangle.$$

Thanks to an integration by parts in the last integral and the property $|m_\sharp| \leqslant 1$, we get

$$\mathcal{E}_\lambda(m(t)) \leqslant \mathcal{E}_\lambda(m_0) + 2\|H_{\rm ext}\|_{L^\infty} + \|\dot{H}_{\rm ext}\|_{L^1}. \tag{4.26}$$

Second step: Estimate on Δm

Multiplying the first equation of (3.15) by Δm we get

$$\frac{1}{2}\frac{\rm d}{{\rm d}t}\|\nabla m\|^2_{L^2} + \frac{\alpha}{\lambda}\|\Delta m\|^2_{L^2} = \frac{\alpha}{\lambda}\|\nabla m\|^4_{L^4} + K_1 + K_2,$$

where

$$\begin{aligned} K_1 &:= \int_\Omega -\alpha\langle H_d(m) - \langle H_d(m), m\rangle m, \Delta m\rangle + \langle m\wedge H_d(m), \Delta m\rangle, \\ K_2 &:= \alpha\int_\Omega |\nabla m|^2\langle H_{\rm ext}, m\rangle. \end{aligned}$$

The inequalities (4.13) and (4.26) give

$$\|\nabla m\|^4_{L^4(\Omega)} \leqslant C^* B_1 \|\Delta m\|^2_{L^2(\Omega)}, \tag{4.27}$$

where B_1 is the left hand side of (4.1). Thanks to (4.1), there exists $\epsilon > 0$ such that $1-\epsilon-C^*B_1 > 0$. Using Proposition 4.2, we can bound K_1 and K_2 by

$$\begin{aligned} |K_1| &\leqslant (\alpha+1)\|H_d(m)\|_{L^2}\|\Delta m\|_{L^2} \\ &\leqslant \frac{\epsilon\alpha}{\lambda}\|\Delta m\|^2_{L^2} + \frac{\lambda}{\epsilon\alpha}(\alpha+1)^2, \\ |K_2| &\leqslant \alpha|H_{\rm ext}|\|\nabla m\|^2_{L^2}. \end{aligned}$$

Thus, we get

$$\frac{1}{2}\frac{\rm d}{{\rm d}t}\|\nabla m(t)\|^2_{L^2} + \frac{\alpha}{\lambda}(1-\epsilon-C^*B_1)\|\Delta m\|^2_{L^2} \leqslant \frac{\lambda}{\epsilon\alpha}(\alpha+1)^2 + \alpha|H_{\rm ext}|B_1.$$

Integrating in time, we get the existence of a constant $B_2 = B_2(\Omega, \alpha, \lambda, \epsilon, \|\nabla m_0\|_{L^2}, \|H_{\rm ext}\|_{L^\infty}, \|\dot{H}_{\rm ext}\|_{L^1}) > 0$ such that

$$\int_0^t \|\Delta m\|^2_{L^2} \leqslant B_2\Big[\|\nabla m_0\|^2_{L^2} + t\Big]. \tag{4.28}$$

Third step: Estimate on $\nabla\Delta m_n$

Let us consider the solutions of (4.16). In the 2D case, it is possible to give better bounds for the right-hand side of (4.19).

Namely, thanks to Cauchy-Schwarz inequality, we have

$$\begin{aligned}|I_1| &\leqslant \frac{\alpha}{\lambda}\int_\Omega 2|D^2m_n||\nabla m_n||m_n||\nabla\Delta m_n| + |\nabla m_n|^3|\nabla\Delta m_n|\\ &\leqslant \frac{\alpha}{\lambda}\Big[2\|m_n\|_{L^\infty}\|\nabla m_n\|_{L^\infty}\|D^2m_n\|_{L^2}\|\nabla\Delta m_n\|_{L^2} + \|\nabla m_n\|^3_{L^6}\|\nabla\Delta m_n\|_{L^2}\Big].\end{aligned}$$

Using (4.10), (4.14) and (4.15), we get $C = C(\Omega) > 0$ such that

$$|I_1| \leqslant \frac{C\alpha}{\lambda}\Big[\|m_n\|_{L^\infty}\|\nabla m_n\|^{1/2}_{L^2}\|\Delta m_n\|_{L^2}\|\nabla\Delta m_n\|^{3/2}_{L^2} + \|\nabla m_n\|_{L^2}\|\Delta m_n\|^2_{L^2}\|\nabla\Delta m_n\|_{L^2}\Big].$$

Thus, there exists $C_1 = C_1(\Omega,\alpha,\lambda) > 0$ such that

$$|I_1| \leqslant \frac{\alpha}{8\lambda}\|\nabla\Delta m_n\|^2_{L^2} + C_1\Big[\|m_n\|^4_{L^\infty}\|\nabla m_n\|^2_{L^2}\|\Delta m_n\|^4_{L^2} + \|\nabla m_n\|^2_{L^2}\|\Delta m_n\|^4_{L^2}\Big]. \tag{4.29}$$

For I_2, we have thanks to Cauchy-Schwarz inequality,

$$\begin{aligned}|I_2| &\leqslant \frac{1}{\lambda}\int_\Omega |\nabla m_n||\Delta m_n||\nabla\Delta m_n|\\ &\leqslant \frac{1}{\lambda}\|\nabla m_n\|_{L^\infty}\|\Delta m_n\|_{L^2}\|\nabla\Delta m_n\|_{L^2},\end{aligned}$$

and using (4.15), we get $C_2 = C_2(\Omega,\alpha,\lambda) > 0$ such that

$$\begin{aligned}|I_2| &\leqslant \frac{C}{\lambda}\|\nabla m_n\|^{1/2}_{L^2}\|\Delta m_n\|_{L^2}\|\nabla\Delta m_n\|^{3/2}_{L^2}\\ &\leqslant \frac{\alpha}{4\lambda}\|\nabla\Delta m_n\|^2_{L^2} + C_2\|\nabla m_n\|^2_{L^2}\|\Delta m_n\|^4_{L^2}.\end{aligned} \tag{4.30}$$

Thanks to Cauchy-Schwarz inequality, we have

$$\begin{aligned}|I_3| &\leqslant \int_\Omega \Big[2\alpha|m_n|+1\Big]|\nabla m_n||H_d(m_n)||\nabla\Delta m_n| + \Big[\alpha|m_n|+1\Big]|m_n||\nabla H_d(m_n)||\nabla\Delta m_n|\\ &\leqslant \Big[2\alpha\|m_n\|_{L^\infty}+1\Big]\Big[\|\nabla m_n\|_{L^\infty}\|H_d(m_n)\|_{L^2}\|\nabla\Delta m_n\|_{L^2} + \|m_n\|_{L^\infty}\|\nabla H_d(m_n)\|_{L^2}\|\nabla\Delta m_n\|_{L^2}\Big]\end{aligned}$$

which, using (4.15) and Proposition 4.2, gives a constant $C = C(\Omega) > 0$ such that

$$|I_3| \leqslant C\Big[2\alpha\|m_n\|_{L^\infty}+1\Big]\Big[\|\nabla m_n\|^{1/2}_{L^2}\|m_n\|_{L^2}\|\nabla\Delta m_n\|^{3/2}_{L^2} + \|m_n\|_{L^\infty}\|m_n\|_{H^1}\|\nabla\Delta m_n\|_{L^2}\Big].$$

Therefore, there exists $C_3 = C_3(\Omega,\alpha,\lambda) > 0$ such that

$$|I_3| \leqslant \frac{\alpha}{8\lambda}\|\nabla\Delta m_n\|^2_{L^2} + C_3\Big[\|m_n\|^4_{L^\infty}+1\Big]\Big[\|\nabla m_n\|^2_{L^2}\|m_n\|^4_{L^2} + \|m_n\|^2_{L^\infty}\|m_n\|^2_{H^1}\Big]. \tag{4.31}$$

The last term is estimated thanks to Cauchy-Schwarz inequality. Indeed, one has

$$\begin{aligned}|I_4| &\leqslant \int_\Omega \Big[2\alpha|m_n|+1\Big]|H_{\text{ext}}||\nabla m_n||\nabla\Delta m_n|\\ &\leqslant \Big[2\alpha\|m_n\|_{L^\infty}+1\Big]|H_{\text{ext}}|\|\nabla m_n\|_{L^2}\|\nabla\Delta m_n\|_{L^2},\end{aligned}$$

and thus, there exists $C_4 = C_4(\Omega,\alpha,\lambda) > 0$ such that

$$|I_4| \leqslant \frac{\alpha}{8\lambda}\|\nabla\Delta m_n\|_{L^2}^2 + C_4[\|m_n\|_{L^\infty}^2 + 1]|H_{\rm ext}|^2\|\nabla m_n\|_{L^2}^2. \tag{4.32}$$

Putting together (4.19), (4.29), (4.30), (4.31) and (4.32), we get the inequality

$$\|\Delta m_n(t)\|_{L^2}^2 + \frac{\alpha}{\lambda}\int_0^t \|\nabla\Delta m_n\|_{L^2}^2 \leqslant \|\Delta m_0\|_{L^2}^2 + \int_0^t F(\|m_n\|_{L^2}, \|m_n\|_{L^\infty}, \|\nabla m_n\|_{L^2}, \|\Delta m_n\|_{L^2}).$$

Considering the liminf when $n \to +\infty$ in both sides (the norms are lower semi continuous for the weak topology), and using (4.26), we get $C = C(\Omega,\alpha,\lambda,\|H_{\rm ext}\|_{L^\infty},\|\dot H_{\rm ext}\|_{L^1}, m_0) > 0$ such that

$$\|\Delta m(t)\|_{L^2}^2 + \frac{\alpha}{\lambda}\int_0^t \|\nabla\Delta m(\tau)\|_{L^2}^2{\rm d}\tau \leqslant \|\Delta m_0\|_{L^2}^2 + C\int_0^t (1 + \|\Delta m(\tau)\|_{L^2}^4){\rm d}\tau. \tag{4.33}$$

Finally, under assumption (4.1), inequality (4.28) holds, thus, Gronwall Lemma applied to (4.33) proves that, for every $T > 0$, $\|\Delta m_n(t)\|_{L^2}$ is bounded uniformly with respect to $t \in [0,T]$. □

4.4. **3D Global smooth solutions on ellipsoidal domains: proof of Theorem 4.3**

This section is dedicated to the proof of Theorem 4.3. This proof involves ideas quite different from those in [5].

Let m be a local solution of (3.15). We prove that, under the assumptions (4.2) and (4.3), m belongs to $L^\infty((0,T),H^2)$ for every $T > 0$, which gives the conclusion.

Let us first notice that it is sufficient to justify (4.4) for every $T > 0$. Indeed, let us assume that (4.4) holds for every $T > 0$. Under the assumption (4.3), by continuity, we have $N(T) < \alpha$ for T small enough. Let

$$T^* := \sup\{T > 0; N(T) < \alpha\}.$$

We assume $T^* < +\infty$. Then, by continuity $N(T^*) = \alpha$, thus (4.4) proves that

$$\|\Delta m(t)\|_{L^2} \leqslant \|\Delta m_0\|_{L^2}, \quad \forall t \in [0,T^*].$$

Then, thanks to (4.3) and the definition of $N(T^*)$, we have $N(T^*) < \alpha$. This is in contradiction with the definition of T^*. Therefore, $T^* = +\infty$ and $m \in L^\infty((0,+\infty),H^2(\Omega))$.

Now, in order to prove (4.4), we go back to the equality (4.19) for the approximate solutions, and as in the preceding section, we improve the bounds on I_1, I_2, I_3, I_4 in order to get the conclusion.

Thanks to Holder inequality, we have

$$\begin{aligned} |I_1| &\leqslant \frac{\alpha}{\lambda}\int_\Omega 2|D^2 m_n||\nabla m_n||m_n||\nabla\Delta m_n| + |\nabla m_n|^3|\nabla\Delta m_n| \\ &\leqslant \frac{C\alpha}{\lambda}\Big[\|D^2 m_n\|_{L^3}\|\nabla m_n\|_{L^6}\|m_n\|_{L^\infty}\|\nabla\Delta m_n\|_{L^2} + \|\nabla m_n\|_{L^6}^3\|\nabla\Delta m_n\|_{L^2}\Big]. \end{aligned}$$

Using (4.12) and (4.9), we get a constant $C = C(\Omega)$ such that

$$|I_1| \leqslant \frac{C\alpha}{\lambda}[\|m_n\|_{L^\infty}\|\Delta m_n\|_{L^2}^{3/2}\|\nabla\Delta m_n\|_{L^2}^{3/2} + \|\Delta m_n\|_{L^2}^3\|\nabla\Delta m_n\|_{L^2}],$$

which, using (4.11), gives the existence of $C_1 = C_1(\Omega) > 0$ such that

$$|I_1| \leqslant \frac{C_1\alpha}{\lambda}\|\nabla\Delta m_n\|_{L^2}^2\Big(\|m_n\|_{L^\infty}\|\Delta m_n\|_{L^2} + \|\Delta m_n\|_{L^2}^2\Big). \tag{4.34}$$

As far as I_2 is concerned, using Holder inequality, we can estimate

$$\begin{aligned} |I_2| &\leqslant \frac{1}{\lambda}\int_\Omega |\nabla m_n||\Delta m_n||\nabla\Delta m_n| \\ &\leqslant \frac{1}{\lambda}\|\nabla m_n\|_{L^6}\|\Delta m_n\|_{L^3}\|\nabla\Delta m_n\|_{L^2} \\ &\leqslant \frac{C}{\lambda}\|\Delta m_n\|_{L^2}^{3/2}\|\nabla\Delta m_n\|_{L^2}^{3/2} \end{aligned}$$

using (4.9) and (4.12), with a constant $C = C(\Omega)$. Thanks to (4.11), we get $C_2 = C_2(\Omega)$ such that

$$|I_2| \leqslant \frac{C_2}{\lambda}\|\Delta m_n\|_{L^2}\|\nabla\Delta m_n\|_{L^2}^2. \tag{4.35}$$

For I_3, Holder inequality leads to

$$\begin{aligned} |I_3| &\leqslant \int_\Omega (2\alpha|m_n|+1)\Big(|\nabla m_n||H_d(m_n)||\nabla\Delta m_n| + |m_n||\nabla H_d(m_n)||\nabla\Delta m_n|\Big) \\ &\leqslant (2\alpha\|m_n\|_{L^\infty}+1)\|\nabla\Delta m_n\|_{L^2}\Big(\|\nabla m_n\|_{L^6}\|H_d(m_n)\|_{L^3} + \|m_n\|_{L^\infty}\|\nabla H_d(m_n)\|_{L^2}\Big). \end{aligned} \tag{4.36}$$

Thanks to the embedding $H^1(\Omega) \to L^3(\Omega)$ and Proposition 4.2, there exists $C = C(\Omega) > 0$ such that

$$\|H_d(m_n)\|_{L^3} \leqslant C\|H_d(m_n)\|_{H^1} \leqslant C\|m_n\|_{H^1}. \tag{4.37}$$

The second part is more subtle. As emphasized in the Remark 4.3, the proof takes advantage of the particular structure of the stray field on an ellipsoid. Indeed, we have

$$H_d(m) = H_d(m_\sharp) + H_d(m - m_\sharp)$$

where $H_d(m_\sharp)$ is constant over Ω, thus thanks to Proposition 4.2 and Poincaré inequality, there exists $C = C(\Omega) > 0$ such that

$$\|\nabla H_d(m)\|_{L^2} = \|\nabla H_d(m - m_\sharp)\|_{L^2} \leqslant C\|m - m_\sharp\|_{H^1} \leqslant C\|\nabla m\|_{L^2}. \tag{4.38}$$

Using (4.36), (4.37) and (4.38), we get a constant $C = C(\Omega) > 0$ such that

$$\begin{aligned} |I_3| &\leqslant C(2\alpha\|m_n\|_{L^\infty}+1)\|\nabla\Delta m_n\|_{L^2}\Big(\|\Delta m_n\|_{L^2}\|m_n\|_{H^1} + \|m_n\|_{L^\infty}\|\nabla m_n\|_{L^2}\Big) \\ &\leqslant C_3(2\alpha\|m_n\|_{L^\infty}+1)(\|m_n\|_{H^1} + \|m_n\|_{L^\infty})\|\nabla\Delta m_n\|_{L^2}^2 \end{aligned} \tag{4.39}$$

using (4.9) and (4.11), and for a constant $C_3 = C_3(\Omega) > 0$. Eventually, the last term is treated as follows

$$|I_4| \leqslant \int_\Omega (2\alpha|m_n|+1)|H_{\text{ext}}||\nabla m_n||\nabla\Delta m_n| \tag{4.40}$$

$$\leqslant (2\alpha\|m_n\|_{L^\infty}+1)|H_{\text{ext}}|\|\nabla m_n\|_{L^2}\|\nabla\Delta m_n\|_{L^2} \tag{4.41}$$

$$\leqslant C_4(2\alpha\|m_n\|_{L^\infty}+1)|H_{\text{ext}}|\|\nabla\Delta m_n\|_{L^2}^2 \tag{4.42}$$

thanks to (4.9) and (4.11) and where $C_4 = C_4(\Omega)$.

Finally, integrating (4.19) between $t=0$ and $t=T$, using (4.34), (4.35), (4.39), (4.42), letting $n\to+\infty$ and using $|m|\equiv 1$, we get $C_\sharp = C_\sharp(\Omega)>0$ such that

$$\|\Delta m(T)\|_{L^2}^2 + \frac{2\alpha}{\lambda}\int_0^T \|\nabla\Delta m(t)\|_{L^2}^2 \mathrm{d}t \leqslant \|\Delta m_0\|_{L^2}^2 + \int_0^T C_\sharp(\alpha+1)\|\nabla\Delta m\|_{L^2}^2\Big(\frac{1}{\lambda}[\|\Delta m\|_{L^2}+\|\Delta m\|_{L^2}^2]+\|\nabla m\|_{L^2}+1+|H_{\text{ext}}|\Big).$$

Let us assume that $\lambda\in(0,1)$ is sufficiently small so that

$$C_\sharp(\Omega)(\alpha+1)(1+\|H_{\text{ext}}\|_{L^\infty}) < \frac{\alpha}{\lambda},$$

which is a consequence of (4.3) if C^{**} is chosen so that $C^{**}>C_\sharp$. Then, we have

$$\|\Delta m(T)\|_{L^2}^2 + \frac{\alpha}{\lambda}\int_0^T \|\nabla\Delta m(t)\|_{L^2}^2 \mathrm{d}t \leqslant \|\Delta m_0\|_{L^2}^2 + \int_0^T C_\sharp(\alpha+1)\|\nabla\Delta m\|_{L^2}^2\Big(\frac{1}{\lambda}\Big[\|\Delta m\|_{L^2}+\|\Delta m\|_{L^2}^2\Big]+\|\nabla m\|_{L^2}\Big).$$

Since $\lambda\in(0,1)$, thanks to (4.9) with $p=2$, there exists $C=C(\Omega)>0$ such that

$$\|\nabla m\|_{L^2}\leqslant \frac{C}{\lambda}\|\Delta m\|_{L^2},$$

and thus, there exists $C^{**}=C^{**}(\Omega)>C_\sharp(\Omega)$ such that

$$\|\Delta m(T)\|_{L^2}^2 + \frac{\alpha}{\lambda}\int_0^T \|\nabla\Delta m(t)\|_{L^2}^2 \mathrm{d}t \leqslant \|\Delta m_0\|_{L^2}^2 + \int_0^T C^{**}\frac{(\alpha+1)}{\lambda}\|\nabla\Delta m\|_{L^2}^2\Big[\|\Delta m\|_{L^2}+\|\Delta m\|_{L^2}^2\Big]\mathrm{d}t.$$

This last estimation leads to (4.4). □

Remark 4.4. The same result holds with the same proof, when Ω is a 2D ellipsoid, instead of a 3D one. However, the result is weaker that the one of Theorem 4.2 because the initial condition needs to be close to constants in $H^2(\Omega)$. In Theorem 4.2 instead, one only needs an initial condition close to constants in $H^1(\Omega)$.

5. Exponential stabilization of uniform magnetizations on ellipsoidal domains

The goal of this subsection is to propose external magnetic fields H_{ext} that produce exponential convergence to global minimizers of the energy $\mathcal{E}_\lambda$. We consider an ellipsoidal domain Ω of $\mathbb{R}^3$ with $|\Omega|=1$, $\alpha>0$ and we study (3.15) and (3.16) with $\lambda>0$. On Ω, the stray field generated by a uniform magnetization is constant, thus, up to a change of coordinates, we may assume that

$$\forall x\in\Omega,\ \forall\widetilde{m}\in S^2,\ H_d(\widetilde{m}\chi_\Omega)(x) = -D\widetilde{m}, \tag{5.1}$$

where

$$D=\begin{pmatrix}\alpha_1 & 0 & 0\\ 0 & \alpha_2 & 0\\ 0 & 0 & \alpha_3\end{pmatrix},\ 0\leqslant\alpha_1\leqslant\alpha_2\leqslant\alpha_3. \tag{5.2}$$

Therefore, for non uniform magnetizations the stray field is given by

$$H_d(m) = -Dm_\sharp + \widetilde{H}_d(m), \tag{5.3}$$

where

$$\widetilde{H}_d(m) := H_d(m - m_\sharp \chi_\Omega). \tag{5.4}$$

Thus, in view of (1.8), we have

$$\|\widetilde{H}_d(m)\|_{L^2} \leqslant \|m - m_\sharp\|_{L^2} \tag{5.5}$$

and Poincaré inequality shows the existence of $C_d = C_d(\Omega) > 0$ such that

$$\|\widetilde{H}_d(m)\|_{L^2} \leqslant C_d \|\nabla m\|_{L^2}, \quad \forall m \in H^1(\Omega). \tag{5.6}$$

We are now in a position to state the results. We begin with the description of the minimizers.

Proposition 5.1. *Let Ω be a 3D ellipsoid. There exists $\lambda^* = \lambda^*(\Omega) > 0$ such that, for every $\lambda \in (0, \lambda^*)$, the micromagnetic energy $\mathcal{E}_\lambda$ has exactly two global minimizers: $m \equiv e_1$ and $m \equiv -e_1$.*

Physically speaking, it is clear that taking H_{ext} parallel to e_1 should force the magnetization to converge to e_1. This is indeed the case, and we even show a slightly stronger result. More precisely, we prove that, for λ small enough (*i.e.* for small domains $\Omega_\lambda = \sqrt{\lambda}\Omega$), the constant external field $H_{\mathrm{ext}} = \beta e_j$ forces, locally around e_j, the exponential convergence of the PDE solutions to e_j, when the parameter $\beta > 0$ is large enough. When $j = 1$, we therefore get the exponential stabilization of the global minimizers of the energy.

In that aim, let us introduce the modified energy $E_{\beta,j}$ defined by

$$E_{\beta,j}(m) := \int_\Omega \frac{1}{2\lambda}|\nabla m|^2 + \beta(1 - m_\sharp^{(j)}) + \frac{1}{2}\int_{\mathbb{R}^3} |H_d(m)|^2 \tag{5.7}$$

that measures the H^1-distance between m and e_j since we have

$$\int_\Omega |m - e_j|^2 = 2(1 - m_\sharp^{(j)}). \tag{5.8}$$

Proposition 5.2. *Let $j \in \{1,2,3\}$. When $H_{\mathrm{ext}}(t) \equiv \beta e_j$ for $t \in [0,T]$, the energy $E_{\beta,j}(m)$ is not increasing on $[0,T]$ along the trajectories of (3.15), and we have the estimate on a smooth solution m of (3.15)*

$$\frac{\mathrm{d}E_{\beta,j}}{\mathrm{d}t} \leqslant -\frac{1}{\alpha}\left(1 - \frac{1}{\sqrt{1+\alpha^2}}\right)\int_\Omega \left|\frac{\partial m}{\partial t}\right|^2 \text{ on } (0,T). \tag{5.9}$$

We now state the main result of this section.

Theorem 5.1. *Let Ω be a 3D ellipsoid and $\alpha > 0$. Let $\alpha_1, \alpha_2, \alpha_3, \beta_1^*, \beta_2^*, \beta_3^*$ be the real numbers defined by (5.1), (5.2) and*

$$\beta_1^* := \alpha_1 + \frac{\alpha_3 - \alpha_2}{2\alpha}, \quad \beta_2^* := \alpha_2 + \frac{\alpha_3 - \alpha_1}{2\alpha}, \quad \beta_3^* := \alpha_3 + \frac{\alpha_2 - \alpha_1}{2\alpha}\cdot \tag{5.10}$$

Let $j \in \{1,2,3\}$ and $\beta > \beta_j^*$. There exists $\lambda^* = \lambda^*(\Omega,\alpha,\beta) > 0$, $\eta = \eta(\Omega,\alpha) > 0$, $\nu = \nu(\Omega,\alpha,\beta,\lambda) > 0$, $K(\Omega,\alpha,\beta,\lambda) > 0$ such that, for every $m_0 \in H^2(\Omega, S^2)$ with $\dfrac{\partial m_0}{\partial \nu} \equiv 0$ on $\partial\Omega$, $E_{\beta,j}(m_0) \leqslant \beta$ and

$$\|\Delta m_0\|_{L^2} < \eta \tag{5.11}$$

there exists a unique global solution of (3.15) and (3.16) with $H_{\mathrm{ext}} \equiv \beta e_j$ which satisfies

$$\|m(t) - e_j\|_{H^1(\Omega)} \leqslant K\|m_0 - e_j\|_{H^1(\Omega)} \mathrm{e}^{-\nu t}. \tag{5.12}$$

Remark 5.1. The same result holds in 2D when Ω is an ellipse and with (5.11) replaced by

$$\|\nabla m_0\|_{L^2} < \eta.$$

Remark 5.2. The constants $\lambda^* = \lambda^*(\Omega,\alpha,\beta) > 0$, $\eta = \eta(\Omega,\alpha) > 0$, $\nu = \nu(\alpha,\beta,\lambda) > 0$, will be explicit in the proof.

Remark 5.3. In view of (4.6), when Ω is a 3D ellipsoid, λ small enough, β large enough and m_0 close enough to a constant in H^2 then the solution $m(t)$ of (3.15), (3.16) with $H_{\text{ext}} = \beta e_j$ converges exponentially to e_j in $H^s(\Omega)$ for every $s < 2$. Indeed, by interpolation, we have for every $\theta \in (0,1]$,

$$\begin{aligned} \|m(t) - e_j\|_{H^{\theta+2(1-\theta)}} &\leqslant \|m(t) - e_j\|_{H^1}^{\theta} \|m(t) - e_j\|_{H^2}^{1-\theta} \\ &\leqslant C(\Omega,\alpha,\beta,\lambda)\|m_0\|_{H^2}\|m(t) - e_j\|_{H^1}^{\theta}. \end{aligned}$$

Remark 5.4. When Ω is a 3D ellipsoid, λ small enough, β large enough, $m_0 \in H^4(\Omega, S^2)$ is in a H^2-neighborhood of constant magnetizations, and when $H_{\text{ext}} \equiv \beta e_j$, then, it can be shown with arguments similar to those used in this article, that smooth solutions belong to

$$C^1([0,+\infty), H^2) \cap C^0([0,+\infty), H^4)$$

and satisfy

$$\begin{aligned} &\forall t > 0,\ \|\Delta^2 m(t)\|_{L^2} \leqslant \|\Delta^2 m_0\|_{L^2}, \\ &\forall t > 0,\ \frac{\mathrm{d}}{\mathrm{d}t}\|\Delta m(t)\|_{L^2}^2 + \frac{\alpha}{\lambda}\|\nabla\Delta m(t)\|_{L^2}^2 \leqslant 0. \end{aligned}$$

Thus, thanks to interpolation theory, the exponential convergence of $m(t)$ to e_j holds in $H^s(\Omega)$, for every $s < 4$. Generalizing the method, one can prove that the convergence in Theorem 5.1 may hold in any $H^s(\Omega)$ for $s > 0$ provided the initial condition is close enough to constant magnetizations in a space $H^{s'}(\Omega)$ where $s' > s$ is well chosen.

The rest of this section, devoted to the proof of Theorem 5.1, is organized as follows. In Section 5.1, we prove Propositions 5.1 and 5.2. In Section 5.2, we prove the exponential convergence to zero of $\|\nabla m(t)\|_{L^2}$. In Section 5.3, we deduce from the previous result the exponential convergence to zero of $|m_\sharp(t) - e_j|$ (or $\|m(t) - e_j\|_{L^2}$ in view of (5.8)). Finally, in Section 5.4, we prove Theorem 5.1 and give explicitly the different constants.

5.1. **Proof of Propositions 5.1 and 5.2**

We give here the proofs of Propositions 5.1 and 5.2.

Proof of Proposition 5.1. From (5.3) and (5.4), one has

$$\|H_d(m)\|_{L^2}^2 = \langle Dm_\sharp, m_\sharp\rangle + \|H_d(m - m_\sharp\chi_\Omega)\|_{L^2}^2 - 2\int_\Omega \langle H_d(m_\sharp\chi_\Omega), H_d(m - m_\sharp\chi_\Omega)\rangle.$$

In the previous equality, the last integral vanishes, since it coincides with

$$\int_\Omega \langle H_d(m_\sharp\chi_\Omega), m - m_\sharp\rangle$$

which is zero because $H_d(m_\sharp\chi_\Omega)$ is constant on Ω. Therefore, we have

$$\begin{aligned}\mathcal{E}_\lambda(m) &= \int_\Omega \frac{1}{2\lambda}\|\nabla m\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}^3}\|H_d(m)\|_{L^2}^2 \\ &\geqslant \frac{1}{2\lambda}\|\nabla m\|_{L^2}^2 + \frac{\langle Dm_\sharp, m_\sharp\rangle}{2}.\end{aligned}$$

Noticing that $|m_\sharp|^2 + \frac{1}{|\Omega|}\|m - m_\sharp\|_{L^2}^2 = 1$, and using Poincare inequality, we get a constant $C_P = C_P(\Omega) > 0$ such that

$$\mathcal{E}_\lambda(m) \geqslant \frac{1}{2}\left(\frac{C_P}{\lambda} - \frac{\alpha_1}{|\Omega|}\right)\|m - m_\sharp\|_{L^2}^2 + \frac{1}{2}\left(\langle Dm_\sharp, m_\sharp\rangle - \alpha_1|m_\sharp|^2\right) + \frac{\alpha_1}{2}.$$

For $\lambda < \lambda^* = \dfrac{C_P|\Omega|}{\alpha_1}$ this is always greater than $\mathcal{E}_\lambda(e_1\chi_\Omega) = \dfrac{\alpha_1}{2}$, with equality if and only if $m = m_\sharp = e_1$. □

Proof of Proposition 5.2. Taking the scalar product in $L^2(\Omega)$ of the first equation of (3.15) with $\dfrac{\partial m}{\partial t}$, we get

$$\int_\Omega \left|\frac{\partial m}{\partial t}\right|^2 = -\alpha\frac{\mathrm{d}E_{\beta,j}}{\mathrm{d}t} + \int_\Omega \left\langle m \wedge H_\lambda(m), \frac{\partial m}{\partial t}\right\rangle.$$

This gives (5.9) because

$$\left|\frac{\partial m}{\partial t}\right|^2 = (1+\alpha^2)|m \wedge H(m)|^2.$$ □

5.2. Exponential convergence of $\|\nabla m\|_{L^2}$

We now pass to exponential convergence results.

Proposition 5.3. (1) *Let Ω be a 2D ellipsoid with $|\Omega| = 1$, $\alpha > 0$, $H_{\rm ext} \in L^\infty(\mathbb{R}_+, \mathbb{R}^3)$ with $\dot{H}_{\rm ext} \in L^1(\mathbb{R}_+, \mathbb{R}^3)$, and $c = c(\Omega) > 0$ be the largest constant such that*

$$c\|\nabla u\|_{L^2(\Omega)} \leqslant \|\Delta u\|_{L^2(\Omega)}, \quad \forall u \in H^2(\Omega) \text{ with } \frac{\partial u}{\partial \nu} \equiv 0 \text{ on } \partial\Omega. \tag{5.13}$$

There exists $\mathcal{C}^* = \mathcal{C}^*(\Omega) > 0$ such that, for every $\lambda > 0$ with

$$0 < \lambda < \frac{\alpha c^2}{4[\|H_{\rm ext}\|_{L^\infty} + \mathcal{C}^*(\alpha+1)]}, \tag{5.14}$$

for every $m_0 \in H^2(\Omega, S^2)$ with $\dfrac{\partial m_0}{\partial \nu} \equiv 0$ on $\partial\Omega$, and

$$\|\nabla m_0\|_{L^2}^2 + \lambda(\|H_d(m_0)\|_{L^2}^2 + 4\|H_{\rm ext}\|_{L^\infty} + 2\|\dot{H}_{\rm ext}\|_{L^1}) < \frac{1}{\mathcal{C}^*}, \tag{5.15}$$

there exists a global solution of (3.15) with initial condition $m(0) = m_0$ and this solution satisfies

$$\|\nabla m(t)\|_{L^2} \leqslant \|\nabla m_0\|_{L^2}\mathrm{e}^{-\frac{\alpha c^2}{2\lambda}t}, \quad \forall t > 0. \tag{5.16}$$

(2) *Let Ω be a 3D ellipsoidal domain with $|\Omega| = 1$, $\alpha > 0$, $H_{\rm ext} \in C^0 \cap L^\infty(\mathbb{R}_+, \mathbb{R}^3)$ and $c = c(\Omega) > 0$ be the largest constant such that (5.13) holds. There exists $\mathcal{C}^{**} = \mathcal{C}^{**}(\Omega, \alpha) > 0$ such that, for every $\lambda \in (0,1)$ with*

$$\mathcal{C}^{**}(1 + \|H_{\rm ext}\|_{L^\infty}) < \frac{\alpha}{\lambda}, \tag{5.17}$$

for every $m_0 \in H^2(\Omega, S^2)$ *with* $\frac{\partial m_0}{\partial \nu} \equiv 0$ *on* $\partial\Omega$, *and*

$$\mathcal{C}^{**}[\|\Delta m_0\|_{L^2} + \|\Delta m_0\|_{L^2}^2] < \alpha, \tag{5.18}$$

there exists a global solution of (3.15) *with initial condition* $m(0) = m_0$ *and this solution satisfies* (5.16).

Remark 5.5. The constants $\mathcal{C}^* = \mathcal{C}^*(\Omega)$ and $\mathcal{C}^{**} = \mathcal{C}^{**}(\Omega, \alpha)$ will be given explicitly.

Remark 5.6. Let us notice that the exponential convergence of $\|\nabla m(t)\|_{L^2}$ to zero holds with any $H_{\text{ext}} = H_{\text{ext}}(t)$ in the suitable functional space. One does not need $H_{\text{ext}} \equiv \beta e_j$ here.

Proof of Proposition 5.3. First, notice that if $\mathcal{C}^* > C^{*4}$ where C^* is as in Theorem 4.2 (resp. $\mathcal{C}^{**} > C^{**}(\alpha+1)$, where C^{**} is as in Thm. 4.3), then, under the assumption (5.15) (resp. (5.17) and (5.18)), then Theorem 4.2 (resp. Thm. 4.3) ensures the existence of global solutions.

The beginning of the proof is the same in the 2D and 3D situations. We estimate the behavior of the energy

$$\widetilde{E}(m) := \frac{1}{2}\int_\Omega |\nabla m|^2$$

by taking the scalar product in $L^2(\Omega)$ of the first equation of (3.15) with Δm. We get

$$\begin{aligned}\frac{\mathrm{d}\widetilde{E}}{\mathrm{d}t} &= -\frac{\alpha}{\lambda}\|\Delta m\|_{L^2}^2 + \frac{\alpha}{\lambda}\|\nabla m\|_{L^4}^4 - \int_\Omega |\nabla m|^2 \langle H_{\text{ext}}, m\rangle \\ &\quad + \int_\Omega -\alpha\langle \Delta m, H_d(m)\rangle - \alpha|\nabla m|^2\langle H_d(m), m\rangle + \langle \Delta m, m\wedge H_d(m)\rangle.\end{aligned} \tag{5.19}$$

Using $|m| \equiv 1$, we have

$$\begin{aligned}-\int_\Omega |\nabla m|^2\langle H_{\text{ext}}, m\rangle &\leqslant |H_{\text{ext}}|\|\nabla m\|_{L^2}^2 \\ &\leqslant \frac{|H_{\text{ext}}|}{c^2}\|\Delta m\|_{L^2}^2.\end{aligned} \tag{5.20}$$

Next, thanks to an integration by part, the property $\frac{\partial m}{\partial \nu} \equiv 0$ on $\partial\Omega$ leads to

$$\int_\Omega \alpha\langle \Delta m, H_d(m)\rangle + \langle \Delta m, m\wedge H_d(m)\rangle = \int_\Omega \alpha\langle \Delta m, \widetilde{H}_d(m)\rangle + \langle \Delta m, m\wedge \widetilde{H}_d(m)\rangle,$$

and using (5.6), (5.13) and $|m| \equiv 1$, we deduce that

$$\begin{aligned}\int_\Omega \alpha\langle \Delta m, H_d(m)\rangle + \langle \Delta m, m\wedge H_d(m)\rangle &\leqslant (\alpha+1)C_d\|\Delta m\|_{L^2}\|\nabla m\|_{L^2} \\ &\leqslant \frac{(\alpha+1)C_d}{c}\|\Delta m\|_{L^2}^2.\end{aligned} \tag{5.21}$$

Eventually, Cauchy-Schwarz inequality, (1.8) and (4.9) implies that there exists $C_1 = C_1(\Omega) > 0$ such that

$$\begin{aligned}\alpha\int_\Omega |\nabla m|^2\langle H_d(m), m\rangle &\leqslant \alpha\int_\Omega |\nabla m|^2|H_d(m)| \\ &\leqslant \alpha\|\nabla m\|_{L^4}^2\|H_d(m)\|_{L^2} \\ &\leqslant \alpha C_1\|\Delta m\|_{L^2}^2.\end{aligned} \tag{5.22}$$

First case: Ω is a 2D ellipse

In the same way as in the first step of the proof of Theorem 4.2, we get

$$\mathcal{E}_\lambda(m(t)) \leqslant \mathcal{E}_\lambda(m_0) + 2\|H_{\text{ext}}\|_{L^\infty} + \|\dot{H}_{\text{ext}}\|_{L^1}. \tag{5.23}$$

Thanks to (4.13) and (5.23), we have

$$\begin{aligned}\frac{\alpha}{\lambda}\|\nabla m\|_{L^4}^4 &\leqslant \frac{\alpha C^{*4}}{\lambda}\|\nabla m\|_{L^2}^2\|\Delta m\|_{L^2}^2 \\ &\leqslant \alpha C^{*4}\Big(2\mathcal{E}_\lambda(m_0) + 4\|H_{\text{ext}}\|_{L^\infty} + 2\|\dot{H}_{\text{ext}}\|_{L^1}\Big)\|\Delta m\|_{L^2}^2. \end{aligned}\tag{5.24}$$

Finally, using (5.19) and (5.20), (5.21), (5.22), (5.24), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{E}(m) \leqslant -\Big[\frac{\alpha}{\lambda} - \alpha C^{*4}\Big(2\mathcal{E}_\lambda(m_0) + 4\|H_{\text{ext}}\|_{L^\infty} + 2\|\dot{H}_{\text{ext}}\|_{L^1}\Big) - \frac{|H_{\text{ext}}|}{c^2} - \frac{(\alpha+1)C_d}{c} - \alpha C_1\Big]\|\Delta m\|_{L^2}^2.$$

Let us assume that (5.14) and (5.15) hold with the constant $\mathcal{C}^*$ defined by

$$\mathcal{C}^* := \max\Big\{4C^{*4}; C_d c + C_1 c^2\Big\}.$$

Then, we have

$$\frac{\mathrm{d}\widetilde{E}}{\mathrm{d}t} \leqslant -\frac{\alpha c^2}{\lambda}\widetilde{E}, \tag{5.25}$$

which gives the conclusion.

Second case: Ω is a 3D ellipsoid

Thanks to (4.9) and (4.6), we get $C_2 = C_2(\Omega) > 0$ such that

$$\begin{aligned}\frac{\alpha}{\lambda}\|\nabla m\|_{L^4}^4 &\leqslant \frac{\alpha C_2}{\lambda}\|\Delta m\|_{L^2}^4 \\ &\leqslant \frac{\alpha C_2}{\lambda}\|\Delta m_0\|_{L^2}^2\|\Delta m\|_{L^2}^2. \end{aligned}\tag{5.26}$$

Finally, using (5.19) and (5.20), (5.21), (5.22), (5.26), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{E}(m) \leqslant -\Big[\frac{\alpha}{\lambda} - \frac{\alpha}{\lambda}C_2\|\Delta m_0\|_{L^2}^2 - \frac{|H_{\text{ext}}|}{c^2} - \frac{(\alpha+1)C_d}{c} - \alpha C_1\Big]\|\Delta m\|_{L^2}^2.$$

Let us assume that (5.17) and (5.18) hold with the constant $\mathcal{C}^{**}$ such that

$$\mathcal{C}^{**} := \max\Big\{C^{**}(\alpha+1), 4\alpha C_2, \frac{4}{c^2}, \frac{4(\alpha+1)C_d}{c} + 4\alpha C_1\Big\}.$$

Then, we have (5.25) which gives the conclusion. □

5.3. Exponential convergence of $m_\sharp$

Now we study the solutions of (3.15) with the external magnetic field $H_{\text{ext}} \equiv \beta e_j$, $j \in \{1, 2, 3\}$.

Proposition 5.4. *Let β_k^*, $k = 1, 2, 3$ be defined by (5.10). Let $j \in \{1, 2, 3\}$ and $\beta > \beta_j^*$. Let m be a global smooth solution of (3.15) with $H_{\text{ext}} \equiv \beta e_j$ such that*

- $E_{\beta,j}(m_0) \leqslant \beta$;
- *there exists $\delta > 0$ such that*

$$\|\nabla m(t)\|_{L^2} \leqslant \|\nabla m_0\|_{L^2} \mathrm{e}^{-\delta t}, \quad \forall t > 0.$$

Then, there exists $C > 0$ (that does not depend on m) such that

$$|m_\sharp(t) - e_j| \leqslant C\Big(|m_\sharp(0) - e_j| + \|\nabla m_0\|_{L^2}\Big)\mathrm{e}^{-\nu t}$$

where $\nu := \min\{\alpha(\beta - \beta_j), \delta\}$.

Proof of Proposition 5.4. We define

$$\begin{aligned} f_j(t) \quad := \quad & \int_\Omega -\frac{\alpha}{\lambda}|\nabla m|^2 m^{(j)} + \alpha\beta m^{(j)}(m^{(j)} - m_\sharp^{(j)}) + \alpha\langle Dm_\sharp, m_\sharp - m\rangle m^{(j)} \\ & - \alpha\Big[\tilde{H}_d(m)^{(j)} - \langle \tilde{H}_d(m), m\rangle m^{(j)}\Big] + (m \wedge \tilde{H}_d(m))^{(j)}. \end{aligned}$$

First case: $j = 1$

Integrating over Ω the first component of the first equality of (3.15) with $H_{\text{ext}} \equiv \beta e_1$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}[1 - m_\sharp^{(1)}] = -\alpha\beta(1 - (m_\sharp^{(1)})^2) - (\alpha_3 - \alpha_2)m_\sharp^{(2)} m_\sharp^{(3)} + \alpha(\alpha_1 - \langle Dm_\sharp, m_\sharp\rangle)m_\sharp^{(1)} + f_1(t).$$

Since $E_{\beta,1}(m_0) \leqslant \beta$ and $t \mapsto E_{\beta,1}[m(t)]$ is not increasing (see Prop. 5.2), we have $m_\sharp^{(1)}(t) \geqslant 0$, $\forall t \geqslant 0$. Thanks to that inequality and $|m_\sharp(t)| \leqslant 1$, we get

$$\begin{aligned} \alpha(\alpha_1 - \langle Dm_\sharp, m_\sharp\rangle)m_\sharp^{(1)} & \leqslant \alpha\alpha_1(1 - |m_\sharp^{(1)}|^2)m_\sharp^{(1)} \\ & \leqslant \alpha\alpha_1[1 - (m_\sharp^{(1)})^2]. \end{aligned}$$

Using $|m_\sharp| \leqslant 1$, we get

$$\begin{aligned} (\alpha_3 - \alpha_2)|m_\sharp^{(2)} m_\sharp^{(3)}| \quad & \leqslant \quad \frac{\alpha_3 - \alpha_2}{2}[(m_\sharp^{(2)})^2 + (m_\sharp^{(3)})^2] \\ & \leqslant \quad \frac{\alpha_3 - \alpha_2}{2}(1 - (m_\sharp^{(1)})^2). \end{aligned}$$

Thanks to the two previous inequalities, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}[1 - m_\sharp^{(1)}] \leqslant -\alpha\beta_1[1 - m_\sharp^{(1)}] + f_1(t), \ \text{ where } \beta_1 := \beta - \beta_1^*.$$

Thanks to (5.5) and Poincaré formula, there exists $C_P = C_P(\Omega) > 0$ such that

$$\begin{aligned} \alpha\langle Dm_\sharp, m - m_\sharp\rangle m^{(1)} \quad & \leqslant \quad \alpha\alpha_3 C_P \|\nabla m\|_{L^2}, \\ \int_\Omega -\alpha[\widetilde{H}_d(m) - \langle \widetilde{H}_d(m), m\rangle m] + (m \wedge \widetilde{H}_d(m))^{(1)} \quad & \leqslant \quad (\alpha + 1)C_d \|\nabla m\|_{L^2(\Omega)}, \\ \alpha\beta \int_\Omega m^{(1)}[m^{(1)} - m_\sharp^{(1)}] \quad & \leqslant \quad \alpha\beta C_P \|\nabla m\|_{L^2(\Omega)}, \end{aligned}$$

thus

$$|f_1(t)| \leqslant F_1 \mathrm{e}^{-\delta t}, \quad \forall t > 0,$$

where

$$F_1 := (\alpha\alpha_3 + \alpha + 1 + \alpha\beta) C_P \|\nabla m_0\|_{L^2} + \frac{\alpha}{\lambda}\|\nabla m_0\|_{L^2}^2.$$

Finally, we get

$$(1 - m_\sharp^{(1)})(t) \leqslant (1 - m_\sharp^{(1)})(0)\mathrm{e}^{-\alpha\beta_1 t} + \frac{F_1}{|\alpha\beta_1 - \delta|}|\mathrm{e}^{-\alpha\beta_1 t} - \mathrm{e}^{-\delta t}|,$$

which gives the conclusion.

Second case: $j = 2$

As in the first case, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[1 - m_\sharp^{(2)}] = -\alpha\beta(1 - (m_\sharp^{(2)})^2) + (\alpha_3 - \alpha_1)m_\sharp^{(1)}m_\sharp^{(3)} + \alpha(\alpha_2 - \langle Dm_\sharp, m_\sharp\rangle)m_\sharp^{(2)} + f_2(t).$$

We have

$$\begin{aligned} \alpha(\alpha_2 - \langle Dm_\sharp, m_\sharp\rangle)m_\sharp^{(2)} &\leqslant \alpha\alpha_2[1 - (m_\sharp^{(2)})^2], \\ (\alpha_3 - \alpha_1)|m_\sharp^{(2)}m_\sharp^{(3)}| &\leqslant \frac{\alpha_3 - \alpha_1}{2}(1 - (m_\sharp^{(2)})^2), \end{aligned}$$

thus

$$\frac{\mathrm{d}}{\mathrm{d}t}[1 - m_\sharp^{(2)}] \leqslant \alpha\beta_2[1 - m_\sharp^{(2)}] + f_2(t) \text{ where } \beta_2 := \beta - \beta_2^*.$$

We conclude in the same way as in the first case.

Third case: $j = 3$

We have

$$\frac{\mathrm{d}}{\mathrm{d}t}[1 - m_\sharp^{(3)}] = -\alpha\beta(1 - (m_\sharp^{(3)})^2) + (\alpha_1 - \alpha_2)m_\sharp^{(1)}m_\sharp^{(2)} + \alpha(\alpha_3 - \langle Dm_\sharp, m_\sharp\rangle)m_\sharp^{(3)} + f_3(t).$$

We have

$$\begin{aligned} \alpha(\alpha_3 - \langle Dm_\sharp, m_\sharp\rangle)m_\sharp^{(3)} &\leqslant \alpha\alpha_3[1 - (m_\sharp^{(3)})^2] \\ (\alpha_2 - \alpha_1)|m_\sharp^{(1)}m_\sharp^{(2)}| &\leqslant \frac{\alpha_2 - \alpha_1}{2}(1 - (m_\sharp^{(3)})^2), \end{aligned}$$

thus

$$\frac{\mathrm{d}}{\mathrm{d}t}[1 - m_\sharp^{(3)}] \leqslant \alpha\beta_3[1 - m_\sharp^{(3)}] + f_3(t) \text{ where } \beta_3 := \beta - \beta_3^*.$$

We conclude in the same way as in the first case. □

5.4. Conclusion: proof of Theorem 5.1

In this section, we deduce from Propositions 5.3 and 5.4 the values of the constants in Theorem 5.1.

2D case:

Let Ω be a 2D ellipsoid domain, $\alpha > 0$, $\lambda > 0$, $j \in \{1, 2, 3\}$, $\beta > \beta_j^*$, $m_0 \in H^2(\Omega, S^2)$ with $\partial m_0/\partial\nu \equiv 0$ on $\partial\Omega$ and $E_{\beta,j}(m_0) \leqslant \beta$. Let $\mathcal{C}^* = \mathcal{C}^*(\Omega)$ and $c = c(\Omega)$ be as in Proposition 5.3. We assume

$$\lambda < \lambda_*(\Omega, \alpha, \beta) := \min\left\{\frac{\alpha c^2}{4[\beta + \mathcal{C}^*(\alpha + 1)]}, \frac{1}{2\mathcal{C}^*[1 + 4\beta]}\right\}$$

and

$$\|\nabla m_0\|_{L^2} \leqslant \eta(\Omega, \alpha) := \frac{1}{\sqrt{2}\mathcal{C}^*}\cdot$$

Then, (5.14) and (5.15) hold, thus (5.16) is satisfied. Applying Proposition 5.4, and using (5.8), we get (5.12) with

$$\nu = \nu(\Omega, \alpha, \beta, \lambda) := \min\left\{\frac{\alpha c^2}{2\lambda}, \alpha(\beta - \beta_j^*)\right\}. \tag{5.27}$$

3D case:

Let Ω be a 3D ellipsoid, $\alpha > 0$, $\lambda \in (0,1)$, $j \in \{1,2,3\}$, $\beta > \beta_j^*$, $m_0 \in H^2(\Omega, S^2)$ with $\partial m_0/\partial \nu \equiv 0$ on $\partial\Omega$ and $E_{\beta,j}(m_0) \leqslant \beta$. Let $\mathcal{C}^{**} = \mathcal{C}^{**}(\Omega, \alpha)$ and $c = c(\Omega)$ be as in Proposition 5.3. We assume

$$\lambda < \lambda_*(\Omega, \alpha, \beta) := \frac{\alpha}{\mathcal{C}^*[1+\beta]}$$

and

$$\|\Delta m_0\|_{L^2} < \eta(\Omega, \alpha) := \min\left\{1, \frac{\alpha}{2\mathcal{C}^{**}}\right\}.$$

Then, (5.17) and (5.18) hold, thus (5.16) is satisfied. Applying Proposition 5.4, and using (5.8), we get (5.12) with (5.27).

6. Magnetization switching on ellipsoidal domains: PDE study

We use the notation β_1^* defined in (5.10), $H_{\mathrm{ext}}(m)$ defined by (2.5).

Proposition 6.1. *Let Ω be a 2D (resp. 3D) ellipsoid domain, $\alpha > 0$, $\beta > \beta_1^*$, $\lambda_* = \lambda_*(\Omega, \alpha, \beta)$ be as in Theorem 5.1, $T > 0$, $m_{\mathrm{ref}} \in H^2((0,T), S^2)$ be such that $-m_{\mathrm{ref}}(0) = m_{\mathrm{ref}}(T) = e_1$. We define $H_{\mathrm{ext}} \in L^\infty(\mathbb{R}_+, \mathbb{R}^3)$ by*

$$H_{\mathrm{ext}}(t) := \begin{cases} H_{\mathrm{ext}}(m_{\mathrm{ref}}(t)) & \text{if } 0 \leqslant t \leqslant T, \\ \beta e_1 & \text{if } t > T, \end{cases}$$

where $H_{\mathrm{ext}}(m_{\mathrm{ref}})$ is defined by (2.5). There exists $\eta > 0$ such that, for every $m_0 \in H^2(\Omega, S^2)$ (resp. $m_0 \in H^3(\Omega, S^2)$) with $\dfrac{\partial m_0}{\partial \nu} \equiv 0$ on $\partial\Omega$ and $\|m_0 + e_1\|_{H^1(\Omega)} < \eta$ (resp. $\|m_0 + e_1\|_{H^2(\Omega)} < \eta$), the solution of (3.15) converges exponentially to e_1 in $H^1(\Omega)$.

Proof. We use the continuity with respect to initial conditions for the $C^0([0,T], H^2)$-topology and we apply Theorem 5.1 on $(T, +\infty)$. □

7. Conclusion

In this paper, we have given a first contribution to the mathematical study of the switching of the magnetization inside a three dimensional small particle. This opens improvements in several directions. First, our results are still restricted to ellipsoidal particles and should be generalized to different shapes. It turns out to be not a simple technical difficulty and we plan to investigate this problem in the near future. Also, the case, particularly relevant in practice of an array of particles should be investigated. Indeed, it is absolutely crucial that the particles behave independently, and switching one particle must not perturb the other ones. Eventually, we plan to consider other types of control like the one that models the spin injection technique.

Let us finish by pointing out several differences between Landau-Lifschitz equations and the harmonic maps heat flow into spheres. Although the equations look quite similar, the gyromagnetic term in the magnetic model make them very different from the point of view of mathematical analysis. Indeed, a lot more results are known in the more geometrical case of the harmonic maps equation, like explicit blow-up solutions in finite time [7,9], global regular solutions if the energy is small [17] or if the energy is non-increasing in time [11], or if the initial condition takes values in a open half-sphere [13], etc. Non-uniqueness results are known for the heat flow of harmonic maps [8] or for Landau-Lifschitz equations but only when the effective magnetic field consists of the exchange term [1]. Such results are still not known for Landau-Lifschitz equations in full generality and seem very challenging.

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