Quadratic terms, Lie brackets and local controllability

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Abstract. We study the small-time local controllability, in the vicinity of an equilibrium, of a nonlinear scalar-input system. A natural strategy consists in proving the controllability of the linearized system at the equilibrium and conclude with a local inversion argument. When this linearized system is not controllable, it is necessary to study the quadratic term. In this article, we link several recent results on this subject. We also propose a unified methodology to prove quadratic obstructions to local controllability, that relies on systematic drift estimates.

For finite-dimensional control systems, quadratic terms only generate obstructions to controllability: they introduce coercive drifts in the dynamics, quantified by negative integer Sobolev norms of the control, linked to Lie brackets.

In infinite dimension, the same obstructions persist, but two new behaviors occur. First, quadratic obstructions can be due to drifts quantified by other norms (for instance fractional Sobolev norms); their geometric interpretation is challenging. Second, and more strikingly, small-time local controllability can sometimes be recovered from the quadratic expansion.

The tools developed for these studies are also used to understand the influence of higher order terms on the small-time local controllability, for single or multi-input systems, and in other research areas.

1 Introduction. Given an evolution system of the form $\dot{x}(t) = f(x(t), u(t))$, *control theory* is the mathematical study of how one can choose the *control* u at each time step to influence the behavior of the *state* x, which can be either finite or infinite dimensional. In this article, we focus on local controllability, which refers to the question of whether, within the vicinity of an equilibrium of the system, one can drive any initial state to any target state using small controls. To simplify the exposition in this introduction, we present the problem in the context of ordinary differential equations (ODEs). The generalization to partial differential equations (PDEs) will be presented in section 4.

1.1 Affine systems, controllability notion. We consider an affine control system

$$\dot{x}(t) = f_0(x(t)) + u(t) f_1(x(t)), \tag{1.1}$$

where f_0, f_1 are analytic vector fields on a neighborhood of 0 in $\mathbb{R}^d$ such that $f_0(0) = 0$ (these assumptions are valid for the whole article). The state is $x(t) \in \mathbb{R}^d$ and the control is $u(t) \in \mathbb{R}$. For each $u \in L^1_{\text{loc}}(\mathbb{R}_+; \mathbb{R})$, there exists a unique mild solution with initial condition $x(0) = 0$ and maximal definition interval, denoted $x(t; u)$. In particular $x(t; 0) = 0$. We are interested in the following infinitesimal question of controllability.

DEFINITION 1.1 (E-STLC). *Let $(E_T, \|.\|_{E_T})$ be a family of normed vector spaces of scalar-valued functions defined on $[0, T]$ for $T > 0$. System (1.1) is E-small-time locally controllable when, for every $T, \rho > 0$, there exists $\delta = \delta(T, \rho) > 0$ such that, for every $x^\star \in B(0, \delta)$, there exists $u \in E_T \cap L^1((0, T); \mathbb{R})$ such that $\|u\|_{E_T} \leq \rho$ and $x(T; u) = x^\star$.*

Any positive answer to the STLC problem may be thought of as a nonlinear local open mapping theorem, which underlines the deepness and intricacy of this problem, when the inverse mapping theorem cannot be used (see [26, section 3.1]). For E, we use the following usual Sobolev spaces.

DEFINITION 1.2 ($W^{m,p}((0, T); \mathbb{R})$ for $m \in \mathbb{N}$). *For $T > 0$ and $p \in [1, \infty]$, the space $W^{m,p}((0, T); \mathbb{R}) = \{u \in L^p((0, T); \mathbb{R}); u^{(j)} \in L^p((0, T); \mathbb{R})$ for $j = 1, \ldots, m\}$ is endowed with the norm $\|u\|_{W^{m,p}} = (\sum_{j=0}^m \|u^{(j)}\|_{L^p}^p)^{1/p}$ if $p \in [1, \infty)$ and $\|u\|_{W^{m,p}} = \sum_{j=0}^m \|u^{(j)}\|_{L^p}$ if $p = \infty$. We also use the notation $H^m((0, T); \mathbb{R})$ for $W^{m,2}((0, T); \mathbb{R})$.*

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DEFINITION 1.3 ($W^{-1,\infty}((0,T);\mathbb{R})$). *The space* $W_0^{1,1}((0,T);\mathbb{R})$ *is the closure of* $C_c^\infty((0,T);\mathbb{R})$ *in* $W^{1,1}((0,T);\mathbb{R})$ *and endowed with the norm* $\|u\|_{W_0^{1,1}} = \|u'\|_{L^1}$. *The space* $W^{-1,\infty}((0,T);\mathbb{R})$ *is the dual of* $W_0^{1,1}((0,T);\mathbb{R})$*, thus endowed with the dual norm* $\|u\|_{W^{-1,\infty}} = \sup\{\langle u, f\rangle; f \in W_0^{1,1}((0,T);\mathbb{R}), \|f\|_{W_0^{1,1}} \leq 1\}$.

A convenient way to estimate the $W^{-1,\infty}$-norm of a control is to consider the L^∞ norm of its primitive. Indeed, with the notation below, one can prove that, for every $u \in L^1((0,T);\mathbb{R})$ then $\|u\|_{W^{-1,\infty}} \leq \|u_1\|_{L^\infty} \leq 2\|u\|_{W^{-1,\infty}}$.

DEFINITION 1.4 (Notation u_j). *For* $T > 0$ *and* $u \in L^1((0,T);\mathbb{R})$*, we define the iterated primitives of* u*, denoted* $u_j : [0,T] \to \mathbb{R}$ *by:* $u_0 = u$ *and* $u_j(t) = \int_0^t u_{j-1}$ *for* $j \in \mathbb{N}^*$.

Then, for every $m \in \mathbb{N}^*$, ($W^{m,\infty}$-STLC) $\Rightarrow$ (L^∞-STLC) $\Rightarrow$ ($W^{-1,\infty}$-STLC), where any reciprocal implication is false (see [9, appendix A.1]). For scalar-input systems, the $W^{-1,\infty}$-STLC is equivalent to the small-state small-time local controllability (see [7, section 8.2]).

In the literature, STLC usually corresponds to L^∞-STLC (see e.g. [26, 46, 67, 68]). Considering other spaces allows to better describe the situation in finite dimension, but also to prepare the transfer from ODEs to PDEs, for which well-posedness may require different control assumptions than smallness in L^∞.

1.2 Illustrative examples. We consider the following systems with initial condition $x(0) = 0$

$$\Sigma_1 \begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = 2ux_1 + x_1^2 \end{cases} \qquad \Sigma_2 \begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_1^2 - x_2^2 \end{cases} \qquad \Sigma_3 \begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2^2 + x_1^3. \end{cases}$$

The system Σ_1 is not locally controllable around 0 because, for any control, the state remains above the parabola. Indeed, by integrating the second equation, we obtain, with the notation of Definition 1.4,

$$x_2(T) - x_1(T)^2 = \int_0^T (x_1(s))^2 \, ds = \int_0^T u_1^2 \geq 0.$$

The system Σ_2 is not locally controllable around zero in time $T \leq 1$ because, for any control, the state remains in the half space $\{x_3 \geq 0\}$. Indeed, using Cauchy-Schwarz inequality,

$$x_3(T) = \int_0^T (x_1^2 - x_2^2) = \int_0^T x_1^2 - \int_0^T \left(\int_0^t x_1\right)^2 dt \geq (1 - T^2) \int_0^T x_1^2 = (1 - T^2) \int_0^T u_1^2 \geq 0.$$

In the example Σ_3, due to Bronislaw Jakubczyk, one could fear that the quadratic term x_2^2 could dominate the cubic one x_1^3 and imply $\dot{x}_3 \geq 0$. This is the case for controls small in strong norms: Σ_3 is not $W^{1,\infty}$-STLC. Indeed, for any control u such that $\|u\|_{W^{1,\infty}} \leq 1$, the state remains on one side of a surface since

$$x_3(T) = \int_0^T (x_2^2 + x_1^3) = (x_2 x_1^2 - x_2^2 u)(T) + \int_0^T (1 + \dot{u}) x_2^2 \geq (x_2 x_1^2 - x_2^2)(T) + (1 - \|\dot{u}\|_{L^\infty}) \int_0^T u_2^2.$$

However, Hector Sussmann proved that Σ_3 is L^∞-STLC (see [68, p. 711-712]). He uses quickly oscillating controls, for which the cubic term dominates the quadratic one because it contains "more derivatives". For instance, for T small, one achieves a motion along $-e_3$ with $u(t) = T^{\frac{1}{2}} \varphi''(t/T)$ where $\varphi \in C_c^\infty((0,1);\mathbb{R})$ and $\alpha = \int_0^1 (\varphi')^3 < 0$ because then $x(T) = (T^6 \|\varphi\|_{L^2}^2 + \alpha T^{\frac{11}{2}}) e_3$. Note that, when $T \to 0$, $\|u\|_{L^\infty} = T^{\frac{1}{2}} \|\varphi''\|_{L^\infty}$ is small, whereas $\|\dot{u}\|_{L^\infty} = T^{-\frac{1}{2}} \|\varphi^{(3)}\|_{L^\infty}$ is large.

In these systems, the quadratic terms do not recover controllability: either they bend the reachable set without creating any new controllable direction (like ux_1 in Σ_1), or they introduce drifts in the dynamics, quantified by negative integer Sobolev norms of the control (see Remark 1.5), like x_1^2 in Σ_1 or x_2^2 in Σ_3. The small-time asymptotics introduces a hierarchy between these quadratic terms (x_1^2 is stronger than x_2^2 in Σ_2). Moreover, the choice of norm for measuring controls is crucial, even for systems in finite dimension: in Σ_3, the "good" term x_1^3 wins the competition with (or neutralizes) the "bad" term x_2^2 for the L^∞-STLC notion, but not for the stronger $W^{1,\infty}$-STLC notion. We shall see that these facts are generic.

Remark 1.5. For $k \in \mathbb{N}^*$, we define $H_*^{-k}((0,T);\mathbb{R})$ as the dual space of $H_*^k((0,T);\mathbb{R}) = \{\varphi \in H^k((0,T);\mathbb{R}); \varphi^{(j)}(T) = 0 \text{ for } j = 0, \ldots k-1\}$, endowed with the norm $\|\varphi\|_{H_*^k} = \|\varphi^{(k)}\|_{L^2}$. Then, for every $u \in L^1((0,T);\mathbb{R})$, we have $\|u_k\|_{L^2} = \|u\|_{H_*^{-k}}$. This justifies the terminology "drift quantified by negative integer Sobolev norms of the control". The equivalence does not hold with the usual H^{-k}-norm, dual norm of H_0^k.

1.3 Lie brackets and conditions for STLC.

Definition 1.6 (Lie bracket of vector fields). *For smooth vector fields f and g on an open subset Ω of $\mathbb{R}^d$, i.e. $f, g \in \mathcal{C}^\infty(\Omega; \mathbb{R}^d)$, we define the smooth vector field $[f,g] = \mathrm{ad}_f(g)$ by $[f,g](x) = Dg(x)f(x) - Df(x)g(x)$ for every $x \in \Omega$. Here $Df(x)$ denotes the differential (or equivalently the Jacobian matrix) of f at the point x.*

The iterated Lie brackets of f and g measure the lack of commutativity between the flows of f and g. For instance, $[f,g] = 0 \Rightarrow e^{t(f+g)} = e^{tf}e^{tg} = e^{tg}e^{tf}$ as long as these flows are defined.

Arthur Krener proved that, if two control systems of the form (1.1) have linearly isomorphic iterated Lie brackets evaluated at 0, then they are diffeomorphic (see [50, Theorem 1]); thus both are either STLC or not STLC. Therefore conditions for local controllability of (1.1) should involve only the evaluations at 0 of the Lie brackets of the vector fields f_0 and f_1.

One knows necessary and sufficient conditions for STLC in particular cases, such as linear [45] or driftless [21, 63] systems, but not in the general case of systems (1.1). Many powerful sufficient conditions for local controllability of (1.1) were obtained (for instance by Hector Sussmann, Andrei Agrachev, Revaz Gamkrelidze, Mikhail Krastanov in [68, 70, 1, 49]), which have iteratively refined the community's vision on which brackets were "good" or "bad", and which competitions could be won (see [43, 44, 46] by Henry Hermes and Matthias Kawski). Comparatively, only very few necessary conditions have been obtained (as in [67] by Gianna Stefani). One still does not know an exhaustive list of "good" and "bad" Lie brackets.

In this article, we show that, for scalar-input systems, at the quadratic level, any bracket is "bad". We identify the brackets likely to neutralize them, depending on the STLC notion chosen. Moreover, these results are proved with a methodology that allows to go further.

1.4 Algebraic notations. To organize the investigation of Lie brackets in a systematic way (which depends neither on the system nor on the choice of coordinates in $\mathbb{R}^d$), a nice approach is to index the terms by an underlying Lie bracket of the free Lie algebra. Thus, we introduce the following notions.

Definition 1.7 (Free Lie algebra). *Let $X = \{X_0, X_1\}$ be a set of two non-commutative indeterminates. Let $\mathcal{A}(X)$ be the algebra of polynomials of the indeterminates X_0 and X_1. For $a, b \in \mathcal{A}(X)$, we define $[a,b] = \mathrm{ad}_a(b) = ab - ba$. This operation satisfies the Jacobi identity $[c,[a,b]] = [[c,a],b] + [a,[c,b]]$. The Free Lie algebra $\mathcal{L}(X)$ is the smallest linear subspace of $\mathcal{A}(X)$ containing X and stable by the Lie bracket $[\cdot,\cdot]$.*

Definition 1.8 ($S_A(X)$). *For $j \in \mathbb{N}$, let $S_j(X)$ be the subspace of $\mathcal{L}(X)$ spanned by monomial brackets of degree j in the indeterminate X_1. For instance $\mathrm{ad}^8_{X_1}(X_0) \in S_8(X)$. For $A \subset \mathbb{N}$, we define $S_A(X) = \oplus_{j \in A} S_j(X)$.*

Definition 1.9 (M_ν, $W_{j,\nu}$). *For $j \in \mathbb{N}^*$ and $\nu \in \mathbb{N}$, we define the Lie brackets*

$$M_\nu = [\dots[X_1, \underbrace{X_0], \dots, X_0]}_{\nu \text{ times}}, \qquad W_{j,\nu} = [\dots[[M_{j-1}, M_j], \underbrace{X_0]\dots, X_0]}_{\nu \text{ times}}. \tag{1.2}$$

The M_ν are called "linear brackets" and belong to $S_1(X)$. The $W_{j,\nu}$ are called "quadratic brackets" and belong to $S_2(X)$. To simplify, we write W_j instead of $W_{j,0}$.

Definition 1.10 (Evaluated Lie bracket). *Let f_0, f_1 be smooth vector fields on an open subset Ω of $\mathbb{R}^d$. We define the set $f = \{f_0, f_1\}$. For $B \in \mathcal{L}(X)$, we define $f_B = \Lambda(B)$, where $\Lambda : \mathcal{L}(X) \to \mathcal{C}^\infty(\Omega; \mathbb{R}^d)$ is the unique homomorphism of Lie algebras such that $\Lambda(X_0) = f_0$ and $\Lambda(X_1) = f_1$. For instance, $f_{W_1} = [f_1, [f_1, f_0]]$. For a subset $\mathcal{N}$ of $\mathcal{L}(X)$ we define $\mathcal{N}(f)(0) = \mathrm{span}\{f_b(0); b \in \mathcal{N}\} \subset \mathbb{R}^d$.*

1.5 Necessary quadratic conditions for STLC. In all the known necessary conditions for STLC, one focuses on a "bad" bracket $\mathfrak{b} \in \mathcal{L}(X)$ and one identifies a subset $\mathcal{N}$ of $\mathcal{L}(X)$ containing all the brackets susceptible to neutralize $\mathfrak{b}$. Then the necessary condition for STLC is $f_\mathfrak{b}(0) \in \mathcal{N}(f)(0)$. For instance, the following necessary condition, concerning the strongest quadratic obstruction, is due to Hector Sussmann (see [68, Proposition 6.3]).

Theorem 1.11. *If system (1.1) is L^∞-STLC then $f_{W_1}(0) \in S_1(f)(0)$.*

In other words, for the system (1.1) to be L^∞-STLC, the quadratic bracket W_1 must be compensated by linear brackets M_ν. This is the case, for example, with the system $\dot{x} = u + x^2$, for which $f_{W_1}(0) = 2f_{M_0}(0)$ and whose L^∞-STLC results from the linear test. Another illustration is given by the system Σ_1, which is not L^∞-STLC because $f_{W_1}(0) = 2e_2 \notin S_1(f)(0) = \mathbb{R}e_1$.

The necessary condition of Theorem 1.11 was generalized at each even order by Gianna Stefani [67], and for the sharp STLC notion in [9, Theorem 1.10]: $W^{-1,\infty}$-STLC $\Rightarrow \mathrm{ad}_{f_1}^{2k}(f_0)(0) \in S_{[\![1,2k-1]\!]}(f)(0)$ for every $k \in \mathbb{N}^*$. In other words, for the system (1.1) to be $W^{-1,\infty}$-STLC, the term of order $2k$ (with respect to the control) associated with the bracket $\mathrm{ad}_{X_1}^{2k}(X_0)$ must be compensated by terms of order $\leq 2k-1$.

The following statement, proved by Frédéric Marbach and the author in [9, Theorem 1.11] extends the previous one to any quadratic bracket.

THEOREM 1.12. *Let $m \in \mathbb{N}$. If system (1.1) is $W^{m,\infty}$-STLC, then for every $(k,\nu) \in \mathbb{N}^* \times \mathbb{N}$, $f_{W_{k,\nu}}(0) \in S_{[\![1,\pi(k,m)]\!]\setminus\{2\}}(f)(0)$ where $\pi(k,m) = 1 + \left\lceil \frac{2k-2}{m+1} \right\rceil$.*

An illustration is given by Σ_3 for which $f_{W_2}(0) = 2e_3 \notin S_1(f)(0) = \mathrm{Span}\{e_1, e_2\}$ thus Σ_3 is not $W^{1,\infty}$-STLC. But Theorem 1.12 gives no obstruction for the L^∞-STLC of Σ_3 since $f_{W_2}(0) = \frac{1}{3} f_P(0)$ where $P = \mathrm{ad}_{X_1}^3(X_0) \in S_3(X)$. More generally, the threshold $\pi(k,m)$ is optimal (see [9, section 6.7]) but the list of brackets in $S_{[\![3,\pi(k,m)]\!]}(X)$ can be shrinked : the minimal list is given in [9, Theorem 1.13] for $m = 0$ and $k = 1,2,3$. Theorem 1.12 also holds for $m = -1$ with $\pi(k,-1) = +\infty$ and a different proof [9, section 10].

In section 2, we introduce algebraic tools necessary to present the proof strategy of Theorem 1.12, in section 3. This strategy is adapted to PDEs in section 4.

2 Preliminaries: algebraic tools, approximate representation formula.

2.1 Hall sets. Unlike $\mathcal{A}(X)$ – whose monomials form a basis – there is no canonical basis of $\mathcal{L}(X)$. In [71], Gérard Viennot theorized an algorithm to construct bases of $\mathcal{L}(X)$, named *Hall sets*, from orders on the free Magma over X, which is defined as follows (see also [64, Chapter 4] by Christophe Reutenauer).

DEFINITION 2.1. $\mathrm{Br}(X)$ *is the set of formal iterated brackets of elements of X, defined by induction: $X_0, X_1 \in \mathrm{Br}(X)$ and, if $a, b \in \mathrm{Br}(X)$, then the ordered pair (a,b) belongs to $\mathrm{Br}(X)$. For instance (X_0, X_0) and $(X_1,(X_1,X_0))$ are elements of $\mathrm{Br}(X)$. Contrary to $\mathcal{L}(X)$, the elements of $\mathrm{Br}(X)$ have well defined left and right factors. There is a natural* evaluation *mapping* E *from $\mathrm{Br}(X)$ to $\mathcal{L}(X)$ defined by induction by $\mathrm{E}(X_j) = X_j$ for $j = 0,1$ and $\mathrm{E}((a,b)) = [\mathrm{E}(a), \mathrm{E}(b)]$. For instance $\mathrm{E}((X_0,X_0)) = 0$ and $\mathrm{E}((X_1,(X_1,X_0))) = W_1$. Through this mapping, $\mathrm{Br}(X)$ spans $\mathcal{L}(X)$.*

DEFINITION 2.2 (Hall set). *A* Hall set *is a subset $\mathcal{B}$ of $\mathrm{Br}(X)$, containing X, with a total order $<$ such that*

- $(\mathfrak{H}1)$*: for $b = (b_1, b_2) \in \mathrm{Br}(X)$, $b \in \mathcal{B}$ iff $b_1, b_2 \in \mathcal{B}$, $b_1 < b_2$ and, either $b_2 \in X$, or $b_2 = (b_2', b_2'')$ with $b_2' \leq b_1$,*
- $(\mathfrak{H}2)$*: for every $b_1, b_2 \in \mathcal{B}$ such that $(b_1, b_2) \in \mathcal{B}$, one has $b_1 < (b_1, b_2)$.*

Then $\mathrm{E}(\mathcal{B})$ is a basis of $\mathcal{L}(X)$ (see [71, Theorem 1.2]).

To visualize this algorithm, we build some elements of a Hall set $\mathcal{B}^\star$ for which $X_1 < X_0$. By $(\mathfrak{H}1)$, $(X_1, X_0) \in \mathcal{B}^\star$, which is not the case for (X_0, X_1): this illustrates how $(\mathfrak{H}1)$ selects formal brackets in $\mathrm{Br}(X)$ whose evaluations in $\mathcal{L}(X)$ are linearly independent (since, in $\mathcal{L}(X)$ we have $[X_0, X_1] = -[X_1, X_0]$). Thus $\mathcal{B}^\star$ contains the 3 elements X_0, X_1 and (X_1, X_0) ordered as follows: $X_1 < X_0$ and $X_1 < (X_1, X_0)$ by $(\mathfrak{H}2)$. To build all the elements of $\mathcal{B}^\star$ with length 3, we need to order (X_1, X_0) and X_0. Let us assume that X_0 is the maximal element i.e. $X_1 < (X_1, X_0) < X_0$. Then, by $(\mathfrak{H}1)$, $(X_1,(X_1,X_0))$ and $((X_1,X_0),X_0)$ are the only elements of $\mathcal{B}^\star$ of length 3. By $(\mathfrak{H}2)$, we have $X_1 < (X_1,X_0) < ((X_1,X_0),X_0)$ and $X_1 < (X_1,(X_1,X_0))$. To built all the elements of $\mathcal{B}^\star$ of length 4, we need an order between $((X_1,X_0),X_0)$ and X_0 and an order between $(X_1,(X_1,X_0))$ and X_0. Let us assume that X_0 is the maximal element i.e.

$$X_1 < (X_1,X_0) < ((X_1,X_0),X_0) < X_0 \qquad \text{and} \qquad X_1 < (X_1,(X_1,X_0)) < X_0.$$

Then, by $(\mathfrak{H}1)$, $(((X_1,X_0),X_0),X_0)$ and $(X_1,(X_1,(X_1,X_0)))$ and $((X_1,(X_1,X_0)),X_0)$ are elements of $\mathcal{B}^\star$ but $(X_1,((X_1,X_0),X_0))$ is not. Indeed, by the Jacobi identity, the following equality holds in $\mathcal{L}(X)$

$$[X_1,[[X_1,X_0],X_0]] = [[X_1,[X_1,X_0]],X_0] + [[X_1,X_0],[X_1,X_0]] = [[X_1,[X_1,X_0]],X_0]. \tag{2.1}$$

By iterating, we see that, if $\mathcal{B}$ is a Hall set, whose maximal element is X_0, then its elements involving one or two X_1's are the ones listed in (1.2); more exactly, these are the formal brackets of $\mathrm{Br}(X)$, obtained by replacing the brackets $[\cdot,\cdot]$ by parentheses $(\cdot,\cdot)$ in these formulas. In particular, $\{M_\nu; \nu \in \mathbb{N}\}$ is a basis of $S_1(X)$ and $\{W_{j,\nu}; j \in \mathbb{N}^*, \nu \in \mathbb{N}\}$ is a basis of $S_2(X)$.

2.2 The formal differential equation. In this work, a fundamental tool is the formal differential equation (2.2). We first introduce notations to define its solution.

Definition 2.3 (Formal series). *For $n \in \mathbb{N}$, let $\mathcal{A}_n(X)$ be the subspace of $\mathcal{A}(X)$ spanned by monomials of degree n over X. The algebra of formal series generated by $\mathcal{A}(X)$ is $\widehat{\mathcal{A}}(X) = \{ \sum_{n\in\mathbb{N}} a_n; \forall n \in \mathbb{N}, a_n \in \mathcal{A}_n(X)\}$.*

For $a \in \widehat{\mathcal{A}}(X)$ with $a_0 = 0$ then $\exp(a) = \sum_{k\in\mathbb{N}} \frac{a^k}{k!}$ and $\log(1-a) = -\sum_{k\in\mathbb{N}^*} \frac{a^k}{k}$ are well-defined in $\widehat{\mathcal{A}}(X)$.

Definition 2.4 (Solution to (2.2)). *For $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$, the solution of the formal differential equation*

$$\begin{cases} \dot{x}(t) = x(t)(X_0 + u(t)X_1), \\ x(0) = 1, \end{cases} \tag{2.2}$$

is the function $x : \mathbb{R}_+ \to \widehat{\mathcal{A}}(X)$, whose components $x_n : \mathbb{R}_+ \to \mathcal{A}_n(X)$ satisfy

$$x_0(t) = 1, \qquad \forall n \in \mathbb{N}^*, \quad x_n(t) = \int_0^t x_{n-1}(s)(X_0 + u(s)X_1)\,\mathrm{d}s. \tag{2.3}$$

Although equation (2.2) is linear, a classical linearization principle allows to recast the study of nonlinear ODEs such as (1.1) driven by vector fields to this setting (see [3, Section 4.1]). A key benefit of this abstract formulation is that it is now independent on f_0 and f_1. By iterating the integral formula (2.3), one obtains the *Chen series*, introduced in [19, 20] and used to prove many necessary conditions of STLC in [67, 68, 47].

2.2.1 A Magnus-type representation formula. In this section, we recall a Magnus-type representation formula, for the solutions of the formal differential equation (2.2), introduced by Frédéric Marbach, Jérémy Le Borgne and the author in [3, Statements 41 and 44]. This formula states that the formal series $\log(\exp(-tX_0)x(t))$ is actually a formal Lie series (thus it can be expanded in bases of $\mathcal{L}(X)$).

Definition 2.5. $\widehat{\mathcal{L}}(X) = \{ a \in \widehat{\mathcal{A}}(X)\,;\, \forall n \in \mathbb{N}\,,\, a_n \in \mathcal{L}(X) \}$ *is the Lie algebra of formal Lie series.*

Proposition 2.6. *For $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$, the solution of the formal differential equation* (2.2) *satisfies*

$$x(t) = \exp(tX_0)\exp(\mathcal{Z}(t,X,u)) \tag{2.4}$$

where $\mathcal{Z}(t,X,u) \in \widehat{\mathcal{L}}(X)$. If $\mathcal{B}$ is a Hall set, there exists a unique family $(\eta_b)_{b\in\mathcal{B}}$ of functionals $\mathbb{R}_+ \times L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}) \to \mathbb{R}$, called coordinates of the pseudo-first kind *associated with $\mathcal{B}$, such that $\mathcal{Z}(t,X,u) = \sum_{b\in\mathcal{B}} \eta_b(t,u)b$.*

Historically, the "coordinates of the first kind" associated with $\mathcal{B}$ are those of $\log(x(t))$, which justifies our terminology. The factorization of $\exp(tX_0)$ ensures a better coordinates decay (see [3]).

2.2.2 Sussmann's infinite product. In this section, we present an expansion for the solution to the formal differential equation (2.2), as a product of exponentials of the elements of a Hall set, multiplied by coefficients that have explicit and simple expressions, given in the following definition.

Definition 2.7. *The* coordinates of the second kind *associated with a Hall set $\mathcal{B}$ is the family $(\xi_b)_{b\in\mathcal{B}}$ of functionals $\mathbb{R}_+ \times L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}) \to \mathbb{R}$ defined by induction by: $\xi_{X_0}(t,u) = t$, $\xi_{X_1}(t,u) = u_1(t)$ (see Definition* 1.4*) and for $b = \mathrm{ad}_{b_1}^m(b_2) \in \mathcal{B} \setminus X$ with $m \in \mathbb{N}^*$ maximal then*

$$\xi_b(t,u) = \frac{1}{m!}\int_0^t \xi_{b_1}(s,u)^m \partial_s \xi_{b_2}(s,u)\,\mathrm{d}s.$$

For instance, if $\mathcal{B}$ is a Hall set that contains the elements of (1.2) (see subsection 2.1) then

$$\forall (j,\nu) \in \mathbb{N}^* \times \mathbb{N}, \qquad \xi_{M_\nu}(t,u) = u_{\nu+1}(t), \qquad \xi_{W_{j,\nu}}(t,u) = \int_0^T \frac{(T-t)^\nu}{\nu!}\frac{u_j(t)^2}{2}\,\mathrm{d}t. \tag{2.5}$$

The following statement is an extension to any Hall set of Sussmann's infinite product on length-compatible Hall sets [69], suggested by Matthias Kawski in [48] and proved in [3, Section 2.5].

THEOREM 2.8. *Let $\mathcal{B}$ be a Hall set. For $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$, the solution of* (2.2) *satisfies*

$$x(t) = \overleftarrow{\prod_{b\in\mathcal{B}}} e^{\xi_b(t,u)\mathrm{E}(b)} \tag{2.6}$$

(in this non-commutative infinite product, the terms are positioned according to the Hall set order $<$, the arrow indicates the increasing order, see [3, Section 2.5] for details).

2.2.3 Magnus expansion expressed with coordinates of the second kind. Using a Hall set $\mathcal{B}$ that contains the elements of (1.2), we deduce from (2.4), (2.5), (2.6) and the Campbell-Baker-Hausdorff formula that the homogeneous components $\mathcal{Z}_j(t,X,u) \in S_j(X)$ of $\mathcal{Z}(t,X,u)$ satisfy

$$\mathcal{Z}_1(t,X,u) = \sum_{j=0}^{\infty} u_{j+1}(t) M_j, \qquad \mathcal{Z}_2(t,X,u) = \sum_{\substack{j\in\mathbb{N}^*\\ \nu\in\mathbb{N}}} \xi_{W_{j,\nu}}(t,u) W_{j,\nu} + \frac{1}{2}\sum_{0\le j<j'} u_{j'+1}(t)u_{j+1}(t)[M_{j'},M_j]. \tag{2.7}$$

By expanding the $[M_{j'},M_j]$ on the basis $\{W_{\ell,\nu};\ell\in\mathbb{N}^*,\nu\in\mathbb{N}\}$, one could compute explicitly the $\eta_{W_{\ell,\nu}}(t,u)$. More generally, for any $b\in\mathcal{B}$, $(\eta_b-\xi_b)(t,u)$ is a linear combination of products of the form $\xi_{b_1}\cdots\xi_{b_q}(t,u)$ where $q\ge 2$ and the b_j are elements of $\mathcal{B}$ whose length is strictly less than that of b.

2.3 An approximate representation formula for affine systems. We present here an approximate representation formula for the solutions of the ODE (1.1) introduced in [3, Proposition 161].

THEOREM 2.9. *Let $M\in\mathbb{N}^*$. The following estimate holds for the solutions of* (1.1) *as $(t,\|u\|_{W^{-1,\infty}})\to 0$,*

$$x(t;u) = \mathcal{Z}_{[\![1,M]\!]}(t,f,u)(0) + O\left(\|u\|^{M+1}_{W^{-1,M+1}} + |x(t;u)|^{1+\frac{1}{M}}\right) \text{ where } \mathcal{Z}_{[\![1,M]\!]}(t,f,u) = \sum_{b\in\mathcal{B}_{[\![1,M]\!]}} \eta_b(t,u) f_b, \tag{2.8}$$

$\mathcal{B}$ is any Hall set, the η_b are defined in Proposition 2.6 *and $\mathcal{B}_{[\![1,M]\!]} = \{b\in\mathcal{B};\mathrm{E}(b)\in S_{[\![1,M]\!]}(X)\}$.*

The vector field $\mathcal{Z}_{[\![1,M]\!]}(t,f,u)$ is constructed as follows: starting from the formal Lie series $\mathcal{Z}(t,X,u)$ defined in Proposition 2.6, first, we remove the monomials of degree $>M$ in the indeterminate X_1, then we replace the indeterminate X_j by the associated vector field f_j for $j=0,1$. The infinite sum in (2.8) converges in the sense of analytic functions, thanks to a growth estimate of the structure constants of Hall sets proved in [6].

A remarkable feature of (2.8) is that, when computing the state $x(t;u)$ as almost $\mathcal{Z}_{[\![1,M]\!]}(t,f,u)(0)$, each term $\eta_b(t,u)f_b(0)$ of the series decouples:

- on one side, a scalar $\eta_b(t,u)\in\mathbb{R}$, which carries the time and control dependency, but in a universal (i.e. system-independent) way,
- on the other side, a vector $f_b(0)\in\mathbb{R}^d$, which encodes the algebraic and geometric dependency on the system, in a coordinate-independent way, as only Lie brackets of f_0, f_1 are involved.

The formula (2.8) is specially well designed for proving controllability results. For instance, if for some $\mathfrak{b}\in\mathcal{B}$, the coordinate $\eta_{\mathfrak{b}}(t,u)$ takes positive values, one may expect the bracket $\mathfrak{b}$ to generate a drift in the dynamics along $f_{\mathfrak{b}}(0)$, when $f_{\mathfrak{b}}(0)\notin \mathrm{Span}\{f_b(0);b\in\mathcal{B}_{[\![1,M]\!]}\setminus\{\mathfrak{b}\}\}$ and when the remainder can be neglected.

A technical drawback of (2.8) is that, unlike the coordinates of the second kind ξ_b, the functionals η_b are not given by nice explicit expressions. Our insight to deal with this difficulty is to rely on the heuristic that, somehow $\eta_b\approx\xi_b$. Precisely, we will use the expansions (2.7) and prove that the contribution of the last sum is negligible.

3 Quadratic obstructions for scalar-input ODEs.

3.1 An interpretation of obstructions to STLC as drifts. In this section, we sketch the proof of Theorem 1.12 in the particular case $m=0$ and $k=2$ (the case $k=1$ is known since Theorem 1.11). The contraposition is a corollary of the following systematic drift estimate, proved in [9, Theorem 6.1].

THEOREM 3.1. *We assume*

$$\forall\nu\in\mathbb{N}, \qquad f_{W_{1,\nu}}(0)\in S_1(f)(0), \qquad f_{W_2}(0)\notin S_{\{1,3\}}(f)(0). \tag{3.1}$$

Let $\mathbb{P}:\mathbb{R}^d\to\mathbb{R}$ giving a component onto $f_{W_2}(0)$, parallel to $S_{\{1,3\}}(f)(0)$. Then, the following estimate holds for the solutions of (1.1)*, as $(T,\|u\|_{W^{-1,\infty}})\to 0$,*

$$\mathbb{P}x(T;u) = \frac{1}{2}\|u_2\|^2_{L^2} + O\left((T+\|u\|_{L^\infty})\|u_2\|^2_{L^2} + |x(T;u)|^{\frac{4}{3}}\right). \tag{3.2}$$

When $(T+\|u\|_{L^\infty})$ is small enough, the estimate (3.2) prevents $x(T;u)$ from reaching targets of the form $x^\star = -\epsilon f_{W_2}(0)$ with $0<\epsilon \ll 1$, because this would entail $-\epsilon = \mathbb{P}x^\star \geq -C|x^\star|^{\frac{4}{3}} = -C'\epsilon^{\frac{4}{3}}$, which fails for ϵ small enough. Thus, estimate (3.2) comes into contradiction with L^∞-STLC.

However this estimate contains more information than one impossible movement. This estimate means that the state $x(T;u)$ drifts systematically (i.e. for any control u) along the direction of $+f_{W_2}(0)$. Moreover the amplitude of this motion is approximately $\frac{1}{2}\|u_2\|^2_{L^2}$. The term $|x(T;u)|^{\frac{4}{3}}$ is necessary : it encodes that the drift can hold with respect to a manifold as in example Σ_1.

To prove Theorem 1.12 for an arbitrary $k \in \mathbb{N}^*$, still by contraposition, one would assume that k is the minimal integer for which $f_{W_k}(0) \notin S_{[\![1,\pi(k,m)]\!]\setminus\{2\}}(f)(0)$ and then prove a drift along the direction $+f_{W_k}(0)$ with amplitude $\frac{1}{2}\|u_k\|^2_{L^2}$. The proof would be similar to the particular case $k=2$ presented here.

Now, we present a unified approach to prove obstructions to STLC introduced in [9]. This approach proves Theorem 3.1, but also Theorem 1.12. It will be adapted to PDEs in section 4.

3.2 Sketch of the proof of the drift estimate. Using (2.8) with $M=3$, we get, as $(T,\|u\|_{W^{-1,\infty}}) \to 0$,

$$\mathbb{P}x(T;u) = \mathbb{P}\mathcal{Z}_2(T,f,u)(0) + O\left(\|u_1\|^4_{L^4} + |x(T;u)|^{\frac{4}{3}}\right).$$

We deduce from (2.7) and (3.1) that

$$\mathbb{P}\mathcal{Z}_2(T,f,u)(0) = \sum_{j\geq 2, \nu\geq 0} \xi_{W_{j,\nu}}(T,u)\mathbb{P}f_{W_{j,\nu}}(0) + \frac{1}{2}\sum_{0\leq j<j'} u_{j'+1}(T)u_{j+1}(T)\mathbb{P}f_{[M_{j'},M_j]}(0). \tag{3.3}$$

By (2.5) and analyticity of f_0, f_1, there exists $C>0$ such that, for every $j\geq 2$, $\nu \in \mathbb{N}$, $b \in \mathrm{Br}(X)$,

$$|u_{j+1}(T)| \leq \frac{(CT)^\ell}{\ell!} T^{-3}\sqrt{T}\|u_2\|_{L^2} \quad \text{where } \ell = |M_j| \geq 3, \tag{3.4}$$

$$|\xi_{W_{j,\nu}}(T,u)| \leq \frac{(CT)^\ell}{\ell!} T^{-5}\|u_2\|^2_{L^2} \quad \text{where } \ell = |W_{j,\nu}| \geq 5, \tag{3.5}$$

$$|f_b(0)| \leq C^{|b|}|b|! \tag{3.6}$$

(see [9, Proposition 3.10] and [3, Lemma 80]). Using also $|u_{j+1}(T)| < |(u_1,u_2)(T)|$ for $j<2$, we deduce

$$\mathbb{P}x(T;u) = \frac{1}{2}\|u_2\|^2_{L^2} + O\left(T\|u_2\|^2_{L^2} + |(u_1,u_2)(T)|^2 + \|u_1\|^4_{L^4} + |x(T;u)|^{\frac{4}{3}}\right). \tag{3.7}$$

To obtain the drift estimate (3.2), we still need to absorb $|(u_1,u_2)(T)|^2$ and $\|u_1\|^4_{L^4}$, which are not negligible compared to $\|u_2\|^2_{L^2}$. To tackle $|(u_1,u_2)(T)|^2$, we introduce new tools, in the following statement.

Proposition 3.2. *The assumption* (3.1) *implies*
- *vectorial relations: the vectors $f_{M_0}(0)$, $f_{M_1}(0)$ are linearly independent,*
- *closed loop estimates: the solutions of* (1.1) *satisfy, when $(T,\|u\|_{W^{-1,\infty}}) \to 0$,*

$$|(u_1,u_2)(T)| = O\left(\sqrt{T}\|u_2\|_{L^2} + \|u_1\|^2_{L^2} + |x(T;u)|\right). \tag{3.8}$$

The essence of closed-loop estimates can be spotted implicitly in previous literature. For instance [67, Lemma 3.2] is a slightly less general version of the closed loop estimate $|u_1(t)| = O(|x(t;u)| + \|u_1\|_{L^1})$ valid when $f_1(0) \neq 0$ (see [9, Lemma 5.4]). In [47, p. 148] a static-state feedback is used to guarantee that $u_1(T) = 0$.

Proof. If $f_{M_0}(0)$ and $f_{M_1}(0)$ are linearly dependent, then, there exists $a \in \mathbb{R}$ such that $f_B(0) = 0$ where $B = M_1 + aM_0$. Since $f_B(0) = f_0(0) = 0$, then $f_{B'}(0) = 0$ where $B' = [B,[B,X_0]] = [M_1 + aM_0, M_2 + aM_1] = W_2 + aW_{1,1} - a^2 W_1$ (see (2.1)) which contradicts (3.1). Thus $f_{M_0}(0)$, $f_{M_1}(0)$ are linearly independent. We deduce from (2.8) with $M=1$, (3.4) and (3.6) the following estimate that ends the proof of Proposition 3.2

$$x(T;u) = u_1(T)f_{M_0}(0) + u_2(T)f_{M_1}(0) + O\left(\sqrt{T}\|u_2\|_{L^2} + \|u_1\|^2_{L^2} + |x(T;u)|^2\right).$$

In (3.7), to deal with the term $\|u_1\|_{L^4}^4$, we use the Gagliardo-Nirenberg inequality (see [36, 61]): there exists $C > 0$ such that, for every $T > 0$ and $u \in L^1((0,T);\mathbb{R})$

$$\|u_1\|_{L^4}^4 \leq C\left(\|u\|_{L^\infty}^2\|u_2\|_{L^2}^2 + T^{-5}\|u_2\|_{L^2}^4\right) \leq 2C\|u\|_{L^\infty}^2\|u_2\|_{L^2}^2. \tag{3.9}$$

Gathering (3.7), (3.8) and (3.9), we obtain the drift estimate (3.2).

Remark 3.3. Theorem 1.12 is written for analytic vector fields f_0, f_1, which allows to work with convergent series (for instance in (3.3)). However, the drift estimate is sufficiently robust to absorb an approximation scheme for non-analytic vector fields. In particular Theorem 1.12 holds for $\mathcal{C}^\infty$ vector fields f_0, f_1. Furthermore, even a finite regularity setting is sufficient for the drift estimate (for instance $f_0 \in \mathcal{C}^4$, $f_1 \in \mathcal{C}^3$ for Theorem 3.1), see [9, section 11].

Finally, we proved Theorem 1.12 by combining
- the approximate representation formula (2.8),
- algebraic arguments, to compute key terms in $\mathcal{Z}_{[\![1,M]\!]}(t,f,u)(0)$,
- geometric arguments: the vectorial relations and closed-loop estimates,
- analytic arguments: the Gagliardo-Nirenberg interpolation inequality, to absorb a high order remainder, by the quadratic drift, when the control is small.

In conclusion, quadratic obstructions to controllability are linked with positive coordinates of the second kind and interpolation inequalities, dictated by the relations between Lie brackets of the vector fields defining the system.

3.3 Extensions to multi-input, higher order, or other areas. In this article, we restrict the study to systems with scalar input and quadratic necessary conditions, but the method adapts well to multi-input systems (see [37] by Théo Gherdaoui) and higher-order terms (see [9, Theorems 1.10 and 1.14]). The Magnus formula (2.8) is also a convenient starting point to prove sufficient conditions for STLC (see [39]).

The formalism of Lie series is also relevant in the order theory of splitting methods, in numerical analysis. By combining it with control arguments, Frédéric Marbach, Adrien Laurent and the author solve open problems in this field [4]. The positivity of the coordinates of the second kind associated with the W_j is crucial (see (2.5)) .

4 Quadratic terms for scalar-input PDEs. As for ODEs, one can deduce the local controllability of a nonlinear PDE from the controllability of its linearization (see, for instance [51, 65, 5, 15]) but the fixed point argument must be adapted for each PDE. When the linearized system is not controllable, a powerful approach is Coron's return method, in which one chooses wisely another trajectory around which the nonlinear PDE is linearized (see for instance [23, 40, 24]); the construction of this trajectory may be complicated [28].

Lie brackets are seldom used for PDEs because they can be hard to define, and it is less clear how to manipulate them than in the finite-dimensional setting. However several particular cases were studied. The behavior found in finite dimension still appears: along a direction lost at the linear order, a quadratic term can create a signed drift, quantified by a negative integer Sobolev norm of the control (see subsection 4.1). But new quadratic obstructions also appear, quantified by other norms and are more robust to perturbations (see subsection 4.3). More strikingly, it is also possible for a quadratic term to recover controllability (see subsection 4.2).

DEFINITION 4.1. *The space $L^2((0,1);\mathbb{C})$ is endowed with the scalar product $\langle \phi, \varphi\rangle = \int_0^1 \phi\overline{\varphi}$. For $m \in \mathbb{N}$, $H^m_N((0,1);\mathbb{C}) = \{\phi \in H^m((0,1);\mathbb{C}); \phi^{(j)}(0) = \phi^{(j)}(1) = 0$ for j odd $< m\}$ is endowed with the norm $\|\phi\|_{H^m_N} = \left(\sum_{j=0}^\infty |\langle j\rangle^m \langle \phi, \varphi_j\rangle|^2\right)^{\frac{1}{2}}$, where $\langle j\rangle = (1+j^2)^{\frac{1}{2}}$ is the Japanese bracket. The dual space of $H^1_N((0,1);\mathbb{C})$ is called $H^{-1}_N((0,1);\mathbb{C})$ and endowed with the norm $\|.\|_{H^m_N}$ for $m = -1$.*

DEFINITION 4.2. *The operator defined by $D(A) = H^2_N((0;1);\mathbb{C})$ and $A\phi = -\phi''$, has eigenvalues $(\lambda_j = (j\pi)^2)_{j\in\mathbb{N}}$ and eigenvectors $\varphi_0 = 1$ and $\varphi_j(x) = \sqrt{2}\cos(j\pi x)$ for $j \in \mathbb{N}^*$.*

4.1 Drifts quantified by negative integer Sobolev norms. We consider the Schrödinger equation

$$\begin{cases} i\partial_t\psi(t,x) = -\partial_x^2\psi(t,x) - u(t)\mu(x)\psi(t,x), & x \in (0,1), \\ \partial_x\psi(t,0) = \partial_x\psi(t,1) = 0. \end{cases} \tag{4.1}$$

where $\mu : (0,1) \to \mathbb{R}$. The control is $u : \mathbb{R}_+ \to \mathbb{R}$ and the state $\psi(t,\cdot)$ lives in the sphere $\mathbb{S} = \left\{\phi \in L^2((0,1);\mathbb{C}); \|\phi\|_{L^2} = 1\right\}$. This system models a non-relativistic charged quantum particle, in an infinite square potential well and a space-uniform electric field with time varying amplitude $u(t)$. The function μ describes the action of this field on the particle. From the modelization point of view, the most common case is $\mu(x) = x - \frac{1}{2}$ (then the electric potential $V(t,x) = -u(t)\mu(x)$ solves $-\Delta V = u$). This particular function μ also occurs after a change of variables for a moving potential well, where $u(t)$ is the acceleration of the box (see [66]). Allowing for a more general space-dependent $\mu(x)$ makes sense (see e.g. [31, eq. (5)]). The Neumann boundary conditions are perhaps less intuitive than Dirichlet ones, from a modeling perspective, but they offer an elegant system for developing a general methodology (see Remark 4.5 for adaptations to the Dirichlet case).

System (4.1) is well-posed in the following sense: if $\mu \in H^2((0,1);\mathbb{R})$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$, then there exists a unique solution $\psi \in \mathcal{C}^0(\mathbb{R}_+; H^2_N)$ of (4.1) such that $\psi(0,\cdot) = \varphi_0$, denoted $\psi(t;u)$. If μ' vanishes at the boundary, this well posedness results from standard arguments. Otherwise, it requires a smoothing effect, because the multiplication by μ does not preserve $H^2_N((0,1);\mathbb{C})$ (see [10, Proposition 1.1]).

We study the small-time local controllability of the nonlinear system (4.1) around the equilibrium φ_0, depending on the function μ. In a power series expansion of the state with respect to the control $\psi = \varphi_0 + \psi_1 + \psi_2 + \dots$ then ψ_1 and ψ_2 solve the linearized and quadratic order systems

$$\begin{cases} i\partial_t \psi_1 = -\partial_x^2 \psi_1 - u(t)\mu(x), \\ \partial_x \psi_1(t,0) = \partial_x \psi_1(t,1) = 0, \end{cases} \qquad \begin{cases} i\partial_t \psi_2 = -\partial_x^2 \psi_2 - u(t)\mu(x)\psi_1, \\ \partial_x \psi_2(t,0) = \partial_x \psi_2(t,1) = 0, \end{cases} \tag{4.2}$$

with initial conditions $\psi_1(0,\cdot) = \psi_2(0,\cdot) = 0$. By the Duhamel formula,

$$\psi_1(t,x) = i \sum_{j\in\mathbb{N}} \langle \mu, \varphi_j \rangle \int_0^t u(s) e^{-i\lambda_j(t-s)} \,\mathrm{d}s\, \varphi_j(x) \tag{4.3}$$

where, for $j \in \mathbb{N}^*$,

$$\langle \mu, \varphi_j \rangle = \sqrt{2} \frac{(-1)^j \mu'(1) - \mu'(0)}{(j\pi)^2} - \frac{\langle \mu'', \varphi_j \rangle}{(j\pi)^2}. \tag{4.4}$$

In particular, if there exists $c > 0$ such that, for every $j \in \mathbb{N}$, $|\langle \mu, \varphi_j \rangle| \geq c\langle j \rangle^{-2}$ (which is generic) then the linearized system is small-time controllable in $H^2_N((0,1);\mathbb{C})$ with controls $u \in L^2((0,T);\mathbb{R})$, thus so does the nonlinear system (4.1) locally around φ_0, see [10, Proposition 1.3].

To ensure that the linearized system is not controllable, from now on, we assume that $\langle \mu, \varphi_J \rangle = 0$ for some $J \in \mathbb{N}$. Then, for the state of the nonlinear system (4.1), the contribution of the control u along the direction φ_J is at least quadratic: $\langle \psi(T;u) - \varphi_0, \varphi_J \rangle = O(\|u\|^2_{L^2})$.

Under appropriate assumptions on μ, we prove a quadratic obstruction, which is the analogue of Theorem 1.12 with $m = 2k-3$ so that $S_{[\![1,\pi(k,m)]\!]\setminus\{2\}}(f)(0) = S_1(f)(0)$ (compensation of the quadratic bracket W_k by linear brackets). We state it directly in the form of a systematic drift estimate (as in Theorem 3.1). Heuristically, we consider $f_0(\psi) = -iA\psi$ and $f_1(\psi) = i\mu\psi$, we fix $k \in \mathbb{N}^*$ as the minimal integer for which $\langle f_{W_k}(\varphi_0), \varphi_J \rangle \neq 0$ and we prove $\langle \psi(T;u) - \varphi_0, \varphi_J \rangle \approx \frac{1}{2}\|u_k\|^2_{L^2} \langle f_{W_k}(0), \varphi_J \rangle$.

In this sentence, it is not clear whether the iterated Lie brackets are well defined (because f_0 can only be applied to functions in $D(A) = H^2_N$). In the particular case $\mu \in C^\infty_c((0,1);\mathbb{R})$, there is no ambiguity: $f_{W_k}(\varphi_0)$ is well defined (see Lemma 4.4) and $\frac{1}{2}\langle f_{W_k}(\varphi_0), \varphi_J \rangle = -ia^k_J$, where a^k_J is the real number defined in (4.5). This number is well defined under the weaker assumption $\mu \in H^{2k+1}_N((0,1);\mathbb{R})$ because integrations by parts prove $\lambda_j^k \langle \mu\varphi_\ell, \varphi_j \rangle = \langle A^k(\mu\varphi_\ell), \varphi_j \rangle = o(1/j)$. This legitimates the following statement (see [10, Theorem A.1]).

THEOREM 4.3. *Let $J \in \mathbb{N}$, $k \in \mathbb{N}^*$, $\mu \in H^{2k+1}_N((0,1);\mathbb{R})$ such that $\langle \mu, \varphi_J \rangle = 0$. We define*

$$\forall n \in [\![1,k]\!], \qquad a^n_J = \sum_{j\in\mathbb{N}} c_j \lambda_j^{n-1} (\lambda_j - \lambda_J)^{n-1} (\lambda_j - \lambda_J/2) \qquad \text{where} \qquad c_j = \langle \mu, \varphi_j \rangle \langle \varphi_j, \mu\varphi_J \rangle \tag{4.5}$$

We assume k is the minimal integer for which $a^k_J \neq 0$. Then, there exist $C > 0$, $\nu \in \mathbb{R}$ such that, for every $T \in (0,1]$ and $u \in L^2((0,T);\mathbb{R})$ with $\|u\|_{L^2} \leq 1$,

$$\left| \langle \psi(T;u) - \varphi_0, \varphi_J \rangle + ia^1_J \|u_1\|^2_{L^2} \right| \leq C \left(T\|u_1\|^2_{L^2} + \|\psi(T;u) - \varphi_0\|^2_{L^2} \right) \text{ if } k = 1, \tag{4.6}$$

$$\left| \langle \psi(T;u) - \varphi_0, \varphi_J \rangle + ia^k_J \|u_k\|^2_{L^2} \right| \leq C \big((T + T^\nu \|u\|_{H^{2k-3}}) \|u_k\|^2_{L^2} + \|\psi(T;u) - \varphi_0\|^2_{L^2} \big) \text{ if } k \geq 2. \tag{4.7}$$

When T is small enough (precisely $CT < |a_J^1|$), estimate (4.6) prevents from reaching targets of the form $\psi^\star = \sqrt{1-\epsilon^2} + i\epsilon\,\mathrm{sign}(a_J^1)\varphi_J$ with $0 < \epsilon \ll 1$ because this would entail $-\epsilon \geq -C\left((\sqrt{1-\epsilon^2}-1)^2+\epsilon^2\right) \sim -C\epsilon^2$. Thus estimate (4.6) denies small-time local controllability with controls bounded in L^2.

Similarly, when T and u are small enough (precisely $C(T+T^\nu\|u\|_{H^{2k-3}}) < |a_J^k|$), estimate (4.7) prevents from reaching targets of the form $\psi^\star = \sqrt{1-\epsilon^2} + i\epsilon\,\mathrm{sign}(a_J^k)\varphi_J$ with $0 < \epsilon \ll 1$. This denies the H^{2k-3}-STLC.

Note that if $\mu \in H^2((0,1);\mathbb{R}) \setminus \{0\}$ and $J = 0$ then $k = 1$ because $a_0^1 = \sum |\langle \mu, \varphi_j\rangle|^2\lambda_j > 0$.

Lemma 4.4. *If $\mu \in C_c^\infty((0,1);\mathbb{R})$, then, for every $(J,n) \in \mathbb{N}\times\mathbb{N}^*$, $2a_J^n = (-1)^{n-1}\langle[\mathrm{ad}_A^{n-1}(\mu), \mathrm{ad}_A^n(\mu)]\varphi_0, \varphi_J\rangle$.*

Proof. The assumption $\mu \in C_c^\infty((0,1);\mathbb{R})$ guarantees that the iterated Lie brackets are well defined: each time A is applied to a function, this function belongs to $D(A) = H_N^2((0,1);\mathbb{C})$. For every $\ell \in \mathbb{N}$ and $m \in \mathbb{N}$ we have

$$\mathrm{ad}_A^m(\mu)\varphi_\ell = \mathrm{ad}_{A-\lambda_\ell}^m(\mu)\varphi_\ell = \sum_{q=0}^{m}(-1)^q\binom{m}{q}(A-\lambda_\ell)^q\mu(A-\lambda_\ell)^{m-q}\varphi_\ell = (-1)^m(A-\lambda_\ell)^m(\mu\varphi_\ell)$$

thus for every $j \in \mathbb{N}$,

$$(\lambda_j-\lambda_\ell)^m\langle\varphi_j, \mu\varphi_\ell\rangle = \langle(A-\lambda_\ell)^m\varphi_j, \mu\varphi_\ell\rangle = \langle\varphi_j, (A-\lambda_\ell)^m(\mu\varphi_\ell)\rangle = (-1)^m\langle\varphi_j, \mathrm{ad}_A^m(\mu)\varphi_\ell\rangle$$

where the second equality results from integrations by parts and the boundary terms vanish because $\mu \in C_c^\infty((0,1);\mathbb{R})$. Therefore, by decomposing $2(\lambda_j - \lambda_J/2) = \lambda_j + (\lambda_j - \lambda_J)$ and using Bessel's equality, we obtain

$$2a_J^n = -\langle\mathrm{ad}_A^n(\mu)\varphi_0, \mathrm{ad}_A^{n-1}(\mu)\varphi_J\rangle - \langle\mathrm{ad}_A^{n-1}(\mu)\varphi_0, \mathrm{ad}_A^n(\mu)\varphi_J\rangle$$

and we conclude thanks to $\mathrm{ad}_A^n(\mu)^* = (-1)^n\,\mathrm{ad}_A^n(\mu)$. ∎

Although the Magnus formula is not established in this infinite dimensional setting, we can extract its main terms from the power series expansion of the solution, and prove the drift estimates with the same key steps as in finite dimension. To lighten the computations, we sketch the proof only in the cases $J = 0$ and $k = 1, 2$.

Sketch of the proof of Theorem 4.3 *when* $(J,k) = (0,1)$. Let $\mu \in H_N^3((0,1);\mathbb{R})$ such that $\langle\mu, \varphi_0\rangle = 0$ and $a_0^1 \neq 0$. To simplify notations, we denote by O estimates that hold uniformly with respect to $T \in (0,1]$ and $u \in L^2((0,T);\mathbb{R})$ such that $\|u\|_{L^2} \leq 1$ (as in Theorem 4.3), for instance

$$\|u_1\|_{L^2} = O(T), \qquad u_1(T) = O(\sqrt{T}). \tag{4.8}$$

Step 1: Coercivity of the quadratic form. We deduce from (4.2) and (4.3) that

$$\langle\psi_2(T), \varphi_0\rangle = i\int_0^T u(t)\langle\mu\psi_1(t), \varphi_0\rangle\,\mathrm{d}t = -\int_0^T u(t)\int_0^t u(s)K(t-s)\,\mathrm{d}s\,\mathrm{d}t$$

where $K(\sigma) = \sum_{j\in\mathbb{N}} c_j e^{-i\lambda_j\sigma}$ for $\sigma \geq 0$. The assumption $\mu \in H_N^3((0,1);\mathbb{R})$ implies $c_j = o(1/\lambda_j^3)$ (see (4.4)) thus $K \in C^2([0,\infty);\mathbb{C})$. To force the presence of the primitive u_1 of u, we perform integrations by parts

$$\langle\psi_2(T), \varphi_0\rangle = -\frac{K(0)}{2}u_1(T)^2 - u_1(T)\int_0^T u_1(s)K'(T-s)\,\mathrm{d}s + \int_0^T u_1(t)\left(u_1(t)K'(0) + \int_0^t u_1(s)K''(t-s)\,\mathrm{d}s\right)\mathrm{d}t.$$

Using the relation $K'(0) = -ia_0^1$ and the Cauchy-Schwarz inequality, we deduce

$$\langle\psi_2(T), \varphi_0\rangle = -ia_0^1\|u_1\|_{L^2}^2 + O\left(u_1(T)^2 + T\|u_1\|_{L^2}^2\right). \tag{4.9}$$

Step 2: Quadratic and cubic estimates. Classical arguments prove

$$\|(\psi-\varphi_0-\psi_1)(T)\|_{L^2} = O\left(\|u_1\|_{L^2}^2 + |u_1(T)|^2\right), \quad |\langle(\psi-\varphi_0-\psi_1-\psi_2)(T), \varphi_J\rangle| = O\left(\|u_1\|_{L^2}^3 + |u_1(T)|^3\right). \tag{4.10}$$

Step 3: Closed loop estimate. Let $j_0 \in \mathbb{N}$ be such that $\langle\mu, \varphi_{j_0}\rangle \neq 0$. We deduce from (4.3) and (4.10) that

$$\langle\psi(T)-\varphi_0, \varphi_{j_0}\rangle = i\langle\mu, \varphi_{j_0}\rangle\int_0^T u(s)e^{-i\lambda_{j_0}(T-s)}\,\mathrm{d}s + O\left(\|u_1\|_{L^2}^2 + u_1(T)^2\right).$$

We force the presence of $u_1(T)$ by performing an integration by part in the integral. With (4.8), this leads to

$$|u_1(T)| = O\left(\sqrt{T}\|u_1\|_{L^2} + \|\psi(T;u) - \varphi_0\|_{L^2}\right) \tag{4.11}$$

Step 4: Drift estimate. Using successively (4.9), (4.10), (4.11) and taking into account (4.8), we obtain

$$\begin{aligned}\langle \psi(T) - \varphi_0, \varphi_0\rangle &= \langle \psi_2(T), \varphi_0\rangle + \langle(\psi - \varphi_0 - \psi_1 - \psi_2)(T), \varphi_0\rangle \\ &= -ia_0^1\|u_1\|_{L^2}^2 + O\left(u_1(T)^2 + T\|u_1\|_{L^2}^2\right) \\ &= -ia_0^1\|u_1\|_{L^2}^2 + O\left(T\|u_1\|_{L^2}^2 + \|\psi(T;u) - \varphi_0\|_{L^2}^2\right).\end{aligned}$$

□

Sketch of the proof for $(J,k) = (0,2)$. When $k \geq 2$, then $K \in C^{2k}([0,\infty);\mathbb{C})$ because $c_j = o(1/\lambda_j^{2k+1})$, thus we can perform more integrations by part in the quadratic form (Step 1) and in the moments (Step 3). For instance, in the particular case $k = 2$, this leads to

$$\langle \psi(T;u) - \varphi_0, \varphi_J\rangle = -ia_J^2\|u_2\|_{L^2}^2 + O\left(T\|u_2\|_{L^2}^2 + \|u_1\|_{L^2}^3 + \|\psi(T;u) - \varphi_0\|_{L^2}^2\right).$$

Thanks to the Gagliardo-Nirenberg inequality

$$\|u_1\|_{L^2}^3 = \|u_2'\|_{L^2}^3 = O\left(\|u_2^{(3)}\|_{L^2}\|u_2\|_{L^2}^2 + T^{-3}\|u_2\|_{L^2}^3\right) = O\left(\left(\|u'\|_{L^2} + T^{-1}\|u\|_{L^2}\right)\|u_2\|_{L^2}^2\right)$$

we obtain (4.7) with $\nu = -1$. □

Remark 4.5. For the bilinear Schrödinger equation with Dirichlet boundary conditions, the small-time local controllability around the ground state is proved via the linear test (for appropriate functions μ) by Camille Laurent and the author in [5], and improved by Mégane Bournissou in [15]. When the linearized system misses a direction, the strongest quadratic obstruction, relying on the Lie bracket W_1 (case $k = 1$), was first proved by Jean-Michel Coron in [25], then improved by Morgan Morancey and the author in [11] and generalized to any quadratic bracket W_k by Mégane Bournissou in [16]. Theorem 4.3 is an adaptation, to Neumann boundary conditions, of this earlier reference. It has been adapted to the multi-input case by Théo Gherdauoi in [38]. See also [55, 56] for the simultanenous exact controllability of several such equations.

4.2 Quadratic terms recovering controllability. In this section, we still consider equation (4.1) but with $\mu \in H^2((0,1);\mathbb{R})$ only. Contrary to Theorem 4.3, μ' is not assumed to vanish at the boundary, thus the best estimates one can use are $\langle \mu, \varphi_j\rangle = O(1/\lambda_j)$ (see (4.4)) and $c_j = O(1/\lambda_j^2)$. In particular the series a_J^1 converges, but the series a_J^2 does not (see (4.5)): our study falls within the grey area between cases $k = 1$ and $k = 2$ of the previous theorem. When $a_J^1 \neq 0$, the drift estimate (4.6) is proved (see [10, Theorem 1.7]) with a more delicate proof, because the kernel K may not be $\mathcal{C}^2$ anymore. When $a_J^1 = 0$, the following positive result is proved by Frédéric Marbach, Thomas Perrin and the author in [10, Theorem 1.9].

THEOREM 4.6. *Let $J \in \mathbb{N}^*$ and $\mu \in H^2((0,1);\mathbb{R})$ such that $\langle \mu, \varphi_J\rangle = 0$ and $a_J^1 = 0$ (see (4.5)). We assume*

$$\exists c > 0, \quad \forall j \in \mathbb{N} \setminus \{J\}, \quad |\langle \mu, \varphi_j\rangle| \geq c\langle j\rangle^{-2}. \tag{4.12}$$

Then, for every $T \in (0,1]$, there exists $\delta > 0$ such that, for every $\psi^\star \in H_N^2((0,1);\mathbb{C}) \cap \mathbb{S}$ with $\|\psi^\star - \varphi_0\|_{H^2} \leq \delta$, there exists $u \in L^2((0,T);\mathbb{R})$ such that $\psi(T;u) = \psi^\star$. Moreover, we have the estimates

$$\|u\|_{L^2} \leq C_T|\langle \psi^\star, \varphi_J\rangle|^{\frac{1}{2}} + C_T\|\psi^\star - \varphi_0\|_{H_N^2}, \tag{4.13}$$

$$\|u_1\|_{L^2} \leq C\left(\frac{|\Re\langle \psi^\star, \varphi_J\rangle|}{T^2}\right)^{\frac{1}{2}} + C\left(\frac{|\Im\langle \psi^\star, \varphi_J\rangle|}{T}\right)^{\frac{1}{2}} + C_T\|\psi^\star - \varphi_0\|_{L^2}, \tag{4.14}$$

where C denotes a time-independent constant and C_T a time-dependent one.

The assumption (4.12) prevents μ' from vanishing at the boundary (see (4.4)). The cost estimate (4.13) involving $|\langle \psi^\star, \varphi_J\rangle|^{\frac{1}{2}}$ is typical of results where the controllability stems from the quadratic order. When $a_J^1 \neq 0$, the amplitude of the drift along the direction $-ia_J^1\varphi_J$ is given by $\|u_1\|_{L^2}^2$ (see (4.6)). The sharp cost estimate (4.14) proves that, when $a_J^1 = 0$, one can move in the directions $\pm i\varphi_J$ with an amplitude scaling like $T\|u_1\|_{L^2}^2$, and in the directions $\pm\varphi_J$ with an amplitude $T^2\|u_1\|_{L^2}^2$. Motions in the other directions are governed by the linear theory thanks to (4.12). In the proof, we will use the following spaces.

Definition 4.7. *For $s \in \mathbb{R}$, $\widetilde{H}^s((0,T);\mathbb{R})$ denotes the closure of $C_c^\infty((0,T);\mathbb{R})$ in $H^s(\mathbb{R};\mathbb{R})$ endowed with*

$$\|u\|_{H^s(\mathbb{R})} = \left(\frac{1}{2\pi}\int_{\mathbb{R}} \langle\omega\rangle^{2s}|\widehat{u}(\omega)|^2\,\mathrm{d}\omega\right)^{\frac{1}{2}} \qquad \text{where} \qquad \widehat{u}(\omega) = \int_{\mathbb{R}} u(t)e^{-i\omega t}\,\mathrm{d}t. \tag{4.15}$$

Then, for $u \in L^2 \cap \widetilde{H}^s((0,T);\mathbb{R})$, we have $\|u\|_{\widetilde{H}^s} = \|\widetilde{u}\|_{H^s(\mathbb{R})}$ where $\widetilde{u}$ denotes the extension of u by 0 on $\mathbb{R}\setminus(0,T)$. If $s \geq 0$ then $\widetilde{H}^{-s}((0,T);\mathbb{R})$ is the dual space of $H^s((0,T);\mathbb{R})$ and $\widetilde{H}^s((0,T);\mathbb{R}) = H_0^s((0,T);\mathbb{R})$ iff $s \notin \frac{1}{2}+\mathbb{N}$ (see [42, Corollary 1.4.4.5] and [54, Theorem 3.30]).

Sketch of the proof of Theorem 4.6. The proof relies on a power series expansion (see [25, Chapter 8]). The key difficulty consists in proving that it is possible to move approximately the state $\psi(T;u)$ in the directions $\pm i\varphi_J$. Hence the core of the proof consists in constructing, for any $T \in (0,1]$, controls $u_\pm \in L^2((0,T);\mathbb{R})$ such that $\psi_1(T;u_\pm) = 0$ and $\langle\psi_2(T;u_\pm),\varphi_J\rangle, \approx \pm iT\|(u_\pm)_1\|_{L^2}^2$. In particular these controls will have zero average, i.e. live in the subspace $\mathcal{H} = \{u \in L^2((0,T);\mathbb{C}); u_1(T) = 0\}$ thus satisfy $\widehat{u}(\omega) = i\omega\widehat{u_1}(\omega)$.

Step 1: Expression of the quadratic form Q_J. Using (4.2) and the function $\rho_J(t) = e^{i\lambda_J t/2}$, we obtain

$$\begin{aligned}\langle\psi_2(T),\varphi_J e^{-i\lambda_J T}\rangle &= -\sum_{j\in\mathbb{N}} c_j \int_0^T u(t)e^{i\lambda_J t}\int_0^t u(s)e^{-i\lambda_j(t-s)}\,\mathrm{d}s\,\mathrm{d}t \\ &= -\sum_{j\in\mathbb{N}} c_j \int_0^T\int_0^t (u\rho_J)(t)(u\rho_J)(s)e^{i\frac{\lambda_J}{2}(t-s)}e^{-i\lambda_j(t-s)}\,\mathrm{d}s\,\mathrm{d}t = -\frac{1}{2}Q_J(u\rho_J,u\overline{\rho_J}),\end{aligned} \tag{4.16}$$

where for any $u,v \in L^2((0,T);\mathbb{C})$,

$$Q_J(u,v) = \int_0^T\int_0^T K_J(t-s)u(s)\bar{v}(t)\,\mathrm{d}s\,\mathrm{d}t \tag{4.17}$$

and for any $\sigma \in \mathbb{R}$,

$$K_J(\sigma) = \exp\left(i\frac{\lambda_J}{2}|\sigma|\right)K(\sigma), \qquad\qquad K(\sigma) = \sum_{j\in\mathbb{N}} c_j e^{-i\lambda_j|\sigma|}.$$

The frequency modulation in the argument of Q_J (i.e. the fact that Q_J is evaluated at $(u\rho_J, u\overline{\rho_J})$ instead of (u,u) in (4.16)) is immaterial (see [10, Lemma 3.6]): there exists $C,\nu > 0$ such that, for every $u \in \mathcal{H}$,

$$|Q_J(u\rho_J,u\overline{\rho_J}) - Q_J(u,u)| \leq C(T^2\|u_1\|_{L^2}^2 + \|u_1\|_{\widetilde{H}^{-\nu}}^2). \tag{4.18}$$

The frequency modulation in the kernel K_J with respect to K has more impact (see [10, Proposition 5.8]): for every $\nu \in [0,1/4)$, there exists $C > 0$ such that, for all $T \in (0,1]$ and $u \in \mathcal{H}$

$$|Q_J(u,u) - Q(u,u) + i\lambda_J K(0)\|u_1\|_{L^2}^2| \leq C(T^2\|u_1\|_{L^2}^2 + \|u_1\|_{\widetilde{H}^{-\nu}}^2). \tag{4.19}$$

where $Q = Q_0$. In (4.18) and (4.19), the first term in the right hand side is negligible with respect to the expected amplitude $\pm iT\|u_1\|_{L^2}^2$ when $T \to 0$. The construction of $u_\pm$ will ensure that the second term also does. Therefore the key point is to understand $Q(u,u)$.

Step 2: Fourier study of Q. The best estimate one can use is $c_j = O(1/\lambda_j^2)$ (see (4.4)), thus one cannot differentiate twice the kernel K (its derivative K' is similar to the Riemann function $\sigma \mapsto \sum_{j\in\mathbb{N}^*}\sin(j^2\sigma)/j^2$ which is differentiable almost nowhere). To avoid integrations by parts, we perform estimates on the Fourier domain. Benefiting from the convolution structure of (4.17), one expects to obtain, from Plancherel's theorem

$$Q(u,u) = \int_0^T (K\star u)(t)\overline{u(t)}\,\mathrm{d}t = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{K}(\omega)|\widehat{u}(\omega)|^2\,\mathrm{d}\omega. \tag{4.20}$$

Since the control u is supported on $[0,T]$ with $T \ll 1$, its Fourier transform involves high frequencies. Thus, one expects to reduce the study of Q to the asymptotic behavior at $\pm\infty$ of $\widehat{K}$. One would for example expect that, if $\Im\widehat{K}(\omega) \sim \omega^{-2}$ as $|\omega| \to \infty$, then $\Im Q(u,u)$ would be equivalent to $\|u\|_{\widetilde{H}^{-1}}^2$, for T small enough.

Things are not so simple because K may not belong to $L^1(\mathbb{R})$. The function K defines a tempered distribution thus its Fourier transform $\mathcal{F}(K)$ is well defined in $\mathcal{S}'(\mathbb{R};\mathbb{C})$. Equality (4.20) can be given the following precise meaning: if $u \in C_c^\infty((0,T);\mathbb{C})$, then $Q(u,u) = \langle \widehat{K}, |\widehat{u}|^2\rangle_{\mathcal{S}',\mathcal{S}}$.

One obtains the following explicit expression in $\mathcal{S}'(\mathbb{R};\mathbb{C})$

$$\mathcal{F}(K) = \pi \sum_{j\in\mathbb{N}} c_j(\delta_{\lambda_j} + \delta_{-\lambda_j}) - 2i \sum_{j\in\mathbb{N}} \frac{\lambda_j c_j}{\lambda_j^2 - \omega^2}$$

where the right-hand side contains a sum of Dirac masses and principal value distributions (at each $\pm\lambda_j$ for which $c_j \neq 0$). A clear example is provided by the particular case $c_0 = 0$ and $c_j = \pi^2/j^4$ for $j \in \mathbb{N}^*$ for which (Mittag-Leffler series for cot and coth)

$$\sum_{j\in\mathbb{N}} \frac{\lambda_j c_j}{\lambda_j^2 - \omega^2} = \sum_{j\in\mathbb{N}^*} \frac{1}{j^2} \frac{1}{j^4 - (\omega/\pi^2)^2} = -\frac{\pi^6}{6\omega^2} - \frac{\pi^6 \cot(\sqrt{\omega})}{4\omega^2\sqrt{\omega}} + \frac{\pi^6 \coth(\sqrt{\omega})}{4\omega^2\sqrt{\omega}}. \tag{4.21}$$

Step 3: Study of the quadratic kernel. In the general case, to mimic the 3 components in (4.21), we prove that

$$\sum_{j\in\mathbb{N}} \frac{\lambda_j c_j}{\lambda_j^2 - \omega^2} = -\frac{a}{\omega^2} + \frac{\Theta_{\text{pv}}(\omega)}{\omega^2} + \frac{\Theta_{\text{reg}}(\omega)}{\omega^2} \tag{4.22}$$

where $a = \sum_{j\in\mathbb{N}^*} \lambda_j c_j$, Θ_{pv} isolates the dominant singularity within each frequency region, and Θ_{reg} is the remainder. The first term in (4.22), scaling like ω^{-2} creates an $\widetilde{H}^{-1}$ drift in the quadratic form Q, when its coefficient $a \neq 0$. The second term contains localized singularities at $\pm\lambda_j$. The third term is estimated in a weaker norm (heuristically the $\widetilde{H}^{-5/4}$ norm in the case (4.21)), and will always be negligible.

Finally, thanks to the frequency modulation (see (4.19)) and the relation $2a - \lambda_J K(0) = 2a_J^1 = 0$, we can heuristically consider $Q_J(u,u)$ given by

$$Q_J(u,u) \approx -\frac{i}{\pi} \langle \Theta_{\text{pv}}, |\widehat{u_1}|^2\rangle_{\mathcal{S}',\mathcal{S}}$$

when $u \in C_c^\infty((0,T);\mathbb{R}) \cap \mathcal{H}$. Now, we can exploit the changes of sign of Θ_{pv} at each singularity, to construct controls achieving $Q_J(u,u) = \pm iT\|u_1\|_{L^2}^2$

Step 4: Quadratic motion along $\pm i\varphi_J$. We want to construct, for any $T \in (0,1]$, controls $u_\pm \in L^2((0,T);\mathbb{R})$ such that $\psi_1(T;u_\pm) = 0$ and $Q_J(u_\pm,u_\pm) \approx \pm iT\|(u_\pm)_1\|_{L^2}^2$.

For the last equality to be possible, we need Θ_{pv} to change sign. Moreover, for $T \ll 1$, since the control is supported in $[0,T]$, its Fourier transform involves high frequencies and have a wide support. So Θ_{pv} not only needs to change sign, but it needs to do so up to $\pm\infty$, and achieve both signs on arbitrarily large intervals. This is the case of the second term in the right hand side of the example (4.21). Heuristically, to achieve a motion of amplitude $T\|u_1\|_{L^2}^2$, we need $\pm\Theta_{\text{pv}} \geq T$ on intervals much larger than $1/T$. To achieve the desired sign, one can then choose a control oscillating at the median frequency of such an interval, say ω_0.

The second requirement on $u_\pm$, which is $\psi_1(T;u_\pm) = 0$, or equivalently $\widehat{u}(\lambda_j) = 0$ for all $j \in \mathbb{N} \setminus \{J\}$ (see (4.3)), is baked in our construction, by choosing ω_0 far from the λ_j. Then correcting the moments does not affect much the quadratic form.

Finally, for the terms $\|u_1\|_{\widetilde{H}^{-\nu}}^2$ to be residual in (4.18) and (4.19), we take ω_0 large enough.

Step 5: Approximate motion along $\pm\varphi_J$. Once the motions in the directions $\pm i\varphi_J$ are established, the motions in the directions $\pm\varphi_J$ follow from a standard argument linked with tangent vectors. For the control-affine system (1.1), it is well-known that if $\pm\xi$ are tangent vectors (i.e. one can move infinitesimally in these directions for small times), then the same holds for $\pm Df_0(0)\xi$ (see [44, Theorem 6] or [14, Property (P2)]). Here $-iA$ plays the role of f_0 and $-iA(\pm i\varphi_J) = \mp\lambda_J\varphi_J$.

The end of the proof relies on a Brouwer fixed point argument. Careful estimates at the fixed point provide the cost estimates (4.13) and (4.14). ∎

4.3 New drifts, specific to PDEs. In this section, we consider the heat equation

$$\begin{cases} \left(\partial_t - \partial_x^2\right) y(t,x) = u(t)\left(\mu(x) + \lambda(x) y(t,x)\right), & x \in (0,1), \\ \partial_x y(t,0) = \partial_x y(t,1) = 0, \end{cases} \tag{4.23}$$

where $\lambda, \mu : (0,1) \to \mathbb{R}$. The state and control are the real valued functions y and u. This abstract system is not intended to model a real-world physical system. We chose it because it lightens the computations. The techniques are applied to more realistic systems in subsection 4.4.

The system (4.23) is well posed in the following sense: for $\lambda, \mu \in H_N^{-1}((0,1);\mathbb{R})$ and $T > 0$, there exists $\eta = \eta(T, \lambda, \mu) > 0$ such that, for every $y_0 \in L^2((0,1);\mathbb{R})$ and $u \in L^\infty((0,T);\mathbb{R})$ such that $\|u\|_{L^\infty} < \eta$, then there exists a unique solution $y \in \mathcal{C}^0([0,T];L^2) \cap L^2((0,T);H_N^1)$ of (4.23) such that $y(0) = y_0$, denoted $y(t;u,y_0)$ (see [8, Section 2.1]). It is this parameter $\eta = \eta(T,\lambda,\mu)$ that will come into play in Theorem 4.9.

Definition 4.8 (E-STLNC). *Let $(E_T, \|.\|_{E_T})$ be a familly of normed vector spaces of real valued functions defined on $[0,T]$ for $T > 0$. System (4.23) is E-small-time locally null controllable if for every $T, \rho > 0$, there exists $\delta > 0$ such that, for every $y_0 \in L^2((0,1);\mathbb{R})$ with $\|y_0\|_{L^2} \leq \delta$, there exists $u \in E_T \cap L^\infty((0,T);\mathbb{R})$ such that $\|u\|_{E_T} \leq \rho$ and $y(T;u,y_0) = 0$.*

In a power series expansion of the state with respect to the control $y = y_1 + y_2 + \dots$ then y_1 and y_2 solve the linearized and quadratic order systems

$$\begin{cases} \left(\partial_t - \partial_x^2\right) y_1(t,x) = u(t)\mu(x), \\ \partial_x y_1(t,0) = \partial_x y_1(t,1) = 0, \end{cases} \qquad \begin{cases} \left(\partial_t - \partial_x^2\right) y_2(t,x) = u(t)\lambda(x) y_1(t,x), \\ \partial_x y_2(t,0) = \partial_x y_2(t,1) = 0, \end{cases} \tag{4.24}$$

with the initial conditions $y_1(0,\cdot) = y_2(0,\cdot) = 0$. By the Duhamel formula,

$$y_1(t,x) = \sum_{j=0}^{\infty} \left(\langle y_{1,0}, \varphi_j \rangle e^{-\lambda_j t} + \langle \mu, \varphi_j \rangle \int_0^t u(\tau) e^{-\lambda_j (t-\tau)} \, \mathrm{d}\tau \right) \varphi_j(x).$$

In particular, if there exists $C, c > 0$ such that for every $j \in \mathbb{N}$, $|\langle \mu, \varphi_j \rangle| \geq C e^{-cj}$ (which is generic) then, for every $m \in \mathbb{N}$, the linearized system is H^m-small-time null controllable, and the nonlinear system (4.23) is H^m-STLNC (see [8, Section 2], the proof relies on a method due to Yuning Liu, Takeo Takahashi and Marius Tucsnak [51]).

To ensure that the linearized system is not controllable, from now on, we assume that $\langle \mu, \varphi_0 \rangle = 0$. For appropriate functions λ, μ, one may obtain quadratic drifts quantified by negative integer Sobolev norms of the control, or quadratic terms recovering controllability, see [8, Theorems 3 and 6]. Here, we focus on different behaviors. Frédéric Marbach and the author prove in [8, Theorem 4] that any negative fractional drift $\|u\|^2_{\widetilde{H}^{-s}}$ with $s > 0$ can be obtained for appropriate λ, μ. Here, to simplify, we only treat the case $s \in (0,1)$.

Theorem 4.9. *Let $s \in (0,1)$, $\alpha > 4s - 1$, $a \in \mathbb{R}^*$, and $\lambda, \mu \in H_N^{-1}((0,1);\mathbb{R})$ be such that $\langle \mu, \varphi_0 \rangle = 0$ and*

$$c_j = \langle \mu, \varphi_j \rangle \langle \lambda, \varphi_j \rangle = \frac{a}{j^{4s-1}} + \underset{j \to \infty}{O}\left(\frac{1}{j^\alpha}\right).$$

Then the system (4.23) is not $H^{2s+\frac{3}{2}}$-STLNC: there exists $C, \beta, \gamma(s) > 0$ such that, for every $T \in (0,1]$, $\delta \in [-1,1]$ and $u \in L^\infty((0,T);\mathbb{R})$ with $\|u\|_{L^\infty} \leq \eta$,

$$\left| \langle y(T;u,\delta\varphi_0), \varphi_0 \rangle - \delta - a\gamma(s) \|u\|^2_{\widetilde{H}^{-s}} \right| \leq C \left(\left(T^{2\beta} + T^{-3} \|u\|_{H^{2s+\frac{3}{2}}} \right) \|u\|^2_{\widetilde{H}^{-s}} + \|u\|_{L^\infty} |\delta| \right). \tag{4.25}$$

This statement applies, for instance, to $\lambda = \mu = \sum_{j \in \mathbb{N}} \langle j \rangle^{\frac{1}{2} - 2s} \varphi_j$, where the series converges in $H_N^{-1}((0,1);\mathbb{R})$ because $s > 0$. When T and u are small enough (precisely $C(T^{2\beta} + T^{-3}\|u\|_{H^{2s+\frac{3}{2}}}) < |a|\gamma(s)$), then the estimate (4.25) prevents from steering to zero initial conditions of the form $y_0 = \delta\varphi_0$ where $\delta \in \mathbb{R}^*$ has the same sign as a. The norm $\|u\|_{\widetilde{H}^{-s}}$ that quantifies the drift is strictly stronger than the norm $\|u_1\|_{L^2}$ involved in the strongest obstruction in finite dimension.

Despite the resemblance between the integer-order and the fractional-order statements, we stress that the nature of the underlying cause might be different. Indeed, the integer-order obstructions occur when a quantity (which is an infinite sum of coefficients) does not vanish, whereas the fractional-order obstructions depend only on the asymptotic behavior of these coefficients; they are more robust to perturbations.

The smallness in $H^{2s+\frac{3}{2}}$ is not the optimal one, but allows a lighter exposition. Actually, quadratic drifts can be quantified by essentially arbitrary weighted negative Sobolev norms (see [8, Section 4.6]).

Sketch of the proof of Theorem 4.9. To simplify notations, we use the notation O for estimates that hold uniformly with respect to $T \in (0,1]$, $\delta \in [-1,1]$ and $u \in L^\infty((0,T);\mathbb{R})$ such that $\|u\|_{L^\infty} \leq \eta$.

Step 1: Coercivity of the quadratic form via a Fourier approach. Using (4.24) and Plancherel theorem, we get

$$\langle y_2(T), \varphi_0 \rangle = \int_0^T u(t) \int_0^t u(\tau) K(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}t = \frac{1}{2}\int_0^T u(K\star u) = \frac{1}{4\pi}\int_{\mathbb{R}} |\widehat{u}|^2 \widehat{K} \text{ where } K(\sigma) = \sum_{j\in\mathbb{N}} c_j e^{-\lambda_j|\sigma|}.$$

An explicit computation and an asymptotic study (with a series-integral comparison) justify that

$$\frac{1}{2}\widehat{K}(\omega) = C\sum_{j=0}^{\infty} \frac{\lambda_j c_j}{\lambda_j^2+\omega^2} = \frac{a\gamma(s)}{|\omega|^{2s}} + \underset{|\omega|\to\infty}{O}\left(\frac{1}{|\omega|^{2(s+\beta)}}\right) \text{ where } \beta>0 \text{ and } \gamma(s) = C\int_0^\infty \frac{y^{3-4s}}{1+y^4}\,\mathrm{d}y. \tag{4.26}$$

Thus, using uncertainty principles, we obtain

$$\langle y_2(T), \varphi_0\rangle = a\gamma(s)\|u\|^2_{\widetilde{H}^{-s}} + O\left(T^{2\beta}\|u\|^2_{\widetilde{H}^{-s}}\right). \tag{4.27}$$

Step 2: Quadratic and cubic estimates. Classical arguments prove that

$$\|y(T;u,\delta\varphi_0) - (\delta\varphi_0 + y_1 + y_2)(T)\|_{L^2} = O\left(\|u\|^3_{L^\infty} + |\delta|\,\|u\|_{L^\infty}\right). \tag{4.28}$$

Step 3: Interpolation. By the fractional Gagliardo-Nirenberg inequality, we have

$$\|u\|^3_{L^\infty} = O\left(T^{-3}\|u\|_{H^{2s+\frac{3}{2}}}\|u\|^2_{\widetilde{H}^{-s}}\right). \tag{4.29}$$

By gathering (4.27), (4.28), (4.29), we obtain (4.25). ∎

Remark 4.10. For a Galerkin approximation of the heat equation (4.23) of finite dimension d, the series in (4.26) would be a finite sum involving at most $d-1$ terms; its asymptotics would be of the form $C\omega^{-2k}$ for some $k \in [\![1, d-1]\!]$: we would recover the usual negative integer Sobolev norms in finite dimension. For PDEs, the infinite dimension of the state space allows richer asymptotic behaviors at $\pm\infty$ for $\widehat{K}$ and, as a consequence, new drift estimates.

4.4 Other equations, bibliographical comments.

4.4.1 The Korteweg-de Vries equation. Many of the previous quadratic behaviors were observed on the KdV equation, which models the propagation of small-amplitude waves on the surface of a uniform channel. We first consider the case of a control on the Neumann condition:

$$\begin{cases} (\partial_t + \partial_x + \partial_x^3)y + y\partial_x y = 0, \quad x\in(0,L), \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = u(t), \end{cases}$$

First, Lionel Rosier proves in [65] the local controllability around zero by linear test, except for a countable set of lengths L, for which the linearized system misses a finite number of directions. Then, Emmanuelle Crépeau and Jean-Michel Coron introduce in [27] the power series expansion method, to take advantage of the nonlinear term $y\partial_x y$. For particular critical lengths (for which the linearized system misses only one direction) they prove the small-time local controllability; their proof relies on the cubic term (the quadratic term vanishes). For any other critical lengths, Emmanuelle Crépeau and Eduardo Cerpa prove in [17] the large-time local controllability, by a power series expansion, relying quadratic terms. Jingrui Niu and Shengquan Xiang prove in [62], for some of these critical lengths, a quadratic obstruction to the small-time local controllability, quantified by $\|u_1\|^2_{L^2}$. Jean-Michel Coron, Armand Koenig and Hoai-Minh Nguyen prove in [30], for all the other critical lengths, a quadratic obstruction quantified by the $H^{-2/3}$-norm of the control; their proof relies on a Fourier-based approach.

For the KdV equation with a control on the Dirichlet boundary condition, (i.e. with $y(t,0) = \partial_x y(t,L) = 0$ and $y(t,L) = u(t)$), Olivier Glass and Sergio Guerrero prove in [41] the local controllability around zero, by linear test, except for a countable set of lengths L (different from the previous one). Hoai-Minh Nguyen proves in [59] that, for any critical length, the linearized system misses only one direction, and for some critical lengths, small-time local controllability does not hold, because of a quadratic drift quantified by the $H^{-1/6}$-norm of the control; his proof relies on a Fourier-based approach.

4.4.2 The Burgers equation. A quadratic obstruction to the small-time local controllability of the Burgers equation

$$\begin{cases} \partial_t y + y\partial_x y - \partial_x^2 y = u(t), & x \in (0,1), \\ y(t,0) = y(t,1) = 0, \end{cases}$$

is proved by Frédéric Marbach in [52]. This is the first result from the literature that involves a drift quantified by a fractional Sobolev norm of the control: the $H^{-5/4}$-norm. The coercivity of the quadratic form is proved via the theory of weakly singular integral kernels. This quadratic obstruction is proved to hold in any time by Hoai-Minh Nguyen in [60], with a Fourier-based approach, similar to [30].

4.4.3 The Saint Venant equation. We consider a water-tank modeled by Saint-Venant equations,

$$\begin{cases} \partial_t H + \partial_x (Hv) = 0, & x \in (0,L), \\ \partial_t v + \partial_x \left(gH + \frac{v^2}{2} \right) = -u(t), & x \in (0,L), \\ v(t,0) = v(t,L) = 0, \\ \ddot{D} = u. \end{cases}$$

Here H denotes the height of the water, v the horizontal velocity field of the water, u is the acceleration imposed on the tank, D is the position of the tank, and g the gravity. The state is $(H, v, \dot{D}, D)$ and the control is u. For any $H_{eq} > 0$, $(H,v) = (H_{eq}, 0)$ is an equilibrium (for $u = 0$). The large time local controllability around any such equilibrium is proved in [24], via Coron's return method, because the linearized system is not controllable. François Dubois, Nicolas Petit and Pierre Rouchon prove in [32] that a traveling time $T_* > 0$ is necessary to bring this linearized system from one position to another for which the water is still at the beginning and at the end. In [29], Jean-Michel Coron, Armand Koenig and Hoai-Minh Nguyen prove that the local controllability of the nonlinear system does hold in time $< 2T^*$. Their proof involves a quadratic drift quantified by the norm $\|u_1\|_{L^2}^2$.

4.4.4 Lie brackets and approximate controllability. Lie brackets techniques are also used to prove small-time global approximate controllability results, for instance for Schrödinger equations with bilinear controls posed on $M = \mathbb{T}^d$ or $\mathbb{R}^d$

$$i\partial_t \psi = -\Delta\psi + \sum_{j=1}^{m} u_j(t)\mu_j(x)\psi(t,x), \quad x \in M. \tag{4.30}$$

The forward flows along the quadratic bracket $+W_1$ are always small-time approximately reachable by the dynamics. This fact is at the root of the small-time controllability of phases (i.e. the capability, for any initial state ψ_0 and phase $\varphi \in L^2(M;\mathbb{R})$ to approximately reach the state $e^{i\varphi}\psi_0$ in arbitrary small time) proved in [33, 34] by Alessandro Duca, Vahagn Nersesyan and Eugenio Pozzoli. More precisely, if a phase φ is small-time approximately reachable, then so does the phase $-|\nabla\varphi|^2$, which corresponds to the bracket $[f_1,[f_1,f_0]]$ with $f_0 = i\Delta$ and $f_1 = i\varphi$. This procedure, initiated with phases in $\text{Span}\{\mu_1,\dots,\mu_m\}$, can be iterated. For appropriate functions $\mu_1,\dots,\mu_m$ (i.e. for which a particular subspace of $\text{Lie}(i\Delta, i\mu_1,\dots,i\mu_m)$ is dense in $L^2(M;i\mathbb{R})$), this allows to prove the control of phases. This saturation argument was introduced in the pioneering article [2] by Andrei Agrachev and Andrey Sarychev.

The flows along the linear bracket M_1 are always small-time approximately reachable by the dynamics. This fact is at the root of the small-time controllability of diffeomorphisms (i.e. the capability, for any initial state ψ_0 and diffeomorphism P of M, to approximately reach the state $\det(DP)^{\frac{1}{2}}\psi_0 \circ P$ in arbitrary small time) proved in [12, 13] by Eugenio Pozzoli and the author. More precisely, if the phases in $\mathbb{R}\varphi$ are small-time approximately reachable, then so do the diffeomorphisms $P_t = e^{t\nabla\varphi}$ (i.e. the flow of the vector field $\nabla\varphi$ on M). Indeed the action on $L^2(M;\mathbb{C})$ of this diffeomorphism is given by the group of unitary operators $e^{t\mathcal{T}}$ associated with the transport operator $\mathcal{T} = \nabla\varphi.\nabla + \frac{1}{2}\text{div}(\varphi)$, which corresponds to the bracket $[f_1,f_0]$ with $f_0 = i\Delta$ and $f_1 = i\varphi$.

Following these ideas, in [12], we prove the small-time global approximate controllability of equations of the form (4.30). The strategy consists in controlling separately the radial part and the angular part of the wavefunction thanks to the control of diffeomorphisms and phases.

We refer to [18, 53, 57, 58, 35] for large time global approximate controllability of Schrödinger equations with bilinear control, proved with different techniques.

4.4.5 Related work. In [22], Shirshendu Chowdhury and Sylvain Ervedoza consider the Navier-Stockes system in a 2D channel. They adapt the power series method to the stabilisation problem: they take advantage of the quadratic term to accelerate convergence towards equilibrium.

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