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LOCAL CONTROLLABILITY OF A ONE-DIMENSIONAL BEAM EQUATION*

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Abstract. We prove that the beam equation with clamped ends is locally controllable in a $H^{5+\epsilon} \times H^{3+\epsilon}((0,1),\mathbb{R})$-neighborhood of a particular trajectory of the free system, with $\epsilon > 0$ and with control functions in $H_0^1((0,T),\mathbb{R})$. Ball,. Marsden, and Slemrod already proved that this equation is not controllable in $H_0^2 \times L^2((0,1),\mathbb{R})$ with control functions in $L^r_{\text{loc}}(\mathbb{R},\mathbb{R})$, $r > 1$. This article justifies that their negative result is due to a choice of functional spaces which does not allow controllability. Our proof uses moment theory and the Nash–Moser theorem.

Key words. control of partial differential equations, bilinear control problem, beam equation, Nash–Moser theorem, trigonometric moment problem

AMS subject classifications. 35B37, 93C10, 93C20, 35Q72, 42A70

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1. Introduction.

1.1. Main result. We consider the beam equation with clamped ends

$$(\Sigma) \begin{cases} u_{tt} + u_{xxxx} + p(t)u_{xx} = 0, (t,x) \in \mathbb{R}_+ \times (0,1), \\ u = u_x = 0 \text{ at } x = 0, 1. \end{cases}$$

It is a nonlinear control system where

- the state is the couple (u, u_t) and
- the control is the real-valued function $t \mapsto p(t)$.

We introduce the operator A defined by

$$D(A) := H^4 \cap H_0^2((0,1),\mathbb{R}), \quad Av := \frac{d^4 v}{dx^4}. \tag{1.1}$$

Let $(\lambda_n)_{n\in\mathbb{N}^*} \subset \mathbb{R}_+^*$ be the increasing sequence of eigenvalues of A and $(\varphi_n)_{n\in\mathbb{N}^*}$ associated orthonormalized eigenvectors. Then, for every $n \in \mathbb{N}^*$, the functions

$$\varphi_n(x)\cos(\sqrt{\lambda_n}t) \text{ and } \varphi_n(x)\sin(\sqrt{\lambda_n}t)$$

are solutions of (Σ), with $p \equiv 0$. For $s > 0$, we introduce the space

$$H^s_{(0)}((0,1),\mathbb{R}) := D(A^{s/4}), \tag{1.2}$$

equipped with the norm

$$\|\varphi\|_{H^s_{(0)}((0,1),\mathbb{C})} := \left(\sum_{k=1}^{\infty} |\lambda_k^{s/4} \langle \varphi, \varphi_k \rangle|^2 \right)^{1/2}. \tag{1.3}$$

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In particular, we have

(1.4)
$$H^s_{(0)}((0,1),\mathbb{R}) = \{\varphi \in H^s((0,1),\mathbb{R}); \varphi = \varphi' = 0 \text{ at } x = 0,1\} \text{ for } s \in \{2,3,4\},$$
$$H^5_{(0)}((0,1),\mathbb{R}) = \{\varphi \in H^5((0,1),\mathbb{R}); \varphi = \varphi' = \varphi^{(4)} = 0 \text{ at } x = 0,1\},$$
$$H^s_{(0)}((0,1),\mathbb{R}) = \{\varphi \in H^s((0,1),\mathbb{R}); \varphi = \varphi' = \varphi^{(4)} = \varphi^{(5)} = 0 \text{ at } x = 0,1\} \text{ for } s = 6,7,8.$$

The main result of this article is the following one.

THEOREM 1. *Let $T := 8/\pi$, $\epsilon > 0$, and*

$$u^{ref}(t,x) := \varphi_2(x)\sin(\sqrt{\lambda_2}t) + \varphi_3(x)\sin(\sqrt{\lambda_3}t). \tag{1.5}$$

There exists a neighborhood V_0 of $(u^{ref}(0), \dot{u}^{ref}(0))$ and a neighborhood V_T of $(u^{ref}(T), \dot{u}^{ref}(T))$ in $H^{5+\epsilon}_{(0)} \times H^{3+\epsilon}_{(0)}((0,1),\mathbb{R})$ such that, for every $(u_0, \dot{u}_0) \in V_0$, for every $(u_T, \dot{u}_T) \in V_T$, there exists $p \in H^1_0((0,T),\mathbb{R})$ such that the solution of (Σ) with $(u(0), \dot{u}(0)) = (u_0, \dot{u}_0)$ and control p satisfies $(u(T), \dot{u}(T)) = (u_T, \dot{u}_T)$.

1.2. A previous noncontrollability result. In [1], Ball, Marsden, and Slemrod discuss the controllability of infinite dimensional bilinear control systems of the form

$$\dot{w}(t) = \mathcal{A}w(t) + p(t)\mathcal{B}w(t). \tag{1.6}$$

Thanks to the Baire lemma, they prove the following noncontrollability result.

THEOREM 2. *Let X be a Banach space with $dim(X) = +\infty$. Let $\mathcal{A}$ generate a C^0-semigroup of bounded linear operators on X and $\mathcal{B} : X \to X$ be a bounded linear operator. Let $w_0 \in X$ be fixed, and let $w(t; p, w_0)$ denote the unique solution of* (1.6) *for $p \in L^1_{loc}([0,+\infty),\mathbb{R})$. The set of states accessible from w_0 defined by*

$$S(w_0) := \{w(t; p, w_0); t \geqslant 0, p \in L^r_{loc}([0,\infty),\mathbb{R}), r > 1\}$$

is contained in a countable union of compact subsets of X and, in particular, has a dense complement.

We can write (Σ) in the first order form (1.6), with

$$w := \begin{pmatrix} u \\ u_t \end{pmatrix}, \quad \mathcal{A} := \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & 0 \\ -\dfrac{d^2}{dx^2} & 0 \end{pmatrix}. \tag{1.7}$$

Let $X := H^2_0((0,1),\mathbb{R}) \times L^2((0,1),\mathbb{R})$, with inner product

$$\langle (u_1,u_2),(v_1,v_2v)\rangle_X := \int_0^1 \left(A^{1/2}u_1 A^{1/2}v_1 + u_2v_2\right) dx.$$

As noticed in [1], Theorem 2 shows that, for every $(u_0, \dot{u}_0) \in X$, the set of (u, u_t) in X accessible from $(u_0, \dot{u}_0)$ with controls in $L^r_{loc}([0,\infty),\mathbb{R})$, $r > 1$, has a dense complement in X. In particular, (Σ) is not controllable in $H^2_0((0,1),\mathbb{R}) \times L^2((0,1),\mathbb{R})$, with control functions p in $L^2_{\text{loc}}([0,+\infty),\mathbb{R})$.

However, this theorem does not give any obstruction for having controllability in other spaces. For example, Theorem 2 does not apply with

$$\widetilde{X} := H^s_{(0)} \times H^{s-2}_{(0)}((0,1),\mathbb{R}) \text{ for } s \in \mathbb{N}^*, s \geqslant 3$$

instead of X. Indeed,

- $\mathcal{A}$ generates a C^0-semigroup of bounded linear operators of $\widetilde{X}$ (we refer to Remark 1 in section 2 for a justification of this statement),
- for $w = (w_1, w_2)^t \in \widetilde{X}$, $\mathcal{B}w = (0, -w_2'')^t$ belongs to $H^s_{(0)} \times H^{s-2}((0,1),\mathbb{R})$,
- but in general w_2'' does not vanish at $x = 0, 1$, so $\mathcal{B}w$ does not belong to $\widetilde{X}$ (see (1.4)), and thus $\mathcal{B}$ does not map $\widetilde{X}$ into $\widetilde{X}$.

Notice that Theorem 2 does not apply either with

$$\overline{X} := H^3_{(0)} \times H^1((0,1),\mathbb{R})$$

(which is a space such that $\mathcal{B}$ maps $\overline{X}$ into $\overline{X}$) instead of X, because $\mathcal{A}$ does not generate a C^0 semigroup of bounded operators of $\overline{X}$ (we refer to Proposition 8 for a proof of this statement).

In this article, we prove a local controllability result in $H^{5+\epsilon}_{(0)} \times H^{3+\epsilon}_{(0)}((0,1),\mathbb{R})$, with $\epsilon > 0$ and with control functions p in $H^1_{loc}(\mathbb{R}_+,\mathbb{R})$. Thus, the noncontrollability result proved by Ball, Marsden, and Slemrod relies on the fact that the choice of functional spaces does not allow controllability. In order to state affirmative controllability results, one must

- either control (u, u_t) in $H^2_0 \times L^2((0,1),\mathbb{R})$ but with a control functions set larger than $\cup_{r>0} L^r_{loc}(\mathbb{R}_+,\mathbb{R})$, for example, $H^{-1}_{loc}(\mathbb{R}_+,\mathbb{R})$,
- or control (u, u_t) using the control functions set $L^2_{loc}(\mathbb{R}_+,\mathbb{R})$ but in a smaller space than $H^2_0 \times L^2((0,1),\mathbb{R})$, for example, $H^3_{(0)} \times H^1_0((0,1),\mathbb{R})$.

1.3. Sketch of the proof. The result of Theorem 1 is a local controllability result, in time $T = 8/\pi$, around the trajectory $(u^{ref}, \dot{u}^{ref}, p \equiv 0)$ for the nonlinear control system (Σ). It is equivalent to a local surjectivity property for the map

$$\Phi_T : (u_0, \dot{u}_0, p) \mapsto (u_0, \dot{u}_0, u(T), u_t(T)), \tag{1.8}$$

where $T = 8/\pi$ and u is a solution of (Σ) with initial condition $u(0) = u_0$, $u_t(0) = \dot{u}_0$. For the proof of such results, there exists a classical approach which consists of

- first proving the global controllability of the linearized system around the trajectory considered (which corresponds to the surjectivity of $d\Phi_T(u^{ref}(0), u_t^{ref}(0), 0))$ and
- then applying the inverse mapping theorem to the nonlinear map Φ_T, which gives a local surjectivity property for Φ_T around $(u_0^{ref}, \dot{u}_0^{ref}, 0)$.

The general strategy adopted in this article is this one, but we will see that the classical inverse mapping theorem is not sufficient to conclude. Thus, we use a more elaborate version of the inverse mapping theorem, namely, the Nash–Moser implicit functions theorem. This strategy has already been used for Schrödinger equations in [3], [4], [2].

First, we consider the linearized system around the trajectory $(u^{ref}, \dot{u}^{ref}, p \equiv 0)$ defined by (1.5), which is

$$(\Sigma_l^{ref}) \begin{cases} U_{tt} + U_{xxxx} + P(t) u^{ref}_{xx} = 0, (t,x) \in (0,\infty) \times \mathbb{R}, \\ U = U_x = 0 \text{ at } x = 0, 1. \end{cases}$$

It is a linear control system where

- the state is the couple (U, U_t) and
- the control is the real-valued function $t \mapsto P(t)$.

We prove the following controllability result for this system.

THEOREM 3. (1) *For every $T > \pi/\sqrt{\lambda_2}$, for every $(U_T, \dot{U}_T) \in H^3_{(0)} \times H^1_0((0,1),\mathbb{R})$, there exists $P \in L^2((0,T),\mathbb{R})$ such that the solution of (Σ_l^{ref}) with $(U(0), \dot{U}(0)) = (0,0)$ and control P satisfies $(U(T), \dot{U}(T)) = (U_T, \dot{U}_T)$.*

(2) *For every $T > 0$, there exists $(U_T, \dot{U}_T) \in H^3_{(0)} \times H^1_0((0,1),\mathbb{R})$, (resp., $(U_T, \dot{U}_T) \in H^2_0 \times L^2((0,1),\mathbb{R})$) such that, for every $P \in H^1((0,T),\mathbb{R})$ (resp., $P \in L^2((0,T),\mathbb{R})$), the solution of (Σ_l^{ref}) with $(U(0), \dot{U}(0)) = (0,0)$ and control P satisfies $(U(T), \dot{U}(T)) \neq (U_T, \dot{U}_T)$.*

Therefore, the linear system (Σ_l^{ref}) is controllable in $H^3_{(0)} \times H^1_0((0,1),\mathbb{R})$ with control functions in $L^2((0,T),\mathbb{R})$, but it is neither controllable in $H^3_{(0)} \times H^1_0((0,1),\mathbb{R})$ with control functions in $H^1((0,T),\mathbb{R})$ nor controllable in $H^2_0 \times L^2((0,1),\mathbb{R})$ with control functions in $L^2((0,T),\mathbb{R})$. As we will see precisely in section 4, statement (2) of Theorem 3 shows that the local controllability around $(u^{ref}, \dot{u}^{ref}, 0)$ cannot be obtained by applying the classical inverse mapping theorem. Roughly speaking, the controls built for the control of the linearized system are not smooth enough for the application of the inverse mapping theorem.

Thus, in order to prove Theorem 1, we use a Nash–Moser theorem. This theorem is an elaborate version of the inverse mapping theorem. It gives the local surjectivity of a nonlinear map Θ around a point x^{ref} thanks to essentially 3 assumptions:

(1) There exist decreasing sequences of spaces $(E_a)_{a \geqslant 0}$, $(F_b)_{b \geqslant 0}$ such that
   - $\Theta : E_a \to F_a$ is C^2 for every a,
   - there exist smoothing linear operators $(S_\theta)_{\theta > 0}$ (resp., $(\tilde{S}_\theta)_{\theta > 0}$) defined on the spaces E_a (resp., F_b) with $S_\theta \to Id$ (resp., $\tilde{S}_\theta \to Id$) when $\theta \to +\infty$,
   - the norms of the operators $Id - S_\theta, S_\theta : E_a \to E_A$ for $a \neq A$ (resp., $Id - \tilde{S}_\theta, \tilde{S}_\theta : F_b \to F_B$ for $b \neq B$) are bounded by an explicit expression in terms of θ, a, A (resp., θ, b, B)

(2) one knows a particular explicit bound on the second differential $d^2\Theta(x)$ for x in a neighborhood of x^{ref};

(3) for every x in a neighborhood of x^{ref}, the differential $d\Theta(x)$ has a right inverse $d\Theta(x)^{-1} : F_{b_1} \to E_{a_1}$, where $a_1 < b_1$, that satisfies particular explicit estimates, called "tame estimates."

We refer to section 5 to see more precisely what the explicit expressions mentioned above look like (the bounds on the smoothing operators mentioned in (1) are given in (5.1), (5.2), (5.3), (5.4); the bound of the second differential mentioned in (2) is given in (5.6); the tame estimates mentioned in (3) are given in (5.7), (5.8)).

The main differences between the inverse mapping theorem and the Nash–Moser theorem are the following:

- The Nash–Moser theorem needs a weaker surjectivity property on $d\Theta(x^{ref})$. Indeed, the inverse mapping theorem needs the surjectivity of the map $d\Theta(x^{ref}) : E_{a_1} \to F_{a_1}$ (with the same index a_1 in both sides; i.e., the nonlinear map Θ has to be C^1 between E_{a_1} and F_{a_1}), whereas the Nash–Moser theorem needs the existence of a right inverse $d\Theta(x^{ref})^{-1}$ defined on a space F_{b_1} strictly included in F_{a_1}, with values in E_{a_1}. Thus $d\Theta(x^{ref})$ does not need to be surjective from E_{a_1} to F_{a_1}.
- The Nash–Moser theorem needs a surjectivity property for all of the differentials $d\Theta(x)$ for x in a neighborhood of x^{ref}, whereas the inverse mapping theorem requires the surjectivity of the differential $d\Theta(x)$ only at the point $x = x^{ref}$. This assumption is often the most difficult to check in the applications of this theorem.

The rest of the paper is organized as follows.

In section 2, we give the definition of solutions for the nonlinear system (Σ), and we recall classical results about the existence, the uniqueness, and the regularity of these solutions. We also prove bounds on those solutions that will be used many times in this article. Finally, we define spaces E_a and F_b such that the map Φ_T defined by (1.8) is a C^1 map from E_a to F_a for every a.

Section 3 is devoted to the proof of Theorem 3. In subsection 3.1, we state and prove some preliminary results about the eigenvalues and eigenvectors of the operator A defined by (1.1). In subsection 3.2, we prove Theorem 3.

In section 4, we explain in detail why Theorem 1 cannot be deduced from the first statement of Theorem 3 by applying the classical inverse mapping theorem to the map Φ_T.

In section 5, we state and prove a Nash–Moser implicit function theorem inspired by [20]. The following sections are dedicated to the verification of each assumption of this theorem, i.e., points (1), (2), and (3) presented above.

In section 6, we present a construction of smoothing operators S_θ, on the spaces E_a and F_b (defined in section 2), with the desired explicit bounds in terms of θ, a, and A for the linear map $S_\theta : E_a \to E_A$, $a < A$ (i.e., point (1)).

In section 7, we prove a bound on the second differential $d^2\Phi_T$ (i.e., point (2)).

In section 8, we prove the existence of a right inverse for $d\Phi_T(u_0^{ref}, \dot{u}_0^{ref}, 0)$ with tame estimates (i.e., the case $x = x^{ref}$ in point (3)).

In section 9, we prove the existence of a right inverse for $d\Phi_T(u_0, \dot{u}_0, p)$ with tame estimates, for every $(u_0, \dot{u}_0, p)$ in a small neighborhood of $(u_0^{ref}, \dot{u}_0^{ref}, 0)$ (i.e., point (3)). This part of the proof is the most technical one.

Finally, in section 10, we give some remarks, conjectures, and prospects.

In this work, we use the same letter C to denote different constants. The value of C can change from one expression to another.

2. Regularity and bound for the solutions of the nonlinear system.

This section is dedicated to the statement of existence, uniqueness, regularity results, and bounds for the solutions of the Cauchy problem

$$(CY) : \begin{cases} u_{tt} + u_{xxxx} + p(t)u_{xx} + f(t) = 0, \ x \in (0,1), t \in \mathbb{R}_+, \\ u = u_x = 0 \text{ at } x = 0, 1, \\ u(0,x) = u_0(x), \dot{w}(0,x) = \dot{u}_0(x). \end{cases}$$

These bounds, presented in subsection 2.1, will be used many times in this article. Then, in subsection 2.2, we deduce the spaces E_a and F_b between which the map Φ_T defined by (1.8) is of class C^1.

All of these results are classical, but we give proofs for the sake of completeness. The reading of the proofs in this section is not necessary for the understanding of the next sections of this article.

2.1. Existence, uniqueness, regularity, and bounds.

We introduce the first order Cauchy problem

$$(\mathcal{CY}) : \begin{cases} \frac{dw}{dt} = -\mathcal{A}w - p(t)\mathcal{B}w + F(t), \\ w(0) = w_0, \end{cases}$$

where $\mathcal{A}$ and $\mathcal{B}$ are linear operators with domains

$$D(\mathcal{A}) := H^4_{(0)} \times H^2_0((0,1), \mathbb{R}), \qquad D(\mathcal{B}) := H^2_0 \times L^2((0,1), \mathbb{R})$$

defined by

$$\mathcal{A}\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} := \begin{pmatrix} -w^2 \\ w^1_{xxxx} \end{pmatrix}, \qquad \mathcal{B}\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} := \begin{pmatrix} 0 \\ w^1_{xx} \end{pmatrix},$$

and $F : (0,T) \to H^2_0 \times L^2((0,1),\mathbb{R})$.

The Cauchy problem (CY) is equivalent to the Cauchy problem $(\mathcal{CY})$ with $w = (u, u_t)$, $w_0 = (u_0, \dot{u}_0)$, and $F = (0, -f)$. Thus, in this section, we work only on $(\mathcal{CY})$.

The operator $\mathcal{A}$ generates a C^0-group of isometries of $H^2_0 \times L^2((0,1),\mathbb{R})$ with the following explicit: expression

$$(2.1)\quad e^{-t\mathcal{A}}\begin{pmatrix} u_0 \\ \dot{u}_0 \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{\infty}\left(\langle u_0,\varphi_k\rangle \cos(\sqrt{\lambda_k}t) + \frac{1}{\sqrt{\lambda_k}}\langle \dot{u}_0,\varphi_k\rangle \sin(\sqrt{\lambda_k}t)\right)\varphi_k \\ \sum_{k=1}^{\infty}\left(-\sqrt{\lambda_k}\langle u_0,\varphi_k\rangle \sin(\sqrt{\lambda_k}t) + \langle \dot{u}_0,\varphi_k\rangle \cos(\sqrt{\lambda_k}t)\right)\varphi_k \end{pmatrix}$$

for every $(u_0, \dot{u}_0) \in H^2_0 \times L^2((0,1),\mathbb{R})$, where $\langle .,. \rangle$ denotes the usual $L^2((0,1),\mathbb{R})$-scalar product.

Remark 1. With this explicit expression of the C^0-group generated by $\mathcal{A}$, it is clear that $e^{-\mathcal{A}t}$ is an isometry of $H^{s+2}_{(0)} \times H^s_{(0)}((0,1),\mathbb{R})$ for every $s \in \mathbb{N}^*$ (with the definition (1.2) and (1.3)) as claimed in section 1.2.

PROPOSITION 1. *Let $T > 0$, $p \in L^1((0,T),\mathbb{R})$, $w_0 \in H^2_0 \times L^2((0,1),\mathbb{R})$, and $F \in L^1((0,T), H^2_0 \times L^2((0,1),\mathbb{R}))$. There exists a unique weak solution of $(\mathcal{CY})$, i.e., a function $w \in C^0([0,T], H^2_0 \times L^2((0,1),\mathbb{R}))$, such that the following equality holds in $H^2_0 \times L^2((0,1),\mathbb{R})$ for every $t \in [0,T]$:*

$$(2.2)\qquad w(t) = e^{-t\mathcal{A}}w_0 + \int_0^t e^{-(t-s)\mathcal{A}}[-p(s)\mathcal{B}w(s) + F(s)]ds.$$

Moreover, it satisfies

$$(2.3)\qquad \|w\|_{C^0([0,T],H^2_0\times L^2)} \leqslant \left(\|w_0\|_{H^2_0\times L^2} + \|F\|_{L^1((0,T),H^2_0\times L^2)}\right)e^{\|p\|_{L^1((0,T),\mathbb{R})}}.$$

Proof of Proposition 1. The existence and uniqueness result comes from a fixed point argument on the map $\Theta : C^0([0,T], H^2_0 \times L^2) \to C^0([0,T], H^2_0 \times L^2)$, $\Theta(\xi) = w$, where w is defined by

$$w(t) = e^{-\mathcal{A}t}w_0 + \int_0^t e^{-\mathcal{A}(t-s)}[-p(s)\mathcal{B}\xi(s) + F(s)]ds.$$

When $\|p\|_{L^1((0,T),\mathbb{R})}$ is small enough, Θ is a contraction of $C^0([0,T], H^2_0 \times L^2)$, and thus it has a unique fixed point $w \in C^0([0,T], H^2_0 \times L^2)$ that satisfies (2.2). If $\|p\|_{L^1((0,T),\mathbb{R})}$ is not small enough, one may use $0 = T_0 < T_1 < \cdots < T_n = T$ where, for $i = 0, \dots, n-1$, $\|p\|_{L^1((T_i,T_{i+1}),\mathbb{R})}$ is small enough so that the previous result holds on $[T_i, T_{i+1}]$ for $i = 0, \dots, n-1$. Then we glue the solutions defined on $[T_0, T_1]$, $[T_1, T_2], \dots, [T_{n-1}, T_n]$. We deduce from the equality (2.2) that

$$\|w(t)\|_{H^2_0\times L^2} \leqslant \|w_0\|_{H^2_0\times L^2} + \int_0^t [|p(s)|\|w(s)\|_{H^2_0\times L^2} + \|F(s)\|_{H^2_0\times L^2}]ds,$$

and thus

$$\|w(t)\|_{H^2_0\times L^2} \leqslant \|w_0\|_{H^2_0\times L^2} + \|F\|_{L^1((0,T),H^2_0\times L^2)} + \int_0^t |p(s)|\|w(s)\|_{H^2_0\times L^2}ds,$$

and Gronwall's Lemma gives (2.3). □

PROPOSITION 2. *Let $T > 0$, $p \in W^{1,1}((0,T),\mathbb{R})$, $w_0 \in H^4_{(0)} \times H^2_0((0,1),\mathbb{R})$, and $F \in W^{1,1}((0,T), H^2_0 \times L^2((0,1),\mathbb{R}))$. The solution w of (2.2) belongs to $C^1([0,T], H^2_0 \times L^2)$ and to $C^0([0,T], H^4_{(0)} \times H^2_0)$. Moreover, for every $r > 0$, there exists $C(r) > 0$ such that, when $\|p\|_{W^{1,1}} \leqslant r$, the quantities*

$$\|w\|_{C^0([0,T],H^4_{(0)}\times H^2_0)} \text{ and } \|w\|_{C^1([0,T],H^2_0\times L^2)} \tag{2.4}$$

are bounded by

$$C(r)[\|w_0\|_{H^4_{(0)}\times H^2_0} + \|F\|_{W^{1,1}((0,T),H^2_0\times L^2)}]. \tag{2.5}$$

Proof of Proposition 2. Under the assumptions of Proposition 2, one can prove, by using the equality (2.2), that $w \in C^1([0,T], H^2_0 \times L^2)$ and that

$$\frac{dw}{dt}(t) = -\mathcal{A}w(t) - p(t)\mathcal{B}w(t) + F(t) \text{ in } H^2_0 \times L^2((0,1),\mathbb{R}) \ \forall t \in [0,T]. \tag{2.6}$$

We also have

$$\begin{aligned}\frac{dw}{dt}(t) &= e^{-\mathcal{A}t}[-\mathcal{A}w_0 - p(0)\mathcal{B}w_0 + F(0)] \\ &\quad + \int_0^t e^{-\mathcal{A}(t-s)}\left[-p(s)\mathcal{B}\frac{dw}{dt}(s) - \dot{p}(s)\mathcal{B}w(s) + \dot{F}(s)ds\right].\end{aligned}$$

By applying Proposition 1 to $\frac{dw}{dt}$, we get $\frac{dw}{dt} \in C^0([0,T], H^2_0 \times L^2)$ and

$$\begin{aligned}\left\|\frac{dw}{dt}\right\|_{C^0([0,T],H^2_0\times L^2)} &\leqslant C\Big[\|w_0\|_{H^4_{(0)}\times H^2_0} + |p(0)|\|w_0\|_{H^2_0\times L^2} + |F(0)|_{H^2_0\times L^2} \\ &\quad + \|\dot{p}\|_{L^1}\|w\|_{C^0([0,T],H^2_0\times L^2)} + \|\dot{F}\|_{L^1((0,T),H^2_0\times L^2)}\Big]e^{\|p\|_{L^1}}.\end{aligned}$$

Therefore, by using (2.3), we get a universal constant $C_1 > 0$ such that

$$\begin{aligned}\left\|\frac{dw}{dt}\right\|_{C^0([0,T],H^2_0\times L^2)} &\leqslant C_1[\|w_0\|_{H^4_{(0)}\times H^2_0} + \|p\|_{W^{1,1}}\|w_0\|_{H^2_0\times L^2} + |F(0)|_{H^2_0\times L^2} \\ &\quad + \|F\|_{W^{1,1}((0,T),H^2_0\times L^2)} + \|p\|_{W^{1,1}}\|F\|_{L^1((0,T),H^2_0\times L^2)}]e^{2\|p\|_{L^1}}.\end{aligned} \tag{2.7}$$

Then by using (2.6), we get $w \in C^0([0,T], H^4_{(0)} \times H^2_0((0,1),\mathbb{R}))$ and

$$\begin{aligned}\|\mathcal{A}w\|_{C^0([0,T],H^2_0\times L^2)} &= \left\|\frac{dw}{dt} + p\mathcal{B}w - F\right\|_{C^0([0,T],H^2_0\times L^2)} \\ &\leqslant \left\|\frac{dw}{dt}\right\|_{C^0([0,T],H^2_0\times L^2)} + \|p\|_{W^{1,1}}\|w\|_{C^0([0,T],H^2_0\times L^2)} \\ &\quad + \|F\|_{C^0([0,T],H^2_0\times L^2)}.\end{aligned}$$

Thanks to (2.7) and (2.3), the quantities in (2.4) are bounded by

$$\begin{aligned}C'(r)[&\|w_0\|_{H^4_{(0)}\times H^2_0} + \|p\|_{W^{1,1}}\|w_0\|_{H^2_0\times L^2} + \|F\|_{W^{1,1}((0,T),H^2_0\times L^2)} \\ &+ \|p\|_{W^{1,1}}\|F\|_{L^1((0,T),H^2_0\times L^2)}]\end{aligned}$$

when $\|p\|_{L^1((0,T),\mathbb{R})} \leqslant r$, where $C'(r) > 0$. Finally, (2.5) is a direct consequence of the previous inequality. ∎

PROPOSITION 3. *Let $T > 0$, $p \in W^{2,1}((0,T),\mathbb{R})$, with $p(0) = p(T) = 0$, $w_0 \in H^6_{(0)} \times H^4_{(0)}((0,1),\mathbb{R})$, and $F \in W^{2,1}((0,T), H^2_0 \times L^2) \cap C^0([0,T], H^4 \times H^2)$, with $F(0), F(T) \in H^4_{(0)} \times H^2_{(0)}((0,1),\mathbb{R})$. The solution w of (2.2) belongs to $C^2([0,T], H^2_0 \times L^2)$, $C^1([0,T], H^4_{(0)} \times H^2_{(0)})$, $C^0([0,T], H^6 \times H^4)$, and $w(T) \in H^6_{(0)} \times H^4_{(0)}$. Moreover, for every $r > 0$, there exists $C(r) > 0$ such that, when $\|p\|_{W^{1,1}} \leqslant r$, the quantities*

$$\|w\|_{C^0([0,T],H^6\times H^4)}, \|w\|_{C^1([0,T],H^4\times H^2)}, \text{ and } \|w\|_{C^2([0,T],H^2_0\times L^2)}$$

are bounded by

$$\begin{aligned} C(r)[\|w_0\|_{H^6_{(0)}\times H^4_{(0)}} + \|p\|_{W^{2,1}}\|w_0\|_{H^2_0\times L^2} + \|F\|_{W^{2,1}((0,T),H^2_0\times L^2)} \\ + \|p\|_{W^{2,1}}\|F\|_{L^1((0,T),H^2_0\times L^2)} + \|F\|_{C^0([0,T],H^4\times H^2)}]. \end{aligned} \tag{2.8}$$

Proof of Proposition 3. By applying Proposition 2 to $\frac{dw}{dt}$, we get $\frac{dw}{dt} \in C^0([0,T], H^4_{(0)} \times H^2_0) \cap C^1([0,T], H^2_0 \times L^2)$. Notice that the assumptions $p(0) = 0$ and $F(0) \in H^4_{(0)} \times H^2_{(0)}((0,1),\mathbb{R}))$ are useful to ensure that the initial condition

$$\frac{dw}{dt}(0) = -\mathcal{A}w_0 - p(0)\mathcal{B}w_0 + F(0)$$

belongs to $H^4_{(0)} \times H^2_0$. Indeed, when $w = (w_1, w_2) \in H^6_{(0)} \times H^4_{(0)}((0,1),\mathbb{R})$, then $w_1'' \in H^2((0,1),\mathbb{R})$, but in general w_1'' does not vanish at $x = 0, 1$, and thus $\mathcal{B}w$ does not belong to $H^4_{(0)} \times H^2_0((0,1),\mathbb{R})$. Proposition 2 also gives the existence of a constant $C(r) > 0$ such that, when $\|p\|_{W^{1,1}((0,T),\mathbb{R})} \leqslant r$, the quantities

$$\left\|\frac{dw}{dt}\right\|_{C^0([0,T],H^4_{(0)}\times H^2_0)} \text{ and } \left\|\frac{dw}{dt}\right\|_{C^1([0,T],H^2_0\times L^2)} \tag{2.9}$$

are bounded by

$$\begin{aligned} &C\left[\left\|\frac{dw}{dt}(0)\right\|_{H^4_{(0)}\times H^2_0} + \|\dot{p}w - \dot{F}\|_{W^{1,1}((0,T),H^2_0\times L^2)}\right] \\ &\leqslant C\Bigg[\|w_0\|_{H^6_{(0)}\times H^4_{(0)}} + |F(0)|_{H^4_{(0)}\times H^2_{(0)}} + \|p\|_{W^{2,1}}\|w\|_{C^0([0,T],H^2_0\times L^2)} \\ &\qquad + \|p\|_{W^{1,1}}\left\|\frac{dw}{dt}\right\|_{C^0([0,T],H^2_0\times L^2)} + \|F\|_{W^{2,1}((0,T),H^2_0\times L^2)}\Bigg]. \end{aligned}$$

By using (2.3) and (2.5), we get the following bounds for the quantities written in (2.9):

$$\begin{aligned} C\Big[&\|w_0\|_{H^6\times H^4} + \|p\|_{W^{2,1}}\|w_0\|_{H^2_0\times L^2} + \|F(0)\|_{H^4_{(0)}\times H^2_0} \\ &\|F\|_{W^{2,1}((0,T),H^2_0\times L^2)} + \|p\|_{W^{2,1}}\|F\|_{L^1((0,T),H^2_0\times L^2)}\Big]. \end{aligned} \tag{2.10}$$

Now, by using (2.6), we deduce that $\mathcal{A}w \in C^0([0,T], H^4 \times H^2)$ and $\mathcal{A}w(T) \in H^4_{(0)} \times H^2_0$ because $p(T) = 0$ and $F(T) \in H^4_{(0)} \times H^2_0$. Moreover,

$$\begin{aligned} \|\mathcal{A}w\|_{C^0([0,T],H^4\times H^2)} \leqslant &\left\|\frac{dw}{dt}\right\|_{C^0([0,T],H^4_{(0)}\times H^2_0)} + \|p\|_{W^{1,1}}\|w\|_{C^0([0,T],H^4_{(0)}\times H^2_0)} \\ &+ \|F\|_{C^0([0,T],H^4\times H^2)}. \end{aligned}$$

Thanks to the previous inequality, (2.10), and (2.5), we get (2.8). ∎

PROPOSITION 4. *Let $T > 0$, $p \in W^{3,1}((0,T),\mathbb{R})$, with $p(0) = p(T) = \dot{p}(0) = \dot{p}(T) = 0$, $w_0 \in H^8_{(0)} \times H^6_{(0)}((0,1),\mathbb{R})$, and $F \in W^{3,1}((0,T), H^2_0 \times L^2) \cap C^1([0,T], H^4 \times H^2) \cap C^0([0,T], H^6 \times H^4)$, with $F(0), F(T) \in H^6_{(0)} \times H^4_{(0)}((0,1),\mathbb{R})$, $\dot{F}(0), \dot{F}(T) \in H^4_{(0)} \times H^2_{(0)}((0,1),\mathbb{R})$. The solution w of (2.2) belongs to $C^3([0,T], H^2_0 \times L^2)$, $C^2([0,T], H^4_{(0)} \times H^2_0)$, $C^1([0,T], H^6 \times H^4)$, $C^0([0,T], H^8 \times H^6)$, and $w(T) \in H^8_{(0)} \times H^6_{(0)}$. Moreover, for every $r > 0$, there exists $C(r) > 0$ such that, when $\|p\|_{W^{1,1}} \leqslant r$, the quantities*

$$\|w\|_{C^0([0,T],H^8\times H^6)}, \|w\|_{C^1([0,T],H^6\times H^4)}, \|w\|_{C^2([0,T],H^4\times H^2)}, \text{ and } \|w\|_{C^3([0,T],H^2_0\times L^2)}$$

are bounded by

(2.11)
$$\begin{aligned} C(r)\Big[&\|w_0\|_{H^8\times H^6} + \|p\|_{W^{2,1}}\|w_0\|_{H^4_{(0)}\times H^2} + \|p\|_{W^{3,1}}\|w_0\|_{H^2_0\times L^2} \\ &+ \|F\|_{W^{3,1}((0,T),H^2_0\times L^2)} + \|p\|_{W^{2,1}}\|F\|_{W^{1,1}((0,T),H^2_0\times L^2)} \\ &+ \|p\|_{W^{3,1}}\|F\|_{L^1((0,T),H^2_0\times L^2)} \\ &\|F\|_{C^0([0,T],H^6\times H^4)} + \|p\|_{W^{2,1}}\|F\|_{C^0((0,T),H^2_0\times L^2)} + \|F\|_{C^1([0,T],H^4\times H^2)}\Big]. \end{aligned}$$

Proof of Proposition 4. We apply Proposition 3 to $\frac{dw}{dt}$. □

2.2. Spaces between which Φ_T is C^1. For $T > 0$ fixed, we introduce the spaces

$$\begin{aligned} E_2 &:= H^2_0((0,1),\mathbb{R}) \times L^2((0,1),\mathbb{R}) \times L^2((0,T),\mathbb{R}), \\ E_4 &:= H^4_{(0)}((0,1),\mathbb{R}) \times H^2_0((0,1),\mathbb{R}) \times H^1((0,T),\mathbb{R}), \\ E_6 &:= H^6_{(0)}((0,1),\mathbb{R}) \times H^4_{(0)}((0,1),\mathbb{R}) \times H^2 \cap H^1_0((0,T),\mathbb{R}), \\ E_8 &:= H^8_{(0)}((0,1),\mathbb{R}) \times H^6_{(0)}((0,1),\mathbb{R}) \times H^3 \cap H^2_0((0,T),\mathbb{R}), \end{aligned} \tag{2.12}$$

and, for $s > 0$,

$$F_s := H^s_{(0)}((0,1),\mathbb{R}) \times H^{s-2}_{(0)}((0,1),\mathbb{R}) \times H^s_{(0)}((0,1),\mathbb{R}) \times H^{s-2}_{(0)}((0,1),\mathbb{R}), \tag{2.13}$$

where $H^s_{(0)}((0,1),\mathbb{R})$ is defined by (1.2). Notice that the spaces E_a depend on T and should be called $E_{a,T}$. However, since no confusion is possible, we omit the subscript T in order to simplify the notations.

PROPOSITION 5. *For every $T > 0$ and for every $a \in \{2,4,6,8\}$, the map Φ_T defined by* (1.8) *is C^1 from E_a to F_a, and, for every $(u_0, \dot{u}_0, p) \in E_a$, $d\Phi_T(u_0,\dot{u}_0,p).(U_0, \dot{U}_0, P) = (U_0, \dot{U}_0, U(T), U_t(T))$, where U is the weak solution of*

$$\begin{cases} U_{tt} + U_{xxxx} + p(t)U_{xx} + P(t)u_{xx} = 0, x \in (0,1), t \in [0,T], \\ U = U_x = 0 \text{ at } x = 0,1, \\ U(0,x) = U_0(x), \\ U_t(0,x) = \dot{U}_0(x), \end{cases} \tag{2.14}$$

and u is the weak solution of

$$\begin{cases} u_{tt} + u_{xxxx} + p(t)u_{xx} = 0, x \in (0,1), t \in [0,T], \\ u = u_x = 0 \text{ at } x = 0,1, \\ u(0,x) = u_0(x), \\ u_t(0,x) = \dot{u}_0(x). \end{cases} \tag{2.15}$$

Proof of Proposition 5. By using Propositions 1, 2, and 3 we see that $\Phi_T : E_a \to F_a$ is continuous for $a = 2,4,6$. Let us prove that $\Phi_T : E_2 \to F_2$ is C^1 (the cases $a =$

$4, 6, 8$ can be treated in the same way). Let $(u_0, \dot{u}_0, p), (U_0, \dot{U}_0, P) \in E_2$, u be the weak solutions of (2.15), U be the weak solutions of (2.14), and $\tilde{u}$ be the weak solution of

$$\begin{cases} \tilde{u}_{tt} + \tilde{u}_{xxxx} + (p+P)(t)\tilde{u}_{xx} = 0, x \in (0,1), t \in [0,T], \\ \tilde{u} = \tilde{u}_x = 0 \text{ at } x = 0, 1, \\ \tilde{u}(0,x) = (u_0 + U_0)(x), \\ \tilde{u}_t(0,x) = (\dot{u}_0 + \dot{U}_0)(x). \end{cases}$$

Then $\Delta := \tilde{u} - u - U$ is the weak solution of

$$\begin{cases} \Delta_{tt} + \Delta_{xxxx} + p(t)\Delta_{xx} + P(t)(\tilde{u} - u)_{xx} = 0, x \in (0,1), t \in [0,T], \\ \Delta = \Delta_x = 0 \text{ at } x = 0, 1, \\ \Delta(0,x) = 0, \\ \Delta_t(0,x) = 0. \end{cases}$$

Thus, Proposition 1 gives

$$\begin{aligned} \|\Delta(T)\|_{H_0^2} + \|\Delta_t(T)\|_{L^2} &\leqslant \|P(t)(\tilde{u} - u)_{xx}\|_{L^1((0,T),L^2)} e^{\|p\|_{L^1}} \\ &\leqslant C\|P\|_{L^1}\|\tilde{u} - u\|_{C^0([0,T],H^2)}. \end{aligned} \tag{2.16}$$

Moreover, $\tilde{u} - u$ is the weak solution of

$$\begin{cases} (\tilde{u} - u)_{tt} + (\tilde{u} - u)_{xxxx} + p(t)(\tilde{u} - u)_{xx} + P(t)\tilde{u}_{xx} = 0, x \in (0,1), t \in [0,T], \\ \tilde{u} - u = (\tilde{u} - u)_x = 0 \text{ at } x = 0, 1, \\ (\tilde{u} - u)(0,x) = U_0(x), \\ (\tilde{u} - u)_t(0,x) = \dot{U}_0(x). \end{cases}$$

Thus, Proposition 1 gives

$$\|\tilde{u} - u\|_{C^0([0,T],H^2)} \leqslant \Big(\|(U_0, \dot{U}_0)\|_{H^2 \times L^2} + \|P\|_{L^1}\|\tilde{u}_{xx}\|_{C^0([0,T],L^2)}\Big) e^{\|p\|_{L^1}}. \tag{2.17}$$

Again, thanks to Proposition 1, we have

$$\|\tilde{u}\|_{C^0([0,T],H^2)} \leqslant \|(u_0 + U_0, \dot{u}_0 + \dot{U}_0)\|_{H^2 \times L^2} e^{\|p+P\|_{L^1}}. \tag{2.18}$$

Therefore, by using (2.16), (2.17), and (2.18), we get

$$\|\Delta(T)\|_{H_0^2} + \|\Delta_t(T)\|_{L^2} = o\Big(\|(U_0, \dot{U}_0, P)\|_{E_2}\Big)$$

when $\|(U_0, \dot{U}_0, P)\|_{E_2} \to 0$. This proves that Φ_T is differentiable at $(u_0, \dot{u}_0, p)$ and that

$$d\Phi_T(u_0, \dot{u}_0, p).(U_0, \dot{U}_0, P) = (U_0, \dot{U}_0, U(T), U_t(T)).$$

The continuity of the map

$$\begin{array}{ccc} E_2 & \to & \mathcal{L}(E_2, F_2) \\ (u_0, \dot{u}_0, p) & \mapsto & d\Phi_T(u_0, \dot{u}_0, p) \end{array}$$

is a consequence of the estimate (2.3). ◻

3. Controllability of the linearized system around $(u^{ref}, \dot{u}^{ref}, p \equiv 0)$. The aim of this section is the proof of Theorem 3. First, in subsection 3.1, we prove some preliminary results, mainly about the eigenvalues and the eigenvectors of the operator A defined by (1.1). The reading of the proofs in this subsection is not necessary for the understanding of the next sections. Then, in subsection 3.2, we prove Theorem 3.

3.1. Preliminaries.

PROPOSITION 6. *The eigenvalues* $(\lambda_n)_{n\in\mathbb{N}^*}$ *of the operator* A *are the numbers*

$$\lambda_n = \nu_n^4, \tag{3.1}$$

where $(\nu_n)_{n\in\mathbb{N}^*}$ *is the increasing sequence of positive solutions of the equation*

$$\cos(\nu_n)\cosh(\nu_n) = 1. \tag{3.2}$$

We have, for every $k \in \mathbb{N}^*$*,*

$$\nu_{2k-1} \in (2k\pi - \pi/2, 2k\pi - \pi/4), \quad \nu_{2k} \in (2k\pi + \pi/4, 2k\pi + \pi/2). \tag{3.3}$$

We have, for every $n \in \mathbb{N}^*$*,*

$$\nu_n = \frac{\pi}{2}(2n+1) - (-1)^n x_n, \text{ where } 0 < x_n < \frac{\pi}{2\cosh(\nu_n)}, \tag{3.4}$$

$$\left|\sqrt{\lambda_n} - \frac{\pi^2}{4}K_n\right| < \frac{\pi^2}{16}, \tag{3.5}$$

where, for every $n \in \mathbb{N}^*$*,*

$$K_n := (2n+1)^2. \tag{3.6}$$

For every $n \in \mathbb{N}^*$*, the function*

$$v_n := \xi_n(\cos(\nu_n x) - \cosh(\nu_n x)) + \zeta_n(\sin(\nu_n x) - \sinh(\nu_n x)), \tag{3.7}$$

where

$$\xi_n := \sin(\nu_n) - \sinh(\nu_n), \quad \zeta_n := -\cos(\nu_n) + \cosh(\nu_n), \tag{3.8}$$

is an eigenvector of A *associated to the eigenvalue* λ_n *and satisfies*

$$v_n(1-x) = (-1)^n v_n(x). \tag{3.9}$$

Moreover, there exists a constant $C \in \mathbb{R}^*$ *such that, when* $n \to +\infty$*,*

$$\|v_n\|_{L^2((0,1),\mathbb{R})} \sim Ce^{\nu_n}. \tag{3.10}$$

Proof of Proposition 6. The relation (3.2) comes from the condition $v = v' = 0$ on $x = 0, 1$ imposed on any eigenvector v of the operator A. The intermediate values theorem gives (3.3). Equation (3.2) provides, for every $n \in \mathbb{N}^*$,

$$\sin(x_n)\cosh(\nu_n) = 1, \tag{3.11}$$

which gives (3.4) thanks to the convexity inequality

$$x \leqslant \frac{\pi}{2}\sin(x) \ \forall x \in [0, \pi/2]. \tag{3.12}$$

For every $n \in \mathbb{N}^*$, we have

$$\sqrt{\lambda_n} - \frac{\pi^2}{4}K_n = \pi(2n+1)x_n - (-1)^n x_n^2.$$

Thus, we just need to justify that, for every $n \in \mathbb{N}^*$,

$$\frac{2(2n+1)}{\cosh \nu_n} + \frac{1}{\cosh(\nu_n)^2} < \frac{1}{4}.$$

This can be proved for $n = 1$ by using $\nu_1 > 3\pi/2$ and for $n \geqslant 2$ by using $\nu_n \geqslant n\pi$. The property (3.9) comes from the explicit expression (3.7) together with (3.2) and the relation $\sin(\nu_n) = (-1)^n\sqrt{1-\cos(\nu_n)^2}$, which is a consequence of (3.3). By using (3.7) and a change of variable, we get

$$\int_0^1 v_n(x)^2 dx = \frac{\xi_n^2}{\nu_n} I_1(\nu_n) + \frac{\zeta_n^2}{\nu_n} I_2(\nu_n) + \frac{2\xi_n\zeta_n}{\nu_n} I_3(\nu_n),$$

where

$$\begin{aligned}
I_1(\nu) &:= \int_0^\nu [\cos(y) - \cosh(y)]^2 dy \\
&= \frac{\sinh(2\nu)}{4} + \nu - \sin(\nu)\cosh(\nu) - \cos(\nu)\sinh(\nu) + \frac{\sin(2\nu)}{4}, \\
I_2(\nu) &:= \int_0^\nu [\sin(y) - \sinh(y)]^2 dy \\
&= \frac{\sinh(2\nu)}{4} - \sin(\nu)\cosh(\nu) + \cos(\nu)\sinh(\nu) - \frac{\sin(2\nu)}{4}, \\
I_3(\nu) &:= \int_0^\nu [\cos(y) - \cosh(y)][\sin(y) - \sinh(y)] dy \\
&= \frac{\cosh(2\nu)}{4} - \sin(\nu)\sinh(\nu) - \frac{\cos(2\nu)}{4}.
\end{aligned}$$

By using the following behaviors, when $n \to +\infty$,

$$\begin{array}{ll} \cos(\nu_n) = O(e^{-\nu_n}), & \sin(\nu_n) = (-1)^n + O(e^{-2\nu_n}), \\ \sin(2\nu_n) = O(e^{-\nu_n}), & \cos(2\nu_n) = -1 + O(e^{-2\nu_n}), \end{array} \tag{3.13}$$

which are consequences of (3.4), we get

$$\int_0^1 v_n(x)^2 dx \sim Ce^{2\nu_n}. \qquad \square$$

The orthonormal basis $(\varphi_n)_{n\in\mathbb{N}^*}$ of $L^2((0,1),\mathbb{R})$ made of eigenvectors of A has been introduced in section 1.1. Up to a change of sign, one may assume that

$$\varphi_n = \frac{v_n}{\|v_n\|_{L^2((0,1),\mathbb{R})}}. \tag{3.14}$$

This equality will be assumed in the remainder of this article.

PROPOSITION 7. *Let $m \in \mathbb{N}^*$. For every $n \in \mathbb{N}^*$, we have*

$$\langle \varphi_m'', \varphi_n \rangle \neq 0 \text{ iff } m \text{ and } n \text{ have the same parity.} \tag{3.15}$$

Moreover, there exists a constant C_m such that, when n tends to $+\infty$ with the same parity as m,

$$\langle \varphi_m'', \varphi_n \rangle \sim \frac{C_m}{n}. \tag{3.16}$$

Proof of Proposition 7. When m and n have different parities, the equality $\langle \varphi_m'', \varphi_n \rangle = 0$ comes from (3.14) and (3.9). We have

$$\langle \varphi_m'', \varphi_m \rangle = -\int_0^1 |\varphi_m'(x)|^2 dx < 0.$$

Let $n \in \mathbb{N}^*$ with the same parity as m and different from m. Thanks to integrations by parts and (3.9), we get

$$(\lambda_n - \lambda_m)\langle v_m'', v_n \rangle = 2(v_m''' v_n'' - v_m'' v_n''')(0).$$

The explicit expression (3.7) leads to

$$(\lambda_n - \lambda_m)\langle v_m'', v_n \rangle = 8\nu_m^2 \nu_n^2 (-\xi_m \nu_n \zeta_n + \zeta_m \nu_m \xi_n). \tag{3.17}$$

The function $f : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$f(x) := \frac{\sinh(x) - \sin(x)}{x(\cosh(x) - \cos(x))}$$

decreases on $\mathbb{R}_+$. If $n < m$ (resp., $n > m$), the inequality $f(\nu_n) > f(\nu_m)$ (resp., $f(\nu_n) < f(\nu_m)$) provides $-\xi_m \nu_n \zeta_n + \zeta_m \nu_m \xi_n < 0$ (resp., > 0). Thus $\langle v_m'', v_n \rangle \neq 0$. Thanks to (3.8), we get

$$\langle v_m'', v_n \rangle \sim -\frac{4\nu_m^2 \xi_m}{\nu_n} e^{\nu_n} \text{ when } n \to +\infty, \text{ with the same parity as } m, \tag{3.18}$$

which together with (3.10) gives the asymptotic behavior (3.16). □

At this step, one can justify the following property, claimed in section 1.2.

PROPOSITION 8. *The operator $\mathcal{A}$ defined by* (1.7) *and* (1.1) *does not generate a C^0-semigroup of bounded operators of $\overline{X} := H^3_{(0)} \times H^1((0,1),\mathbb{R})$.*

Proof of Proposition 8. We argue by contradiction. Let us assume that $\mathcal{A}$ generates a C^0-semigroup of bounded operators of $\overline{X}$. Then there exists $m > 0$ such that, for every $t \in [0,1]$, for every $(u_0, \dot{u}_0) \in H^3_{(0)} \times H^1((0,1),\mathbb{R})$,

$$e^{-t\mathcal{A}} \begin{pmatrix} u_0 \\ \dot{u}_0 \end{pmatrix} \in H^3_{(0)} \times H^1((0,1),\mathbb{R})$$

and

$$\left\| e^{-t\mathcal{A}} \begin{pmatrix} u_0 \\ \dot{u}_0 \end{pmatrix} \right\|_{H^3_{(0)} \times H^1} \leqslant m \|(u_0, \dot{u}_0)\|_{H^3_{(0)} \times H^1}. \tag{3.19}$$

Let us consider $(u_0, \dot{u}_0) \in H^3_{(0)} \times H^1((0,1),\mathbb{R})$ defined by $u_0 := 0$ and $\dot{u}_0 := \varphi_1''$. By using (2.1), we get

$$e^{-t\mathcal{A}} \begin{pmatrix} u_0 \\ \dot{u}_0 \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{\infty} \frac{1}{\sqrt{\lambda_k}} \langle \varphi_1'', \varphi_k \rangle \sin(\sqrt{\lambda_k} t) \varphi_k \\ \sum_{k=1}^{\infty} \langle \varphi_1'', \varphi_k \rangle \cos(\sqrt{\lambda_k} t) \varphi_k \end{pmatrix}.$$

Thanks to (3.19), the $H^3_{(0)}((0,1),\mathbb{R})$-norm of the first component of the right-hand side is bounded by $m\|\varphi_1''\|_{H^1}$ for every $t \in [0,1]$; i.e., (see (1.3) for the definition of the $H^3_{(0)}((0,1),\mathbb{R})$-norm)

$$\sum_{k=1}^{\infty} |\lambda_k^{1/4} \langle \varphi_1'', \varphi_k \rangle \sin(\sqrt{\lambda_k} t)|^2 \leqslant m^2 \|\varphi_1''\|_{H^1}^2 \ \forall t \in [0,1]. \tag{3.20}$$

Thanks to (3.1), (3.3), and Proposition 7, there exists an odd integer $N_0 \in \mathbb{N}^*$ and $C_0 > 0$ such that

$$|\lambda_k^{1/4}\langle\varphi_1'', \varphi_k\rangle|^2 \geqslant C_0 \ \forall k \geqslant N_0, k \text{ odd}. \tag{3.21}$$

Thus, (3.20) and (3.21) imply

$$C_0 \sum_{k=N_0, k \text{ odd}}^{\infty} |\sin(\sqrt{\lambda_k}t)|^2 \leqslant m^2\|\varphi_1''\|_{H^1}^2 \ \forall t \in [0,1]. \tag{3.22}$$

Let $N \in \mathbb{N}^*$ be such that $N \geqslant N_0$ and $\pi/(2\sqrt{\lambda_N}) \leqslant 1$. Let $t_N := \pi/(2\sqrt{\lambda_N})$. For every $k \in \mathbb{N}^*$, odd with $N_0 \leqslant k \leqslant N$, we have $\sqrt{\lambda_k}t_N \in (0, \pi/2)$, and thus $\sin(\sqrt{\lambda_k}t_N) \geqslant (2/\pi)\sqrt{\lambda_k}t_N$. We also have $t_N \in [0,1]$, so we deduce from (3.22) that

$$\frac{1}{2}C_0\left(\frac{2}{\pi}\right)^2 \sum_{k=N_0}^{N} \lambda_k t_N^2 \leqslant m^2\|\varphi_1''\|_{H^1}^2.$$

Thanks to (3.1), (3.3), and the definition of t_N, there exists $C_1 > 0$ such that

$$\sum_{k=N_0}^{N} \lambda_k t_N^2 \geqslant \frac{C_1}{N^4}\sum_{k=N_0}^{N} k^4 \geqslant \frac{C_1}{N^4}\int_{N_0}^{N} x^4 dx \geqslant \frac{C_1}{5N^4}(N^5 - N_0^5).$$

We have proved that, for every $N \in \mathbb{N}^*$ such that $N \geqslant N_0$ and $\pi/(2\sqrt{\lambda_N}) \leqslant 1$, the following inequality holds:

$$\frac{1}{2}C_0\left(\frac{2}{\pi}\right)^2 \frac{C_1}{5N^4}(N^5 - N_0^5) \leqslant m^2\|\varphi_1''\|_{H^1}^2.$$

We get a contradiction by considering large enough N. ∎

LEMMA 1. *The frequencies*

$$0, 2\sqrt{\lambda_2}, 2\sqrt{\lambda_3}, \sqrt{\lambda_3} \pm \sqrt{\lambda_1}, \sqrt{\lambda_4} \pm \sqrt{\lambda_2}, \sqrt{\lambda_{2k}} \pm \sqrt{\lambda_2}, \sqrt{\lambda_{2k-1}} \pm \sqrt{\lambda_3} \quad \text{for } k \in \mathbb{N}^*, k \geqslant 3,$$

are all different.

Proof of Lemma 1. First, we claim that the nonnegative integers

$$0, 2K_2, 2K_3, K_3 \pm K_1, K_4 \pm K_2, K_{2k-1} \pm K_3, K_{2k} \pm K_2 \text{ for } k \in \mathbb{N}^*, k \geqslant 3,$$

where K_n is defined by (3.6), are all different. First, for every $k \geqslant 5$, we have

$$K_{2k-1} + K_3 < K_{2k} - K_2 < K_{2k} + K_2 < K_{2k+1} - K_3.$$

Indeed, for $k \geqslant 5$, we have

$$(K_{2k} - K_2) - (K_{2k-1} + K_3) = (4k+1)^2 - (4k-1)^2 - 25 - 49 \geqslant 16k - 74 \geqslant 6,$$
$$(K_{2k+1} - K_3) - (K_{2k} + K_2) = (4k+3)^2 - (4k+1)^2 - 25 - 49 \geqslant 16k - 66 \geqslant 14.$$

Moreover, we have

$$\begin{array}{llll} 2K_2 = 50, & 2K_3 = 98, & K_3 - K_1 = 40, & K_3 + K_1 = 58, \\ K_4 - K_2 = 56, & K_4 + K_2 = 106, & K_5 - K_3 = 72, & K_5 + K_3 = 170, \\ K_6 - K_2 = 144, & K_6 + K_2 = 194, & K_7 - K_3 = 176, & K_7 + K_3 = 274, \\ K_8 - K_2 = 264, & K_8 + K_2 = 314, & K_9 - K_3 = 312, & \end{array}$$

and thus the claim is proved. We deduce Lemma 1 from the previous result and the inequality (3.5). ∎

3.2. Proof of Theorem 3.

Proof of statement (1) *of Theorem* 3. Since the system (Σ_l^{ref}) is linear and $e^{-T\mathcal{A}}$ is a bounded operator of $H^3_{(0)} \times H^1_0((0,1),\mathbb{R})$, it is sufficient to prove Theorem 3 with $U_0 = \dot{U}_0 = 0$.

Let $T > 0$ and U be a solution of (Σ_l^{ref}) for some $P \in L^2((0,T),\mathbb{R})$, with $U(0) = \dot{U}(0) = 0$. Then $(U, U_t) \in C^0([0,T], H^2_0 \times L^2((0,1),\mathbb{R}))$. The family $(\varphi_k)_{k\in\mathbb{N}^*}$ is an orthonormal basis of $L^2((0,1),\mathbb{R})$, and thus, for every $t \in [0,T]$,

$$U(t) = \sum_{k=1}^{\infty} x_k(t)\varphi_k, \text{ where } x_k(t) := \int_0^1 U(t,x)\varphi_k(x)dx.$$

By recalling that u^{ref} is given by (1.5), the partial differential equation satisfied by U provides, for every $k \in \mathbb{N}^*$, the following explicit expression:

$$x_k(t) = -\frac{1}{\sqrt{\lambda_k}} \int_0^t P(\tau)[b_k \sin(\sqrt{\lambda_2}\tau) + c_k \sin(\sqrt{\lambda_3}\tau)] \sin[\sqrt{\lambda_k}(t-\tau)]d\tau,$$

where, for every $k \in \mathbb{N}^*$,

$$b_k := \langle \varphi_2'', \varphi_k \rangle, \quad c_k := \langle \varphi_3'', \varphi_k \rangle.$$

The equality $(U(T), \dot{U}(T)) = (U_T, \dot{U}_T)$ is equivalent to

$$\begin{aligned}\int_0^T & P(t)[b_k \sin(\sqrt{\lambda_2}t) + c_k \sin(\sqrt{\lambda_3}t)]e^{-i\sqrt{\lambda_k}t}dt \\ &= -e^{-i\sqrt{\lambda_k}T}\left(\langle \dot{U}_T, \varphi_k\rangle + i\sqrt{\lambda_k}\langle U_T, \varphi_k\rangle\right) \ \forall k \in \mathbb{N}^*.\end{aligned} \tag{3.23}$$

Thanks to Proposition 7, the equality (3.23) is satisfied if P solves the following moment problem:

$$\begin{cases} \int_0^T P(t)\sin(\sqrt{\lambda_3}t)e^{-i\sqrt{\lambda_{2k-1}}t}dt = d_{2k-1}(U_T, \dot{U}_T) \ \forall k \in \mathbb{N}^*, \\ \int_0^T P(t)\sin(\sqrt{\lambda_2}t)e^{-i\sqrt{\lambda_{2k}}t}dt = d_{2k}(U_T, \dot{U}_T) \ \forall k \in \mathbb{N}^*, \end{cases} \tag{3.24}$$

where, for every $k \in \mathbb{N}^*$,

$$d_{2k-1}(U_T, \dot{U}_T) := -\frac{e^{-i\sqrt{\lambda_{2k-1}}T}}{c_{2k-1}}\left(\langle \dot{U}_T, \varphi_{2k-1}\rangle + i\sqrt{\lambda_{2k-1}}\langle U_T, \varphi_{2k-1}\rangle\right),$$

$$d_{2k}(U_T, \dot{U}_T) := -\frac{e^{-i\sqrt{\lambda_{2k}}T}}{b_{2k}}\left(\langle \dot{U}_T, \varphi_{2k}\rangle + i\sqrt{\lambda_{2k}}\langle U_T, \varphi_{2k}\rangle\right).$$

The moment problem (3.24) is satisfied, in particular, when

$$\begin{cases} \int_0^T P(t)dt = 0, \\ \int_0^T P(t)e^{-i2\sqrt{\lambda_3}t}dt = -2id_3, \\ \int_0^T P(t)e^{i(-\sqrt{\lambda_{2k-1}}\pm\sqrt{\lambda_3})t}dt = \pm i d_{2k-1}(U_T, \dot{U}_T) \ \forall k \in \mathbb{N}^*, k \neq 2, \\ \int_0^T P(t)e^{-i2\sqrt{\lambda_2}t}dt = -2id_2, \\ \int_0^T P(t)e^{i(-\sqrt{\lambda_{2k}}\pm\sqrt{\lambda_2})t}dt = \pm i d_{2k}(U_T, \dot{U}_T) \ \forall k \in \mathbb{N}^*, k \neq 1. \end{cases} \tag{3.25}$$

Let $(\omega_n)_{n\in\mathbb{N}}$ be the nondecreasing sequence of the frequencies appearing in the previous moment problem, writt en in the following way:

$$\int_0^T P(t)e^{i\omega_n t}dt = \delta_n \ \forall n \in \mathbb{N}.$$

For large enough indexes, the successive terms of the sequence $(\omega_n)_{n\in\mathbb{N}}$ are

$$\sqrt{\lambda_{2k-1}}-\sqrt{\lambda_3} < \sqrt{\lambda_{2k-1}}+\sqrt{\lambda_3} < \sqrt{\lambda_{2k}}-\sqrt{\lambda_2} < \sqrt{\lambda_{2k}}+\sqrt{\lambda_2} < \sqrt{\lambda_{2k+1}}-\sqrt{\lambda_3} < \cdots .$$

The gap between the first and the second terms (resp., the third and the fourth terms) is $2\sqrt{\lambda_3}$ (resp., $2\sqrt{\lambda_2}$). The gap between the second and the third terms (resp., the fourth and the fifth terms) tends to $+\infty$ when $k \to +\infty$; indeed, by using (3.1) and (3.3), we have

$$\begin{aligned}
(\sqrt{\lambda_{2k}} - \sqrt{\lambda_2}) - (\sqrt{\lambda_{2k-1}} + \sqrt{\lambda_3}) &= \nu_{2k}^2 - \nu_{2k-1}^2 - \nu_2^2 - \nu_3^2 \\
&= (\nu_{2k} - \nu_{2k-1})(\nu_{2k} + \nu_{2k-1}) - \nu_2^2 - \nu_3^2 \\
&\geqslant \frac{\pi}{2}\left(4k\pi - \frac{\pi}{4}\right) - \nu_2^2 - \nu_3^2, \\
(\sqrt{\lambda_{2k+1}} - \sqrt{\lambda_3}) - (\sqrt{\lambda_{2k}} + \sqrt{\lambda_2}) &= \nu_{2k+1}^2 - \nu_{2k}^2 - \nu_2^2 - \nu_3^2 \\
&= (\nu_{2k+1} - \nu_{2k})(\nu_{2k+1} + \nu_{2k}) - \nu_2^2 - \nu_3^2 \\
&\geqslant \pi\left(4k\pi + \frac{7\pi}{4}\right) - \nu_2^2 - \nu_3^4.
\end{aligned}$$

Thus,

$$\text{liminf}_{n\to+\infty}(\omega_{n+1} - \omega_n) = 2\sqrt{\lambda_2}.$$

The frequencies appearing in (3.25) are all different (see Lemma 1), and thus the moment problem (3.25) has a solution $P \in L^2((0,T),\mathbb{R})$ when $T > \pi/\sqrt{\lambda_2}$ and $(d_n(U_T, \dot{U}_T))_{n\in\mathbb{N}^*}$ belongs to $l^2(\mathbb{N}^*,\mathbb{C})$ (see [22, Chap. 1.2]). Thanks to (3.16), the assumption that $(U_T, \dot{U}_T) \in H^3_{(0)} \times H^1_0((0,1),\mathbb{R})$ guarantees that $(d_n(U_T, \dot{U}_T))_{n\in\mathbb{N}^*}$ belongs to $l^2(\mathbb{N}^*,\mathbb{C})$. ∎

Proof of statement (2) *of Theorem* 3. We assume that the system (Σ_l^{ref}) is controllable in $H^3_{(0)} \times H^1_0((0,1),\mathbb{R})$ with control functions in $H^1((0,T),\mathbb{R})$. Thanks to the equivalence between the controllability of (Σ_l^{ref}) and the solvability of the moment problem (3.24), we deduce that, for every $d = (d_k)_{k\in\mathbb{N}^*} \in l^2(\mathbb{N}^*,\mathbb{C})$, there exists $P \in H^1((0,T),\mathbb{R})$ such that

$$\int_0^T P(t)\sin(\sqrt{\lambda_3}t)e^{-i\sqrt{\lambda_{2k-1}}t}dt = d_k \ \forall k \in \mathbb{N}^*. \tag{3.26}$$

However, thanks to (3.1) and (3.3), for any $P \in H^1((0,T),\mathbb{R})$, an integration by parts shows that

$$\left|\int_0^T P(t)\sin(\sqrt{\lambda_3}t)e^{-i\sqrt{\lambda_{2k-1}}t}dt\right| \leqslant \frac{C}{k^2}\|P\|_{H^1((0,T),\mathbb{R})}.$$

Thus, for every $d = (d_k)_{k\in\mathbb{N}^*} \in l^2(\mathbb{N}^*,\mathbb{C})$, there exists $C > 0$ such that

$$|d_k| \leqslant \frac{C}{k^2}.$$

We get a contradiction by considering, for example, $d_k = 1/k$. Therefore, the system (Σ_l^{ref}) is not controllable in $H^3_{(0)} \times H^1_0((0,1),\mathbb{R})$ with control functions in $H^1((0,T),\mathbb{R})$.

We assume that the system (Σ_l^{ref}) is controllable in $H^2_0 \times L^2((0,1),\mathbb{R})$ with control functions in $L^2((0,T),\mathbb{R})$. Then, for every $d = (d_k)_{k\in\mathbb{N}^*} \in h^{-1}(\mathbb{N}^*,\mathbb{C})$, i.e.,

$$\sum_{k=1}^{\infty} \left|\frac{1}{k} d_k\right|^2 < \infty,$$

there exists $P \in L^2((0,T),\mathbb{R})$ such that (3.26) holds. However, for any $P \in L^2((0,T),\mathbb{R})$, the Cauchy–Shwarz inequality shows that

$$\left|\int_0^T P(t)\sin(\sqrt{\lambda_3}t)e^{-i\sqrt{\lambda_{2k-1}}t}dt\right| \leqslant \sqrt{T}\|P\|_{L^2((0,T),\mathbb{R})}.$$

Thus, for every $d \in h^{-1}(\mathbb{N}^*,\mathbb{C})$, there exists $C > 0$ such that $|d_k| \leqslant C$ for every $k \in \mathbb{N}^*$. We get a contradiction by considering, for example, $d_k = k^{1/4}$. Therefore, the system (Σ_l^{ref}) is not controllable in $H^2_0 \times L^2((0,1),\mathbb{R})$ with control functions in $L^2((0,T),\mathbb{R})$. □

4. Why the inverse mapping theorem does not apply. As we have seen in Proposition 5, for every $T > 0$, the map Φ_T defined by (1.8) is C^1 between the following spaces:

$$H^2_0((0,1),\mathbb{R}) \times L^2((0,1),\mathbb{R}) \times L^2((0,T),\mathbb{R}) \to H^2_0 \times L^2 \times H^2_0 \times L^2((0,1),\mathbb{R}).$$

In the same way we proved that Φ_T is C^1 from $H^4_{(0)}((0,1),\mathbb{R}) \times H^2_{(0)}((0,1),\mathbb{R}) \times H^1((0,T),\mathbb{R})$ to $H^4_{(0)} \times H^2_{(0)} \times H^4_{(0)} \times H^2_{(0)}((0,1),\mathbb{R})$, one can prove that Φ_T is C^1 between the following spaces:

$$H^3_{(0)}((0,1),\mathbb{R}) \times H^1_0((0,1),\mathbb{R}) \times H^1((0,T),\mathbb{R}) \to H^3_{(0)} \times H^1_0 \times H^3_{(0)} \times H^1_0((0,1),\mathbb{R}).$$

Thus, in order to apply the inverse mapping theorem on the map Φ_T, one needs to prove the surjectivity of $d\Phi_T(u_0^{ref}, \dot{u}_0^{ref}, 0)$

- from $H^2_0((0,1),\mathbb{R}) \times L^2((0,1),\mathbb{R}) \times L^2((0,T),\mathbb{R})$ to $H^2_0 \times L^2 \times H^2_0 \times L^2((0,1),\mathbb{R})$
- or from $H^3_{(0)}((0,1),\mathbb{R}) \times H^1_0((0,1),\mathbb{R}) \times H^1((0,T),\mathbb{R})$ to $H^3_{(0)} \times H^1_0 \times H^3_{(0)} \times H^1_0((0,1),\mathbb{R})$.

It means that one needs to prove the controllability of the linear system (Σ_l^{ref})

- in $H^2_0 \times L^2$ with controls in L^2
- or in $H^3_{(0)} \times H^1_0$ with controls in H^1.

However, as seen in statement (2) of Theorem 3, this is impossible. Thus the inverse mapping theorem cannot be applied with these spaces.

Let us emphasize that the inverse mapping theorem could be applied if we could prove that the map Φ_T is well-defined and C^1 between the following spaces:

$$H^3_{(0)}((0,1),\mathbb{R}) \times H^1_0((0,1),\mathbb{R}) \times L^2((0,T),\mathbb{R}) \to H^3_{(0)} \times H^1_0 \times H^3_{(0)} \times H^1_0((0,1),\mathbb{R}).$$

With further developments, one can prove that the map Φ_T is well-defined between these spaces but it is not C^1. Thus, we have the same pathology as in [8], [3], [4], [2]. The strategy developed in this article to solve this pathology is different from the one of [8] and similar to the one of [3], [4], [2].

5. The Nash–Moser theorem used. In order to get local controllability for the nonlinear system (Σ) around $(u^{ref}, \dot{u}^{ref}, p \equiv 0)$, we use a Nash–Moser theorem given by Hörmander in [20]. In this section, we recall the context and the statement of this theorem. We repeat the proof in order to justify that, in our situation, we need only a finite number of bounds on the right inverse of the differential map.

We consider a family of Hilbert spaces $(E_a)_{a\in[2,8]}$ with continuous injections $E_b \to E_a$ of norm $\leqslant 1$ when $b \geqslant a$. We suppose that we have linear operators $S_\theta : E_2 \to E_8$ for $\theta \geqslant 1$. We also assume that there exists a constant $K > 0$ such that, for every $a, b \in [2,8]$ and for every $u \in E_a$,

$$\|S_\theta u\|_b \leqslant K\|u\|_a \text{ when } b \leqslant a, \tag{5.1}$$

$$\|S_\theta u\|_b \leqslant K\theta^{b-a}\|u\|_a \text{ when } b > a, \tag{5.2}$$

$$\|u - S_\theta u\|_b \leqslant K\theta^{b-a}\|u\|_a \text{ when } b < a, \tag{5.3}$$

$$\left\|\frac{d}{d\theta}S_\theta u\|_b \leqslant K\theta^{b-a-1}\right\|u\|_a. \tag{5.4}$$

We fix a sequence $(\theta_j)_{j\in\mathbb{N}}$ of the form $\theta_j := (j+1)^\delta$ where $0 < \delta$, and we set, for every $j \in \mathbb{N}$, $\Delta_j := \theta_{j+1} - \theta_j$. For every $u \in E_a$, we have a decomposition

$$u = \sum_{j=0}^{\infty} \Delta_j R_j u$$

with convergence in E_b when $b < a$, where

$$R_j u := \frac{1}{\Delta_j}(S_{\theta_{j+1}} - S_{\theta_j})u \text{ if } j > 0 \text{ and } R_0 u := \frac{1}{\Delta_0} S_{\theta_1} u.$$

Moreover there exists a constant K' such that, for every $b \in [2,8]$,

$$\|R_j u\|_b \leqslant K' \theta_j^{b-a-1}\|u\|_a.$$

From (5.2) and (5.3), we get the logarithmic convexity of the norms: There exists a constant $c \geqslant 1$ such that, for every $a, b \in [2,8]$ with $a < b$, $\lambda \in [0,1]$, and $u \in E_b$,

$$\|u\|_{\lambda a+(1-\lambda)b} \leqslant c\|u\|_a^\lambda \|u\|_b^{1-\lambda}. \tag{5.5}$$

We refer to [20] for the proof of the two previous properties.

We have another family $(F_a)_{a\in[2,8]}$ with the same properties as above, and we use the same notations for the smoothing operators. Moreover, we assume that the injection $F_b \to F_a$ is compact when $b > a$.

THEOREM 4. *Let β be a real number such that*

$$5 < \beta < 6.$$

Let V be a E_4-neighborhood of zero and Φ a map from V to F_4 which is twice differentiable and satisfies

$$\|\Phi''(u; v, w)\|_6 \leqslant C \sum (1 + \|u\|_m)\|v\|_{m'}\|w\|_{m''}, \tag{5.6}$$

where the sum is taken over the following values:

m	m'	m''
6	2	2
4	6	2
4	2	6

We assume that $\Phi : E_4 \to F_4$ is continuous for every $a \in [2, 8]$. We assume that, for every $v \in V \cap E_8$, $\Phi'(v)$ has a right inverse $\psi(v) : F_7 \to E_6$, that $(v, g) \mapsto \psi(v, g)$ is continuous from $(V \cap E_6) \times F_7 \to E_6$, and that there exists a constant C such that, for every $(v, g) \in (V \cap E_6) \times F_7$,

$$\|\psi(v)g\|_2 \leqslant C\|g\|_3, \tag{5.7}$$

$$\|\psi(v)g\|_6 \leqslant C[\|g\|_7 + \|v\|_8\|g\|_3]. \tag{5.8}$$

Then, for every $f \in F_\beta$ with a sufficiently small norm, there exists $u \in E_4$ such that $\Phi(u) = \Phi(0) + f$.

Remark 2. The differences between the previous statement and Hörmander's statement are the following ones:

- First, here we do not use the weak spaces E'_a defined by Hörmander, which simplifies the statement a little bit;
- then, here, the number of tame estimates to be proved is finite, which is more practical for the application of the theorem.

Proof of Theorem 4. Let $g \in F_\beta$. We have a decomposition

$$g = \sum_{j=0}^{\infty} \Delta_j g_j, \text{ with } \|g_j\|_b \leqslant K'\|g\|_\beta \theta_j^{b-\beta-1} \ \forall b \in [2, 8], \tag{5.9}$$

where $g_j := R_j g$ for every $j \in \mathbb{N}$. We claim that, when $\|g\|_\beta$ is small enough, we can define a sequence $(u_j)_{j\in\mathbb{N}}$ with $u_0 = 0$ and the recursive formula

$$u_{j+1} := u_j + \Delta_j \dot{u}_j, \quad \dot{u}_j := \psi(v_j)g_j, \quad v_j := S_{\theta_j} u_j.$$

We also claim that there exist constants C_1, C_2, C_3, C_4 such that, for every $j \in \mathbb{N}^*$,

$$\|\dot{u}_j\|_a \leqslant C_1\|g\|_\beta \theta_j^{a-\beta} \ \forall a \in \{2, 4, 6\}, \tag{5.10}$$

$$\|u_j\|_4 \leqslant C_2\|g\|_\beta \text{ and } \|u_j\|_6 \leqslant C_2\|g\|_\beta \theta_j^{7-\beta}, \tag{5.11}$$

$$\|v_j\|_4 \leqslant C_3\|g\|_\beta \ , \ \|v_j\|_a \leqslant C_3\|g\|_\beta \theta_j^{a-\beta+1} \forall a \in \{6, 8\}, \tag{5.12}$$

$$\|u_j - v_j\|_a \leqslant C_4\|g\|_\beta \theta_j^{a-\beta+1} \ \forall a \in \{2, 4, 6\}. \tag{5.13}$$

More precisely, we prove by induction on $k \in \mathbb{N}$ the following property:

$\mathcal{P}_k$: u_j is well-defined for $j = 0, \ldots, k + 1$,
(5.10) is satisfied for $j = 0, \ldots, k$,
(5.11), (5.12), and (5.13) are satisfied for $j = 0, \ldots, k + 1$.

We introduce $r > 0$ such that, for every $u \in E_4$, $\|u\|_\alpha < r$ implies $u \in V$.

Property $\mathcal{P}_0$ is obvious. Let $k \in \mathbb{N}^*$. We assume that property $\mathcal{P}_{k-1}$ is satisfied. Let us prove $\mathcal{P}_k$.

The vector u_{k+1} is well-defined if $v_k \in V$. By using (5.1) and (5.10), we get

$$\|v_k\|_4 \leqslant KC_1\|g\|_\beta \sum_{j=0}^{k-1} \Delta_j \theta_j^{4-\beta}.$$

Since $\beta < 5$, the sum

$$S := \sum_{j=0}^{\infty} \Delta_j \theta_j^{4-\beta}$$

is convergent. Therefore, when

$$\|g\|_\beta \leqslant \frac{r}{C_1KS},$$

$v_k \in V$ and u_{k+1} is well-defined.

Let us prove (5.10) for $j = k$. By using (5.7) and (5.9), we get

$$\|\dot{u}_k\|_2 \leqslant CK'\|g\|_\beta \theta_k^{2-\beta}.$$

By using (5.8), (5.12), and (5.9), we get

$$\|\dot{u}_k\|_6 \leqslant CK'\|g\|_\beta[\theta_k^{6-\beta} + C_3\|g\|_\beta \theta_k^{9-\beta}\theta_k^{2-\beta}] \leqslant 2CK'\|g\|_\beta \theta_k^{6-\beta},$$

when $\|g\|_\beta < 1/C_3$. The convexity of the norm (5.5) provides

$$\|\dot{u}_k\|_4 \leqslant c\sqrt{2}CK'\|g\|_\beta \theta_k^{4-\beta}.$$

Therefore, we have (5.10) for $j = k$, when $\|g\|_\beta < 1/C_3$ for $C_1 = \max\{2, c\sqrt{2}\}CK'$.

Let us prove (5.11) for $j = k + 1$. As noticed at the beginning of the proof by induction, we have

$$\|u_{k+1}\|_4 \leqslant C_1\|g\|_\beta \sum_{j=0}^{k} \Delta_j \theta_j^{4-\beta} \leqslant C_1\|g\|_\beta S.$$

Thanks to (5.10), we have

$$\|u_{k+1}\|_6 \leqslant C_1\|g\|_\beta \sum_{j=0}^{k} \Delta_j \theta_j^{6-\beta} \leqslant C_1\|g\|_\beta \frac{\theta_{k+1}^{7-\beta}}{7-\beta}.$$

Therefore, we have (5.11) for $j = k + 1$, with

$$C_2 := C_1 \max\left\{S, \frac{1}{7-\beta}\right\}.$$

We get (5.12) for $j = k + 1$ thanks to (5.1) and (5.2), with $C_3 := KC_2$. We get (5.13) for $j = k + 1$ thanks to (5.11) and (5.12) for $a = 6$ and thanks to (5.3) and (5.11) for $a \in \{2, 4\}$, with $C_4 := \max\{C_2 + C_3; KC_2\}$.

Inequality (5.10) proves that (u_k) converges in E_4 toward

$$u := \sum_{j=0}^{\infty} \Delta_j \dot{u}_j.$$

The continuity of the map $\Phi : E_4 \to F_4$ implies that $\Phi(u_k)$ converges to $\Phi(u)$ in F_4.

Let us study the limit of the sequence $(\Phi(u_k))_{k\in\mathbb{N}}$ in a different way. We have

$$\Phi(u_{j+1}) - \Phi(u_j) = \Delta_j(e_j' + e_j'' + g_j),$$

where

$$\begin{aligned} e_j' &:= \frac{1}{\Delta_j}\left(\Phi(u_j + \Delta_j \dot{u}_j) - \Phi(u_j) - \Phi'(u_j)\Delta_j \dot{u}_j\right) \\ &= \Delta_j \int_0^1 (1-t)\Phi''(u_j + t\Delta_j \dot{u}_j; \dot{u}_j, \dot{u}_j)dt, \\ e_j'' &:= (\Phi'(u_j) - \Phi'(v_j))\dot{u}_j = \int_0^1 \Phi''(v_j + t(u_j - v_j); u_j - v_j, \dot{u}_j). \end{aligned}$$

Thanks to (5.6), we have

$$\begin{aligned} \|e_j'\|_6 &\leqslant C\sum(1 + \|u_j\|_m + \Delta_j\|\dot{u}_j\|_m)\|\dot{u}_j\|_{m'}\|\dot{u}_j\|_{m''} \\ &\leqslant C[(1 + (C_1 + C_2)\|g\|_\beta \theta_j^{7-\beta})C_1^2\|g\|_\beta^2\theta_j^{4-2\beta} \\ &\quad + 2(1 + (C_1 + C_2)\|g\|_\beta)C_1^2\|g\|_\beta^2\theta_j^{8-2\beta}] \\ &\leqslant C\|g\|_\beta^2\theta_j^{8-2\beta}, \\ \|e_j''\|_6 &\leqslant C\sum(1 + \|v_j\|_m + \|u_j - v_j\|_m)\|u_j - v_j\|_{m'}\|\dot{u}_j\|_{m''} \\ &\leqslant C[(1 + (C_3 + C_4)\|g\|_\beta\theta_j^{7-\beta})C_1C_4\|g\|_\beta^2\theta_j^{5-2\beta} \\ &\quad + 2(1 + (C_3 + C_4)\|g\|_\beta)C_1C_4\|g\|_\beta^2\theta_j^{9-2\beta}] \\ &\leqslant C\|g\|_\beta^2\theta_j^{9-2\beta}. \end{aligned}$$

Since $9 - 2\beta < -1$, then $\sum \Delta_j(e_j' + e_j'')$ converges in F_6 and

$$\left\| \sum_{j=0}^{\infty} \Delta_j(e_j' + e_j'') \right\|_6 \leqslant C\|g\|_\beta^2.$$

The uniqueness of the limit of the sequence $(\Phi(u_k))_{k\in\mathbb{N}}$ gives the following equality in F_4:

$$\Phi(u) = g + \mathcal{T}(g),$$

where $\mathcal{T}(g) \in F_6$ and

$$\|\mathcal{T}(g)\|_6 \leqslant C\|g\|_\beta^2.$$

We conclude by applying the Leray–Schauder fixed point theorem. □

Remark 3. The proof can also be done thanks to the Banach fixed point theorem, instead of the Leray–Schauder fixed point theorem, provided one adds new assumptions (see [3, Appendix C]). In this situation, one does not need any longer the compactness of the injections $F_b \to F_a$ for $b > a$.

6. Smoothing operators. In this section, we build smoothing operators on the spaces E_a and F_b defined by (2.12) and (2.13). First, we smooth the functions in $H^a_{(0)}((0,1),\mathbb{R})$ for every integer $a \in \{2, \dots, 8\}$. Let $s \in C^\infty(\mathbb{R},\mathbb{R})$ be such that

$$s = 1 \text{ on } [0,1], 0 \leqslant s \leqslant 1, s = 0 \text{ on } [2,+\infty).$$

We define

$$S_\theta u := \sum_{k=1}^{\infty} s\left(\frac{k}{\theta}\right)\langle u, \varphi_k\rangle \varphi_k.$$

The proof of the following proposition is easy.

PROPOSITION 9. *There exists a constant $K>0$ such that, for every integer $a\in\{2,\dots,8\}$, for every $u\in H^a_{(0)}((0,1),\mathbb{R})$ and and for every $\theta \geqslant 1$, we have*

$$\begin{aligned}
\|S_\theta u\|_{H^b((0,1),\mathbb{R})} &\leqslant K\|u\|_{H^a((0,1),\mathbb{R})} \text{ for every } b\in\{2,\dots a\},\\
\|S_\theta u\|_{H^b((0,1),\mathbb{R})} &\leqslant K\theta^{b-a}\|u\|_{H^a((0,1),\mathbb{R})} \text{ for every } b\in\{a+1,\dots,8\},\\
\|u - S_\theta u\|_{H^b((0,1),\mathbb{R})} &\leqslant K\theta^{b-a}\|u\|_{H^a((0,1),\mathbb{R})} \text{ for every } b\in\{2,\dots,a-1\},\\
\left\|\frac{d}{d\theta}S_\theta u\right\|_{H^b((0,1),\mathbb{R})} &\leqslant K\theta^{b-a-1}\|u\|_{H^a((0,1),\mathbb{R})} \text{ for every } b\in\{2,\dots,8\}.
\end{aligned}$$

Suitable smoothing operators for the control function p can be built with convolution products and truncations with C^∞-function having compact support as in [3, sect. 3.3.2]. This construction is inspired by [19].

7. Bound on Φ_T''. The aim of this section is the proof of inequality (5.6) on the map Φ_T defined by (1.8). More precisely, we prove the following proposition.

PROPOSITION 10. *Let $T>0$. The map $\Phi_T : E_6 \mapsto F_6$ is twice differentiable, and, for every $(u_0,\dot u_0,p)$, $(\lambda_0,\dot\lambda_0,\rho)$, $(\mu_0,\dot\mu_0,\theta)\in E_6$,*

$$\Phi_T''(u_0,\dot u_0,p).((\lambda_0,\dot\lambda_0,\rho),(\mu_0,\dot\mu_0,\theta)) = (0,0,h(T),h_t(T)),$$

where

$$\begin{cases} u_{tt}+u_{xxxx}+p(t)u_{xx}=0,\\ u=u_x=0 \text{ at } x=0,1,\\ u(0)=u_0, \dot u(0)=\dot u_0,\end{cases} \qquad \begin{cases} \lambda_{tt}+\lambda_{xxxx}+p(t)\lambda_{xx}+\rho(t)u_{xx}=0,\\ \lambda=\lambda_x=0 \text{ at } x=0,1,\\ \lambda(0)=\lambda_0, \dot\lambda(0)=\dot\lambda_0,\end{cases}$$

$$\begin{cases} \mu_{tt}+\mu_{xxxx}+p(t)\mu_{xx}+\theta(t)u_{xx}=0,\\ \mu=\mu_x=0 \text{ at } x=0,1,\\ \mu(0)=\mu_0, \dot\mu(0)=\dot\mu_0,\end{cases}$$

$$\begin{cases} h_{tt}+h_{xxxx}+p(t)h_{xx}+\rho(t)\mu_{xx}+\theta(t)\lambda_{xx}=0,\\ h=h_x=0 \text{ at } x=0,1,\\ h(0)=0, \dot h(0)=0.\end{cases}$$

For every $r>0$, there exists a constant $C(r)>0$ such that, for every $(u_0,\dot u_0,p)$, $(\lambda_0,\dot\lambda_0,\rho)$, $(\mu_0,\dot\mu_0,\theta)\in E_6$, with $\|(u_0,\dot u_0,p)\|_4\leqslant r$,

$$\begin{aligned}
&\|\Phi_T''(u_0,\dot u_0,p).((\lambda_0,\dot\lambda_0,\rho),(\mu_0,\dot\mu_0,\theta))\|_6\\
&\qquad\leqslant C(r)\sum(1+\|(u_0,\dot u_0,p)\|_m)\|(\lambda_0,\dot\lambda_0,\rho)\|_{m'}\|(\mu_0,\dot\mu_0,\theta)\|_{m''}
\end{aligned}\tag{7.1}$$

where the sum is taken over the following values:

m	m'	m''
6	2	2
4	6	2
4	2	6

Proof of Proposition 10. The regularity C^2 of the map Φ_T can be proved thanks to Propositions 1, 2, and 3 in a very similar way as for the regularity C^1 in Proposition 5. Here we only justify (7.1). By applying Proposition 3 (which is possible because $\|p\|_{W^{1,1}((0,T),\mathbb{R})} \leqslant \sqrt{T}\|p\|_{H^1((0,T),\mathbb{R})} \leqslant \sqrt{T}r$) we get $C = C(r) > 0$ such that

$$\begin{aligned} &\|(h,h_t)\|_{C^0([0,T],H^6\times H^4)} \\ &\quad\leqslant C\Big[\|f\|_{W^{2,1}((0,T),L^2)} + \|p\|_{W^{2,1}}\|f\|_{L^1((0,T),L^2)} + \|f\|_{C^0([0,T],H^2)}\Big], \end{aligned} \tag{7.2}$$

where $f(t) := \rho(t)\mu_{xx}(t) + \theta(t)\lambda_{xx}(t)$. Let us define $f_1(t) := \rho(t)\mu_{xx}(t)$ and $f_2(t) := \theta(t)\lambda_{xx}(t)$. We work only on f_1, and we deduce the same results for f_2 just by exchanging (μ,ρ) and (λ,θ). By using Propositions 1, 2, nad 3, we get constants $C = C(r)$ such that, when $\|(u_0,\dot{u}_0,p)\|_{E_4} \leqslant r$,

$$\begin{aligned} \|f_1\|_{L^1((0,T),L^2)} &\leqslant C\|\rho\|_{L^1}\|\mu\|_{C^0((0,T),H^2)}, \\ \|f_1\|_{W^{2,1}((0,T),L^2)} &\leqslant C[\|\rho\|_{H^2}\|\mu\|_{C^0([0,T],H^2)} + \|\rho\|_{H^1}\|\mu\|_{C^1([0,T],H^2)} \\ &\qquad + \|\rho\|_{L^2}\|\mu\|_{C^2([0,T],H^2)}], \\ \|f_1\|_{C^0([0,T],H^2)} &\leqslant C\|\rho\|_{H^1}\|\mu\|_{C^0([0,T],H^4)}, \end{aligned}$$

where

$$\begin{aligned} \|\mu\|_{C^0([0,T],H^2)} &\leqslant C[\|(\mu_0,\dot{\mu}_0)\|_{H_0^2\times L^2} + \|\theta u_{xx}\|_{L^1((0,T),L^2)}] \\ &\leqslant C[\|(\mu_0,\dot{\mu}_0)\|_{H_0^2\times L^2} + \|\theta\|_{L^2}\|(u_0,\dot{u}_0)\|_{H_0^2\times L^2}] \\ &\leqslant C\|(\mu_0,\dot{\mu}_0,\theta)\|_{E_2}, \\ \|\mu\|_{C^0([0,T],H^4)}, \|\mu\|_{C^1([0,T],H^2)} &\leqslant C[\|(\mu_0,\dot{\mu}_0)\|_{H^4_{(0)}\times H_0^2} + \|\theta u_{xx}\|_{W^{1,1}((0,T),L^2)}] \\ &\leqslant C[\|(\mu_0,\dot{\mu}_0)\|_{H^4_{(0)}\times H_0^2} + \|\theta\|_{H^1}\|(u_0,\dot{u}_0)\|_{H_0^2\times L^2} \\ &\qquad + \|\theta\|_{L^1}\|(u_0,\dot{u}_0)\|_{H^4_{(0)}\times H_0^2}] \\ &\leqslant C\|(\mu_0,\dot{\mu}_0,\theta)\|_{E_4}, \\ \|\mu\|_{C^2([0,T],H^2)} &\leqslant C[\|(\mu_0,\dot{\mu}_0)\|_{H^6_{(0)}\times H^4_{(0)}} + \|p\|_{W^{2,1}}\|(\mu_0,\dot{\mu}_0)\|_{H_0^2\times L^2} \\ &\qquad + \|\theta u_{xx}\|_{W^{2,1}((0,T),L^2)} + \|p\|_{W^{2,1}}\|\theta u_{xx}\|_{L^1((0,T),L^2)} \\ &\qquad + \|\theta u_{xx}\|_{C^0([0,T],H^2)}]. \end{aligned}$$

We have

$$\begin{aligned} \|\theta u_{xx}\|_{W^{2,1}((0,T),L^2)} &\leqslant C[\|\theta\|_{H^2}\|u\|_{C^0([0,T],H^2)} + \|\theta\|_{H^1}\|u\|_{C^1([0,T],H^2)} \\ &\qquad + \|\theta\|_{L^2}\|u\|_{C^2([0,T],H^2)}] \\ &\leqslant C\{\|\theta\|_{H^2}\|(u_0,\dot{u}_0)\|_{H_0^2\times L^2} + \|\theta\|_{H^1}\|(u_0,\dot{u}_0)\|_{H^4_{(0)}\times H_0^2} \\ &\qquad + \|\theta\|_{L^2}[\|(u_0,\dot{u}_0)\|_{H^6_{(0)}\times H^4_{(0)}} + \|p\|_{W^{2,1}}\|(u_0,\dot{u}_0)\|_{H_0^2\times L^2}]\} \\ &\leqslant C[\|\theta\|_{H^2} + \|\theta\|_{L^2}\|(u_0,\dot{u}_0,p)\|_{E_6}], \\ \|\theta u_{xx}\|_{L^1((0,T),L^2)} &\leqslant C\|\theta\|_{L^2}\|(u_0,\dot{u}_0)\|_{H_0^2\times L^2} \\ &\leqslant C\|\theta\|_{L^2}, \\ \|\theta u_{xx}\|_{C^0([0,T],H^2)} &\leqslant C\|\theta\|_{H^1}\|(u_0,\dot{u}_0)\|_{H^4_{(0)}\times H_0^2} \\ &\leqslant C\|\theta\|_{H^1}, \end{aligned}$$

and thus

$$\|\mu\|_{C^2([0,T],H^2)} \leqslant C[\|(\mu_0,\dot{\mu}_0,\theta)\|_{E_6} + \|(u_0,\dot{u}_0,p)\|_{E_6}\|(\mu_0,\dot{\mu}_0,\theta)\|_{E_2}].$$

We deduce from the previous computations that

$$
\begin{aligned}
\|f_1\|_{L^1((0,T),L^2)} &\leqslant C\|\rho\|_{L^2}\|(\mu_0,\dot\mu_0,\theta)\|_{E_2},\\
\|f_1\|_{W^{2,1}((0,T),L^2)} &\leqslant C\{\|\rho\|_{H^2}\|(\mu_0,\dot\mu_0,\theta)\|_{E_2}+\|\rho\|_{H^1}\|(\mu_0,\dot\mu_0,\theta)\|_{E_4}\\
&\quad +\|\rho\|_{L^2}\|(\mu_0,\dot\mu_0,\theta)\|_{E_6}\\
&\quad +\|\rho\|_{L^2}\|(u_0,\dot u_0,p)\|_{E_6}\|(\mu_0,\dot\mu_0,\theta)\|_{E_2}\},\\
\|f_1\|_{C^0([0,T],H^2)} &\leqslant C\|\rho\|_{H^1}\|(\mu_0,\dot\mu_0,\theta)\|_{E_4}.
\end{aligned}
\tag{7.3}
$$

The inequalities (7.2) and (7.3) give the conclusion. □

8. Controllability of the linearized system around $(u^{ref},\dot u^{ref},p\equiv 0)$ with Nash–Moser bounds. In all of this section, $T:=8/\pi$. We recall that the spaces E_a and F_b are defined by (2.12) and (2.13). The goal of this section is the proof of the following result.

PROPOSITION 11. *There exists $C>0$ such that the map $d\Phi_T(u_0^{ref},\dot u_0^{ref},0)$ has a right inverse*

$$d\Phi_T(u_0^{ref},\dot u_0^{ref},0)^{-1}:F_7\to E_6$$

and, for every $(U_0,\dot U_0,U_T,\dot U_T)\in F_7$,

$$\|d\Phi_T(u_0^{ref},\dot u_0^{ref},0)^{-1}.(U_0,\dot U_0,U_T,\dot U_T)\|_{E_2}\leqslant C\|(U_0,\dot U_0,U_T,\dot U_T)\|_{F_3},$$

$$\|d\Phi_T(u_0^{ref},\dot u_0^{ref},0)^{-1}.(U_0,\dot U_0,U_T,\dot U_T)\|_{E_6}\leqslant C\|(U_0,\dot U_0,U_T,\dot U_T)\|_{F_7}.$$

We introduce the linear map M, defined on $L^2((0,T),\mathbb{R})$ by $M(P)=(M(P)_k)_{k\in\mathbb{N}^*}$,

$$
\begin{aligned}
M(P)_{2k-1}&:=\frac{1}{T}\int_0^T P(t)\sin(\sqrt{\lambda_3}t)e^{-i\sqrt{\lambda_{2k-1}}t}dt\ \forall k\in\mathbb{N}^*,\\
M(P)_{2k}&:=\frac{1}{T}\int_0^T P(t)\sin(\sqrt{\lambda_2}t)e^{-i\sqrt{\lambda_{2k}}t}dt\ \forall k\in\mathbb{N}^*.
\end{aligned}
$$

For $s>0$, we introduce the space

$$h^s(\mathbb{N}^*,\mathbb{C}):=\left\{d=(d_n)_{n\in\mathbb{N}^*};\sum_{n=1}^{\infty}|n^s d_n|^2<+\infty\right\}.$$

Thanks to the equivalence between the controllability of the linearized system (Σ_l^{ref}) and the solvability of the moment problem (3.24) (which is explained in the proof of statement (1) of Theorem 3, in section 3.2), Proposition 11 is a consequence of the following result.

PROPOSITION 12. *There exists $C>0$ such that the map M has a right inverse*

$$M^{-1}:h^4(\mathbb{N}^*,\mathbb{C})\to H_0^2((0,T),\mathbb{R})$$

and, for every $d\in h^4(\mathbb{N}^*,\mathbb{C})$,

$$\|M^{-1}(d)\|_{L^2((0,T),\mathbb{R})}\leqslant C\|d\|_{l^2(\mathbb{N}^*,\mathbb{C})},$$

$$\|M^{-1}(d)\|_{H_0^2((0,T),\mathbb{R})}\leqslant C\|d\|_{h^4(\mathbb{N}^*,\mathbb{C})}.$$

Proposition 11 is a consequence of Proposition 12, but they are not equivalent. Indeed, Proposition 12 provides a right inverse $d\Phi_T(u_0, \dot{u}_0, 0)^{-1}$ defined on F_7 with values in $H^7_{(0)}((0,1),\mathbb{R}) \times H^5_{(0)}((0,1),\mathbb{R}) \times H^2_0((0,T),\mathbb{R})$ which is stricly smaller than E_6 (see (2.12) for a definition of E_6).

We will prove Proposition 12 by using an auxiliary linear map $\widetilde{M}$, which is easier to deal with than M and close enough to M so that the surjectivity of $\widetilde{M}$ implies the surjectivity of M. Let us introduce the map $\widetilde{M} : L^2((0,T),\mathbb{R}) \to l^2(\mathbb{N}^*,\mathbb{C})$ defined by

$$\begin{aligned} \widetilde{M}(P)_{2k-1} &:= \frac{1}{T}\int_0^T P(t)\sin\left(\frac{\pi^2}{4}K_3 t\right) e^{-i\frac{\pi^2}{4}K_{2k-1}t}dt \ \forall k \in \mathbb{N}^*, \\ \widetilde{M}(P)_{2k} &:= \frac{1}{T}\int_0^T P(t)\sin\left(\frac{\pi^2}{4}K_2 t\right) e^{-i\frac{\pi^2}{4}K_{2k}t}dt \ \forall k \in \mathbb{N}^*, \end{aligned} \tag{8.1}$$

where $(K_n)_{n\in\mathbb{N}^*}$ is defined by (3.6). The linear map $\widetilde{M}$ maps $L^2((0,T),\mathbb{R})$ into $l^2(\mathbb{N}^*,\mathbb{C})$, $H^1_0((0,T),\mathbb{R})$ into $h^2(\mathbb{N}^*,\mathbb{C})$, and $H^2_0((0,T),\mathbb{R})$ into $h^4(\mathbb{N}^*,\mathbb{C})$. Indeed, each term of the sequence is the sum of two Fourier coefficients of P. Note that $T = 8/\pi$ is chosen so that, for every $n \in \mathbb{N}^*$, $e^{i\frac{\pi^2}{4}K_n t}$ is T-periodic. Thus, the previous statement is a consequence of Bessel Parseval equality and integrations by parts.

For technical reasons, the space $h^s(\mathbb{N}^*,\mathbb{C})$ is equipped with the unusual norm

$$\|d\|_{h^s(\mathbb{N}^*,\mathbb{C})} := \left(\sum_{n=1}^{\infty} |K_n^{s/2} d_n|^2\right)^{1/2}.$$

On the spaces $L^2((0,T),\mathbb{R})$, $H^1_0((0,T),\mathbb{R})$, and $H^2_0((0,T),\mathbb{R})$ we use

$$\|f\|_{L^2((0,T),\mathbb{R})} := \left(\frac{1}{T}\int_0^T |f(t)|^2 dt\right)^{1/2},$$
$$\|f\|_{H^1_0((0,T),\mathbb{R})} := \|f'\|_{L^2((0,T),\mathbb{R})} \ , \ \|f\|_{H^2_0((0,T),\mathbb{R})} := \|f''\|_{L^2((0,T),\mathbb{R})}.$$

More precisely, we prove Proposition 12 by applying the next proposition to $\mathcal{M} = M$ and $\widetilde{\mathcal{M}} = \widetilde{M}$.

PROPOSITION 13. *Let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ be continuous linear maps from $L^2((0,T),\mathbb{R})$ to $l^2(\mathbb{N}^*,\mathbb{C})$, from $H^1_0((0,T),\mathbb{R})$ to $h^2(\mathbb{N}^*,\mathbb{C})$, and from $H^2_0((0,T),\mathbb{R})$ to $h^4(\mathbb{N}^*,\mathbb{C})$. We assume that there exist positive constants C_0, C_1, C_2, C such that $\widetilde{\mathcal{M}}$ has a right inverse*

$$\widetilde{M}^{-1} : h^4(\mathbb{N}^*,\mathbb{C}) \to H^2_0((0,T),\mathbb{R})$$

which satisfies, for every $d \in h^4(\mathbb{N}^*,\mathbb{C})$,

$$\|\widetilde{\mathcal{M}}^{-1}(d)\|_{L^2((0,T),\mathbb{R})} \leqslant C_0\|d\|_{l^2(\mathbb{N}^*,\mathbb{C})},$$
$$\|\widetilde{\mathcal{M}}^{-1}(d)\|_{H^1_0((0,T),\mathbb{R})} \leqslant C_1\|d\|_{h^2(\mathbb{N}^*,\mathbb{C})} + C\|d\|_{l^2(\mathbb{N}^*,\mathbb{C})},$$
$$\|\widetilde{\mathcal{M}}^{-1}(d)\|_{H^2_0((0,T),\mathbb{R})} \leqslant C_2\|d\|_{h^4(\mathbb{N}^*,\mathbb{C})} + C\|d\|_{h^2(\mathbb{N}^*,\mathbb{C})}.$$

We also assume that there exist positive constants $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{C}_5$ such that, for *every* $P \in H_0^2((0,T),\mathbb{R})$,

$$\|(\widetilde{\mathcal{M}} - \mathcal{M})(P)\|_{l^2(\mathbb{N}^*,\mathbb{C})} \leqslant \mathcal{C}_0 \|P\|_{L^2((0,T),\mathbb{R})},$$

$$\|(\widetilde{\mathcal{M}} - \mathcal{M})(P)\|_{h^2(\mathbb{N}^*,\mathbb{C})} \leqslant \mathcal{C}_1 \|P\|_{H_0^1((0,T),\mathbb{R})} + \mathcal{C}_3 \|P\|_{L^2((0,T),\mathbb{R})},$$

$$\|(\widetilde{\mathcal{M}} - \mathcal{M})(P)\|_{h^4(\mathbb{N}^*,\mathbb{C})} \leqslant \mathcal{C}_2 \|P\|_{H_0^2((0,T),\mathbb{R})} + \mathcal{C}_4 \|P\|_{H_0^1((0,T),\mathbb{R})} + \mathcal{C}_5 \|P\|_{L^2((0,T),\mathbb{R})}.$$

We assume that $C_0\mathcal{C}_0$*,* $C_1\mathcal{C}_1$*, and* $C_2\mathcal{C}_2$ *are* < 1*. Then there exists a constant* $m > 0$ *such that* $\mathcal{M}$ *has a right inverse*

$$\mathcal{M}^{-1} : h^4(\mathbb{N}^*,\mathbb{C}) \to H_0^2((0,T),\mathbb{R})$$

which satisfies, for every $d \in h^4(\mathbb{N}^*,\mathbb{C})$*,*

$$\begin{aligned}
\|\mathcal{M}^{-1}(d)\|_{L^2((0,T),\mathbb{R})} &\leqslant m\|d\|_{l^2(\mathbb{N}^*,\mathbb{C})},\\
\|\mathcal{M}^{-1}(d)\|_{H_0^1((0,T),\mathbb{R})} &\leqslant m\|d\|_{h^2(\mathbb{N}^*,\mathbb{C})},\\
\|\mathcal{M}^{-1}(d)\|_{H_0^2((0,T),\mathbb{R})} &\leqslant m\|d\|_{h^4(\mathbb{N}^*,\mathbb{C})}.
\end{aligned}$$

Remark 4. One of the interests of this proposition, when we apply it, is that the constants C and $\mathcal{C}_j$ for $j = 3, 4, 5$ can be large. In the application of this proposition to the maps M and $\widetilde{M}$, we will put all of the possible terms in the factor with C or $\mathcal{C}_j$ for $j = 3, 4, 5$ in order to reduce $\mathcal{C}_1$ and $\mathcal{C}_2$ and ensure $C_1\mathcal{C}_1 < 1$ and $C_2\mathcal{C}_2 < 1$.

Proof of Proposition 13. We introduce $\Delta_i := C_i\mathcal{C}_i$ for $i = 0, 1, 2$. Let $d \in h^4(\mathbb{N}^*,\mathbb{C})$. We define by induction the sequence $(w_n)_{n\in\mathbb{N}}$ in $H_0^2((0,T),\mathbb{R})$ by

$$\begin{cases} w_0 := \widetilde{\mathcal{M}}^{-1}(d),\\ w_{n+1} := \widetilde{\mathcal{M}}^{-1}[(\widetilde{\mathcal{M}} - \mathcal{M})(w_n)]. \end{cases}$$

Then we have, for every $n \in \mathbb{N}^*$,

$$\|w_n\|_{L^2((0,T),\mathbb{R})} \leqslant C_0 \Delta_0^n \|d\|_{l^2(\mathbb{N}^*,\mathbb{C})},$$

$$\|w_n\|_{H_0^1((0,T),\mathbb{R})} \leqslant C_1 \Delta_1^n \|d\|_{h^2(\mathbb{N}^*,\mathbb{C})} + x_n \|d\|_{l^2(\mathbb{N}^*,\mathbb{C})},$$

$$\|w_n\|_{H_0^2((0,T),\mathbb{R})} \leqslant C_2 \Delta_2^n \|d\|_{h^4(\mathbb{N}^*,\mathbb{C})} + y_n \|d\|_{h^2(\mathbb{N}^*,\mathbb{C})} + z_n \|d\|_{l^2(\mathbb{N}^*,\mathbb{C})},$$

where $x_0 = y_0 = C$, $z_0 = 0$, and, for every $n \in \mathbb{N}^*$,

$$x_{n+1} = \Delta_1 x_n + XC_0\Delta_0^n, \quad y_{n+1} = \Delta_2 y_n + YC_1\Delta_1^n, \quad z_{n+1} = \Delta_2 z_n + Yx_n + ZC_0\Delta_0^n,$$

$$X := C\mathcal{C}_0 + C_1\mathcal{C}_3,\ Y := C_2\mathcal{C}_4 + C\mathcal{C}_1,\ Z := C_2\mathcal{C}_5 + C\mathcal{C}_3.$$

Under the assumption that $\Delta_i < 1$ for $i = 0, 1, 2$, the sums $\sum x_n$, $\sum y_n$, and $\sum z_n$ converge and

$$(1 - \Delta_1)\sum_{n=0}^{\infty} x_n = C + \frac{XC_0}{1 - \Delta_0}, \qquad (1 - \Delta_2)\sum_{n=0}^{\infty} y_n = C + \frac{YC_1}{1 - \Delta_1},$$
$$(1 - \Delta_2)\sum_{n=0}^{\infty} z_n = Y\sum_{n=0}^{\infty} x_n + \frac{ZC_0}{1 - \Delta_0}.$$

Thus $\sum w_n$ converges in $H_0^2((0,T),\mathbb{R})$ to a function w which satisfies

$$\begin{aligned}
\|w\|_{L^2((0,T),\mathbb{R})} &\leqslant \frac{C_0}{1-\Delta_0}\|d\|_{l^2},\\
\|w\|_{H_0^1((0,T),\mathbb{R})} &\leqslant \frac{C_1}{1-\Delta_1}\|d\|_{h^2} + \frac{1}{1-\Delta_1}\left(C + \frac{XC_0}{1-\Delta_0}\right),\\
\|w\|_{H_0^2((0,T),\mathbb{R})} &\leqslant \frac{C_2}{1-\Delta_2}\|d\|_{h^4} + \frac{1}{1-\Delta_1}\left(C + \frac{YC_1}{1-\Delta_1}\right)\|d\|_{h^2}\\
&\quad + \frac{1}{1-\Delta_2}\left[\frac{Y}{1-\Delta_1}\left(C + \frac{XC_0}{1-\Delta_0}\right) + \frac{ZC_0}{1-\Delta_0}\right]\|d\|_{l^2}.
\end{aligned}$$

For every $n \in \mathbb{N}^*$, we have

$$\mathcal{M}\left(\sum_{k=0}^{n} w_k\right) = d + (\mathcal{M} - \widetilde{\mathcal{M}})(w_n),$$

where $(\mathcal{M} - \widetilde{\mathcal{M}})(w_n) \to 0$ in l^2 because $w_n \to 0$ in L^2 and $(\mathcal{M} - \widetilde{\mathcal{M}})$ is continuous from L^2 to l^2, and thus $\mathcal{M}(w) = d$. ☐

In the next proposition, we check the first assumption of Proposition 13 with $\widetilde{\mathcal{M}} = \widetilde{M}$.

PROPOSITION 14. *The linear map* $\widetilde{M}$ *defined by* (8.1) *has a right inverse* $\widetilde{M}^{-1} : h^4(\mathbb{N}^*,\mathbb{C}) \to H_0^2((0,T),\mathbb{R})$ *such that, for every* $d \in h^4(\mathbb{N}^*,\mathbb{C})$,

$$\|\widetilde{M}^{-1}(d)\|_{L^2((0,T),\mathbb{R}} \leqslant C_0\|d\|_{l^2(\mathbb{N}^*,\mathbb{C})}, \tag{8.2}$$

$$\|\widetilde{M}^{-1}(d)\|_{H_0^1((0,T),\mathbb{R}} \leqslant C_1\|d\|_{h^2(\mathbb{N}^*,\mathbb{C})} + C\|d\|_{l^2(\mathbb{N}^*,\mathbb{C})}, \tag{8.3}$$

$$\|\widetilde{M}^{-1}(d)\|_{H_0^2((0,T),\mathbb{R}} \leqslant C_2\|d\|_{h^4(\mathbb{N}^*,\mathbb{C})} + C\|d\|_{h^2(\mathbb{N}^*,\mathbb{C})}, \tag{8.4}$$

where $C_0 := 2\sqrt{3}$, $C_1 := \frac{\sqrt{3}}{2\sqrt{2}}\pi^2$, $C_2 := \frac{\sqrt{3}}{8\sqrt{2}}\pi^4$, *and* C *is a positive constant.*

Proof of Proposition 14. Let $d \in h^4(\mathbb{N}^*,\mathbb{C})$. For a function $P \in H_0^2((0,T),\mathbb{R})$, the equality $\widetilde{M}(P) = d$ is satisfied in particular when

$$\begin{cases}
\frac{1}{T}\int_0^T P(t)dt = 0,\\
\frac{1}{T}\int_0^T P(t)e^{-i\frac{\pi^2}{4}2K_3t}dt = -2id_3,\\
\frac{1}{T}\int_0^T P(t)e^{-i\frac{\pi^2}{4}(K_{2k-1}-K_3)t}dt = id_{2k-1}\ \forall k \in \mathbb{N}^* \text{ such that } k \neq 2,\\
\frac{1}{T}\int_0^T P(t)e^{-i\frac{\pi^2}{4}(K_{2k-1}+K_3)t}dt = -id_{2k-1}\ \forall k \in \mathbb{N}^* \text{ such that } k \neq 2,\\
\frac{1}{T}\int_0^T P(t)e^{-i\frac{\pi^2}{4}2K_2t}dt = -2id_2,\\
\frac{1}{T}\int_0^T P(t)e^{-i\frac{\pi^2}{4}(K_{2k}-K_2)t}dt = id_{2k}\ \forall k \in \mathbb{N}^* \text{ such that } k \neq 2,\\
\frac{1}{T}\int_0^T P(t)e^{-i\frac{\pi^2}{4}(K_{2k}+K_2)t}dt = -id_{2k}\ \forall k \in \mathbb{N}^* \text{ such that } k \neq 2.
\end{cases} \tag{8.5}$$

Let us consider the candidate

$$\begin{aligned}
P(t) := 2\Re\Bigg[&-2id_3e^{i\frac{\pi^2}{4}2K_3t} - 2id_2e^{i\frac{\pi^2}{4}2K_2t}\\
&+\sum_{k=1,k\neq 2}^{\infty} id_{2k-1}\left(e^{i\frac{\pi^2}{4}(K_{2k-1}-K_3)t} - e^{i\frac{\pi^2}{4}(K_{2k-1}+K_3)t}\right)\\
&+\sum_{k=2}^{\infty} id_{2k}\left(e^{i\frac{\pi^2}{4}(K_{2k}-K_2)t} - e^{i\frac{\pi^2}{4}(K_{2k}+K_2)t}\right)\Bigg]\frac{1}{2}(1-e^{i\frac{\pi^2}{4}t})(1-e^{-i\frac{\pi^2}{4}t}).
\end{aligned}$$

Then $P \in H_0^2((0,T),\mathbb{R})$ because $d \in h^4(\mathbb{N}^*,\mathbb{C})$ and $t \mapsto (1-e^{i\pi^2 t/4})(1-e^{-i\pi^2 t/4})$ belongs to $H_0^2((0,T),\mathbb{R})$. The positive integers

$$2K_3, 2K_3 \pm 1, 2K_2, 2K_2 \pm 1, K_3 \pm K_1, K_3 \pm K_1 \pm 1, K_4 \pm K_2, K_4 \pm K_2 \pm 1,$$
$$K_{2k-1} \pm K_3, K_{2k-1} \pm K_3 \pm 1, K_{2k} \pm K_2, K_{2k} \pm K_2 \pm 1, \text{ for } k \in \mathbb{N} \text{ such that } k \geqslant 3,$$

are all different except $K_3 + K_1 - 1 = K_4 - K_2 + 1$ (which concerns the coefficients d_1 and d_4 in P) and $K_8 + K_2 - 1 = K_9 - K_3 + 1$ (which concerns the coefficients d_8 and d_9 in P). This statement can be proved in the same way as the claim in the proof of Lemma 1. Thanks to the orthogonality of the different terms in P, the function P solves the moment problem (8.5). We have

$$\begin{aligned} &2id_3 e^{i\frac{\pi^2}{4}2K_3 t}\frac{(1-e^{i\frac{\pi^2}{4}t})(1-e^{-i\frac{\pi^2}{4}t})}{2} \\ &\quad = 2id_3\left[-\frac{1}{2}e^{i\frac{\pi^2}{4}(2K_3+1)t} + e^{i\frac{\pi^2}{4}2K_3 t} - \frac{1}{2}e^{i\frac{\pi^2}{4}(2K_3-1)t}\right], \end{aligned}$$

where all of the terms are orthogonal, and thus

$$\begin{aligned} \left\| 2\Re[2id_3 e^{i\frac{\pi^2}{4}2K_3 t}]\frac{(1-e^{i\frac{\pi^2}{4}t})(1-e^{-i\frac{\pi^2}{4}t})}{2}\right\|^2_{L^2((0,T),\mathbb{R})} &= 4|d_3|^2\left(\frac{1}{4}+1+\frac{1}{4}\right)*2 \\ &= 12|d_3|^2. \end{aligned}$$

For the same reason, we have

$$\left\| 2\Re[2id_2 e^{i\frac{\pi^2}{4}2K_2 t}]\frac{(1-e^{i\frac{\pi^2}{4}t})(1-e^{-i\frac{\pi^2}{4}t})}{2}\right\|^2_{L^2((0,T),\mathbb{R})} = 12|d_2|^2.$$

We have

$$\begin{aligned} &id_{2k}\left(e^{i\frac{\pi^2}{4}(K_{2k}-K_2)t} - e^{i\frac{\pi^2}{4}(K_{2k}+K_2)t}\right)\frac{(1-e^{i\frac{\pi^2}{4}t})(1-e^{-i\frac{\pi^2}{4}t})}{2} \\ &= id_{2k}\left[-\frac{1}{2}e^{i\frac{\pi^2}{4}(K_{2k}-K_2-1)t} + e^{i\frac{\pi^2}{4}(K_{2k}-K_2)t} - \frac{1}{2}e^{i\frac{\pi^2}{4}(K_{2k}-K_2+1)t}\right] \\ &\quad - id_{2k}\left[-\frac{1}{2}e^{i\frac{\pi^2}{4}(K_{2k}+K_2-1)t} + e^{i\frac{\pi^2}{4}(K_{2k}+K_2)t} - \frac{1}{2}e^{i\frac{\pi^2}{4}(K_{2k}+K_2+1)t}\right], \end{aligned}$$

where the terms are orthogonal, and thus

$$\begin{aligned} &\left\| 2\Re\left[id_{2k}\left(e^{i\frac{\pi^2}{4}(K_{2k}-K_2)t} - e^{i\frac{\pi^2}{4}(K_{2k}+K_2)t}\right)\right]\frac{(1-e^{i\frac{\pi^2}{4}t})(1-e^{-i\frac{\pi^2}{4}t})}{2}\right\|^2_{L^2((0,T),\mathbb{R})} \\ &\quad = |d_{2k}|^2\left(\frac{1}{4}+1+\frac{1}{4}\right)*4 = 6|d_{2k}|^2. \end{aligned}$$

For the same reason, we have

$$\begin{aligned} &\left\| 2\Re\left[id_{2k-1}\left(e^{i\frac{\pi^2}{4}(K_{2k-1}-K_3)t} - e^{i\frac{\pi^2}{4}(K_{2k-1}+K_3)t}\right)\right]\frac{(1-e^{i\frac{\pi^2}{4}t})(1-e^{-i\frac{\pi^2}{4}t})}{2}\right\|^2_{L^2((0,T),\mathbb{R})} \\ &\quad = 6|d_{2k-1}|^2. \end{aligned}$$

Since $K_8 + K_2 - 1 = K_9 - K_3 + 1$, we have

$$\begin{aligned}
&\Big[id_8\Big(e^{i\frac{\pi^2}{4}(K_8-K_2)t} - e^{i\frac{\pi^2}{4}(K_8+K_2)t}\Big) \\
&\qquad + id_9\Big(e^{i\frac{\pi^2}{4}(K_9-K_3)t} - e^{i\frac{\pi^2}{4}(K_9+K_3)t}\Big)\Big]\frac{(1-e^{i\frac{\pi^2}{4}t})(1-e^{-i\frac{\pi^2}{4}t})}{2} \\
&= id_8\left[-\frac{1}{2}e^{i\frac{\pi^2}{4}(K_8-K_2-1)t} + e^{i\frac{\pi^2}{4}(K_8-K_2)t} - \frac{1}{2}e^{i\frac{\pi^2}{4}(K_8-K_2+1)t}\right] \\
&\qquad - id_8\left[e^{i\frac{\pi^2}{4}(K_8+K_2)t} - \frac{1}{2}e^{i\frac{\pi^2}{4}(K_8+K_2+1)t}\right] \\
&\qquad + id_9\left[-\frac{1}{2}e^{i\frac{\pi^2}{4}(K_9-K_3-1)t} + e^{i\frac{\pi^2}{4}(K_9-K_3)t}\right] \\
&\qquad - id_9\left[-\frac{1}{2}e^{i\frac{\pi^2}{4}(K_9+K_3-1)t} + e^{i\frac{\pi^2}{4}(K_9+K_3)t} - \frac{1}{2}e^{i\frac{\pi^2}{4}(K_9+K_3+1)t}\right] \\
&\qquad + i(d_8-d_9)\frac{1}{2}e^{i\frac{\pi^2}{4}(K_8+K_2-1)t},
\end{aligned}$$

where all of the terms are orthogonal, and thus

$$\begin{aligned}
&\Bigg\|2\Re\Big[id_8\Big(e^{i\frac{\pi^2}{4}(K_8-K_2)t} - e^{i\frac{\pi^2}{4}(K_8+K_2)t}\Big) \\
&\qquad + id_9\Big(e^{i\frac{\pi^2}{4}(K_9-K_3)t} - e^{i\frac{\pi^2}{4}(K_9+K_3)t}\Big)\Big]\frac{(1-e^{i\frac{\pi^2}{4}t})(1-e^{-i\frac{\pi^2}{4}t})}{2}\Bigg\|^2_{L^2((0,T),\mathbb{R})} \\
&= [|d_8|^2 + |d_9|^2] * \frac{11}{4} * 2 + |d_8 - d_9|^2 * \frac{1}{4} * 2 \\
&\leqslant \frac{13}{2}[|d_8|^2 + |d_9|^2].
\end{aligned}$$

For the same reasons, we have

$$\begin{aligned}
&\Bigg\|2\Re\Big[id_4\Big(e^{i\frac{\pi^2}{4}(K_4-K_2)t} - e^{i\frac{\pi^2}{4}(K_4+K_2)t}\Big) \\
&\qquad + id_1\Big(e^{i\frac{\pi^2}{4}(K_1-K_3)t} - e^{i\frac{\pi^2}{4}(K_1+K_3)t}\Big)\Big]\frac{(1-e^{i\frac{\pi^2}{4}t})(1-e^{-i\frac{\pi^2}{4}t})}{2}\Bigg\|^2_{L^2((0,T),\mathbb{R})} \\
&\leqslant \frac{13}{2}[|d_1|^2 + |d_4|^2].
\end{aligned}$$

Thus, we have

$$\|P\|^2_{L^2} \leqslant 12 \sum_{n\in\{1,2,3,4,8,9\}} |d_n|^2 + 6 \sum_{n\in\mathbb{N}^*, n\notin\{1,2,3,4,8,9\}} |d_n|^2$$

which justifies (8.2) with $C_0 = 2\sqrt{3}$. For the computation of $\|P'\|^2_{L^2}$, we use the same kind of arguments together with

$$\begin{aligned}
&(K_n - K_j)^2 + (K_n + K_j)^2 \\
&\qquad + \frac{(K_n - K_j - 1)^2 + (K_n + K_j + 1)^2 + (K_n - K_j + 1)^2 + (K_n + K_j - 1)^2}{4} \\
&= 3\left(K_n^2 + K_j^2 + \frac{1}{3}\right).
\end{aligned}$$

This gives (8.3) with $C_1 = \pi^2\sqrt{3}/(2\sqrt{2})$. For the computation of $\|P''\|^2_{L^2}$, we also use the same kind of arguments, together with

$$\begin{aligned}
&(K_n - K_j)^4 + (K_n + K_j)^4 \\
&\quad + \frac{(K_n - K_j - 1)^4 + (K_n + K_j + 1)^4 + (K_n - K_j + 1)^4 + (K_n + K_j - 1)^4}{4} \\
&= 3K_n^4 + 6K_n^2(3K_j^2 + 1) + 3K_j^4 + 6K_j^2 + 1.
\end{aligned}$$

This gives (8.4) with $C_2 = \pi^4\sqrt{3}/(8\sqrt{2})$. □

In the next proposition, we check the second assumption of Proposition 13 with $\widetilde{\mathcal{M}} = \widetilde{M}$ and $\mathcal{M} = M$.

PROPOSITION 15. *For every $P \in H^2_0((0,T),\mathbb{R})$, we have*

$$\|(\widetilde{M} - M)(P)\|_{l^2(\mathbb{N}^*,\mathbb{C})} \leqslant \mathcal{C}_0 \|P\|_{L^2((0,T),\mathbb{R})},$$

$$\|(\widetilde{M} - M)(P)\|_{h^2(\mathbb{N}^*,\mathbb{C})} \leqslant \mathcal{C}_1 \|P\|_{H^1_0((0,T),\mathbb{R})} + \mathcal{C}_3 \|P\|_{L^2((0,T),\mathbb{R})},$$

$$\|(\widetilde{M} - M)(P)\|_{h^4(\mathbb{N}^*,\mathbb{C})} \leqslant \mathcal{C}_2 \|P\|_{H^2_0((0,T),\mathbb{R})} + \mathcal{C}_4 \|P\|_{H^1_0((0,T),\mathbb{R})} + \mathcal{C}_5 \|P\|_{L^2((0,T),\mathbb{R})},$$

where

$$\begin{aligned}
&\mathcal{C}_0 := \sqrt{S_0} + \frac{T}{\sqrt{2}} \left(\left| \sqrt{\lambda_3} - \frac{\pi^2}{4} K_3 \right|^2 + \left| \sqrt{\lambda_2} - \frac{\pi^2}{2} K_2 \right|^2 \right)^{1/2}, \quad \mathcal{C}_2 := \frac{4}{\pi^2}\mathcal{C}_1, \\
&\mathcal{C}_1 := \frac{2\sqrt{2}}{\pi^2} T \left(\left| \sqrt{\lambda_3} - \frac{\pi^2}{4} K_3 \right| + \left| \sqrt{\lambda_2} - \frac{\pi^2}{2} K_2 \right| \right), \\
&S_0 := 2 \sum_{n=1}^{\infty} \left(1 - \operatorname{sinc}\left[T \left(\sqrt{\lambda_n} - \frac{\pi^2}{4} K_n \right) \right] \right),
\end{aligned}$$

$\operatorname{sinc}(x) := \sin(x)/x$, *and $\mathcal{C}_j$ is a positive constant for $j = 3, 4, 5$.*

Proof of Proposition 15. By using decompositions of the form

$$\begin{aligned}
(M - \widetilde{M})(P)_{2k-1} &= \frac{1}{T} \int_0^T P(t) \sin(\sqrt{\lambda_3} t)(e^{-i\sqrt{\lambda_{2k-1}}t} - e^{-i\frac{\pi^2}{4}K_{2k-1}t}) dt \\
&\quad + \frac{1}{T} \int_0^T P(t) \left[\sin(\sqrt{\lambda_3} t) - \sin\left(\frac{\pi^2}{4} K_3 t \right) \right] e^{-i\frac{\pi^2}{4}K_{2k-1}t} dt,
\end{aligned} \tag{8.6}$$

the Cauchy–Schwarz inequality, and Bessel Parseval inequality, we get

$$\begin{aligned}
\|(M - \widetilde{M})(P)\|_{l^2(\mathbb{N}^*,\mathbb{C})} \leqslant{}& \left(\frac{1}{2} \left\| P(t) \left[\sin(\sqrt{\lambda_3} t) - \sin\left(\frac{\pi^2}{4} K_3 t \right) \right] \right\|^2_{L^2} \right. \\
&\left. + \frac{1}{2} \left\| P(t) \left[\sin(\sqrt{\lambda_2} t) - \sin\left(\frac{\pi^2}{4} K_2 t \right) \right] \right\|^2_{L^2} \right)^{1/2} \\
&+ \|P\|_{L^2} \left(\sum_{n=1}^{\infty} \frac{1}{T} \int_0^T |e^{-i\sqrt{\lambda_n}t} - e^{-i\frac{\pi^2}{4}K_n t}|^2 dt \right)^{1/2}
\end{aligned}$$

(the factor $\frac{1}{2}$ comes from the fact that we sum only positive frequencies of a real-valued function), which gives the first bound of the proposition. By using decompositions of the form (8.6), the triangular inequality, the Cauchy–Schwarz inequality, and integrations by parts, we get

$$\|(M-\widetilde{M})(P)\|_{h^2} \leqslant \|P\|_{L^2}\sqrt{S_1} + \frac{2\sqrt{2}}{\pi^2}\left(\left\|\frac{d}{dt}\left\{P(t)\left[\sin(\sqrt{\lambda_3}t) - \sin\left(\frac{\pi^2}{4}K_3 t\right)\right]\right\}\right\|_{L^2} + \left\|\frac{d}{dt}\left\{P(t)\left[\sin\left(\sqrt{\lambda_2}t\right) - \sin\left(\frac{\pi^2}{4}K_2 t\right)\right]\right\}\right\|_{L^2}\right),$$

where

$$S_1 := \sum_{n=1}^{\infty} K_n^2 \frac{1}{T}\int_0^T \left|e^{-i\sqrt{\lambda_n}t} - e^{-i\frac{\pi^2}{4}K_n t}\right|^2 dt.$$

In the same way, we get

$$\|(M-\widetilde{M})(P)\|_{h^4} \leqslant \|P\|_{L^2}\sqrt{S_2} + \frac{8\sqrt{2}}{\pi^4}\left(\left\|\frac{d^2}{dt^2}\left\{P(t)\left[\sin(\sqrt{\lambda_3}t) - \sin\left(\frac{\pi^2}{4}K_3 t\right)\right]\right\}\right\|_{L^2} + \left\|\frac{d^2}{dt^2}\left\{P(t)\left[\sin(\sqrt{\lambda_2}t) - \sin\left(\frac{\pi^2}{4}K_2 t\right)\right]\right\}\right\|_{L^2}\right),$$

where

$$S_2 := \sum_{n=1}^{\infty} K_n^4 \frac{1}{T}\int_0^T \left|e^{-i\sqrt{\lambda_n}t} - e^{-i\frac{\pi^2}{4}K_n t}\right|^2 dt. \qquad \square$$

Finally, in the next proposition, we check the last assumption of Proposition 13 with $\widetilde{\mathcal{M}} = \widetilde{M}$ and $\mathcal{M} = M$.

PROPOSITION 16. *The constants C_0, C_1, C_2, $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$ defined in Propositions* 14 *and* 15 *satisfy $C_1\mathcal{C}_1 = C_2\mathcal{C}_2$, $C_0\mathcal{C}_0 < 1$, and $C_1\mathcal{C}_1 < 1$.*

Proof of Proposition 16. First, let us give a bound on the last term in $\mathcal{C}_0$. We have

$$\cos\left(\frac{5\pi}{2}\right)\cosh\left(\frac{5\pi}{2}\right) = 0 < 1,$$
$$\cos\left(\frac{5\pi}{2} - \frac{1}{1000}\right)\cosh\left(\frac{5\pi}{2} - \frac{1}{1000}\right) = 1.286\ldots > 1,$$

and thus, thanks to the intermediate values theorem, $x_2 \in (0, 1/1000)$. We have

$$\sqrt{\lambda_2} - \frac{\pi^2}{4}K_2 = \left(\frac{5\pi}{2} - x_2\right)^2 - \frac{\pi^2}{4}K_2 = -5\pi x_2 + x_2^2.$$

The two terms of the right-hand side have different signs and $x_2 < 5\pi$, and thus

$$\left(\sqrt{\lambda_2} - \frac{\pi^2}{4}K_2\right)^2 < (5\pi x_2)^2 < 25\pi^2 10^{-6}. \tag{8.7}$$

In the same way, we deduce from

$$\cos\left(\frac{7\pi}{2}\right)\cosh\left(\frac{7\pi}{2}\right) = 0 < 1$$
$$\text{and } \cos\left(\frac{7\pi}{2}+\frac{1}{10000}\right)\cosh\left(\frac{7\pi}{2}+\frac{1}{10000}\right) = 2.98\ldots > 1$$

that $x_3 \in (0, 1/10000)$. Moreover,

$$\sqrt{\lambda_3} - \frac{\pi^2}{4}K_3 = \left(\frac{7\pi}{2}+x_3\right)^2 - \frac{\pi^2}{4}K_3 = 7\pi x_3 + x_3^2,$$

and thus

$$\left(\sqrt{\lambda_3} - \frac{\pi^2}{4}K_3\right)^2 < (7\pi 10^{-4} + 10^{-8})^2 < \pi^2 10^{-6}.$$

Therefore

(8.8)
$$\frac{T}{\sqrt{2}}\left[\left(\sqrt{\lambda_2} - \frac{\pi^2}{4}K_2\right)^2 + \left(\sqrt{\lambda_3} - \frac{\pi^2}{4}K_3\right)^2\right]^{1/2} < \frac{1}{\sqrt{2}}\frac{8}{\pi}(26\pi^2 10^{-6})^{1/2} = 8\sqrt{13}*10^{-3}.$$

Now let us give a bound on S_0. We have

$$S_0 \leqslant 2\sum_{n=1}^{\infty}\frac{1}{6}\left[T\left(\sqrt{\lambda_n} - \frac{\pi^2}{4}K_n\right)\right]^2 \leqslant \frac{T^2}{3}\sum_{n=1}^{\infty}\left(\sqrt{\lambda_n} - \frac{\pi^2}{4}K_n\right)^2. \tag{8.9}$$

First, we study the cases where n is odd. We have

$$\nu_{2k-1} = 2k\pi - \frac{\pi}{2} + x_{2k-1} = (4k-1)\frac{\pi}{2} + x_{2k-1},$$

and thus

$$\left(\sqrt{\lambda_{2k-1}} - \frac{\pi^2}{4}K_{2k-1}\right)^2 = \left(\pi(4k-1)x_{2k-1} + x_{2k-1}^2\right)^2.$$

For every $k \geqslant 2$, we have $x_{2k-1} \leqslant \pi(4k-1)$, and thus, by using (3.3) and (3.4), we get

$$\begin{aligned}\left(\sqrt{\lambda_{2k-1}} - \frac{\pi^2}{4}K_{2k-1}\right)^2 &\leqslant [2\pi(4k-1)x_{2k-1}]^2\\ &\leqslant 4\Big[\pi(4k-1)\frac{\pi}{2\cosh(\nu_{2k-1})}\Big]^2\\ &\leqslant \pi^2\frac{[(4k-1)\pi]^2}{\cosh(2k\pi-\pi/2)^2}\\ &\leqslant \frac{\pi^2}{a}\frac{1}{\cosh(2k\pi-\pi/2)}\\ &\leqslant \frac{2\pi^2}{a}e^{-2k\pi+\frac{\pi}{2}},\end{aligned}$$

where

$$a := \frac{1}{2(7\pi)^2}\left(e^{\frac{7\pi}{2}} - 1 - \frac{7\pi}{2}\right).$$

Indeed, for $k \geqslant 2$, we have

$$\begin{aligned}\cosh\left(2k\pi - \frac{\pi}{2}\right) &= \cosh\left((4k-1)\frac{\pi}{2}\right)\\ &\geqslant \frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n!}\left((4k-1)\frac{\pi}{2}\right)^2\left(\frac{7\pi}{2}\right)^{n-2}\\ &\geqslant \frac{1}{2}\left(\frac{2}{7\pi}\right)^2\left(e^{\frac{7\pi}{2}} - 1 - \frac{7\pi}{2}\right)\left((4k-1)\frac{\pi}{2}\right)^2\\ &\geqslant a[(4k-1)\pi]^2.\end{aligned}$$

Thus,

$$\frac{T^2}{3}\sum_{n=1, n \text{ odd}}^{\infty}\left(\sqrt{\lambda_n} - \frac{\pi^2}{4}K_n\right)^2 \leqslant \frac{T^2}{3}\left\{\left(\sqrt{\lambda_1} - \frac{\pi^2}{4}K_1\right)^2 + \frac{2\pi^2}{a}\int_1^{\infty} e^{-2\pi x + \frac{\pi}{2}}\,dx\right\},$$

We have

$$\cos\left(\frac{3\pi}{2} + \frac{1}{55}\right)\cosh\left(\frac{3\pi}{2} + \frac{1}{55}\right) = 1.030\ldots > 1,$$

and thus $x_1 \in (0, 1/55)$. Therefore

$$\frac{T^2}{3}\sum_{n=1, n \text{ odd}}^{\infty}\left(\sqrt{\lambda_n} - \frac{\pi^2}{4}K_n\right)^2 \leqslant \frac{1}{3}\left(\frac{8}{\pi}\right)^2\left\{\left(\frac{3\pi}{55} + \left(\frac{1}{55}\right)^2\right)^2 + \frac{\pi}{a}e^{-3\pi/2}\right\} \tag{8.10}$$

Now we deal with even integers in the sum S_0. We have

$$\nu_{2k} = 2k\pi + \frac{\pi}{2} - x_{2k} = (4k+1)\frac{\pi}{2} - x_{2k},$$

and thus, for $k \geqslant 2$, by using (3.4), we get

$$\begin{aligned}\left(\sqrt{\lambda_{2k}} - \frac{\pi^2}{4}K_{2k}\right)^2 &= \left(\pi(4k+1)x_{2k} - x_{2k}^2\right)^2\\ &\leqslant (\pi(4k+1)x_{2k})^2\\ &\leqslant \frac{\pi^2}{4}\frac{[(4k+1)\pi]^2}{\cosh(2k\pi + \pi/4)^2}\\ &\leqslant \frac{\pi^2}{4b}\frac{1}{\cosh(2k\pi + \pi/4)}\\ &\leqslant \frac{\pi^2}{2b}e^{-2k\pi - \pi/4},\end{aligned}$$

where

$$b := \frac{2}{(18\pi)^2}\left(e^{17\pi/4} - 1 - \frac{17\pi}{4}\right).$$

Indeed, for $k \geqslant 2$, we have

$$\begin{aligned}
&\cosh\left[2k\pi + \frac{\pi}{4}\right] \cosh\left[\left(4k+\frac{1}{2}\right)\frac{\pi}{2}\right] \\
&\quad \geqslant \frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n!}\left[\left(4k+\frac{1}{2}\right)\frac{\pi}{2}\right]^2\left(\frac{17\pi}{4}\right)^{n-2} \\
&\quad \geqslant \frac{1}{2}\left(\frac{4}{17\pi}\right)^2\left\{e^{\frac{17\pi}{4}} - 1 - \frac{17\pi}{4}\right\}\left(\frac{4k+\frac{1}{2}}{4k+1}\right)^2\left((4k+1)\frac{\pi}{2}\right)^2 \\
&\quad \geqslant \frac{1}{2}\left(\frac{4}{17\pi}\right)^2\left\{e^{\frac{17\pi}{4}} - 1 - \frac{17\pi}{4}\right\}\left(\frac{17}{18}\right)^2\left((4k+1)\frac{\pi}{2}\right)^2 \\
&\quad \geqslant b[(4k+1)\pi]^2.
\end{aligned}$$

Thus,

$$\frac{T^2}{3}\sum_{n=2, n \text{ even}}^{\infty}\left(\sqrt{\lambda_n} - \frac{\pi^2}{4}K_n\right)^2 \leqslant \frac{T^2}{3}\left\{\left(\sqrt{\lambda_2} - \frac{\pi^2}{4}K_2\right)^2 + \frac{\pi^2}{2b}\int_1^{\infty} e^{-2\pi x - \frac{\pi}{2}}\, dx\right\}. \tag{8.11}$$

Thanks to (8.11), we get

$$\frac{T^2}{3}\sum_{n=2, n \text{ even}}^{\infty}\left(\sqrt{\lambda_n} - \frac{\pi^2}{4}K_n\right)^2 \leqslant \frac{1}{3}\left(\frac{8}{\pi}\right)^2\left\{25\pi^2 10^{-6} + \frac{\pi}{4b}e^{-5\pi/2}\right\}. \tag{8.12}$$

Thanks to the explicit expression of C_0 and $\mathcal{C}_0$ and the inequalities (8.8), (8.9), (8.10), and (8.12), we get

$$C_0\mathcal{C}_0 = 0.9847\ldots < 1.$$

Finally, we have

$$\begin{aligned}
C_2\mathcal{C}_2 = C_1\mathcal{C}_1 &= \frac{\sqrt{3}}{2\sqrt{2}}\pi^2\frac{2\sqrt{2}}{\pi^2}\frac{8}{\pi}\left(5\pi x_2 + 7\pi x_3 + x_3^2\right) \\
&\leqslant \sqrt{3}\frac{8}{\pi}\left(\frac{5\pi}{1000} + \frac{14\pi}{10000}\right) = 0.0886\ldots < 1.
\end{aligned}$$

In this proof, there are two numerical values. They were computed thanks to the software Maple, with a precision that guarantees the validity of the first decimals. ∎

9. Controllability of the linearized system around (u, u_t, p). In all of this section, $T := 8/\pi$. The goal of this section is the proof of the following result, which is the only assumption of Theorem 4 which is missing, for its application to the map Φ_T, defined by (1.8).

PROPOSITION 17. *There exist $\delta^* > 0$, $C > 0$ such that*

- *for every $(u_0, \dot{u}_0, p) \in E_8 \cap V$, where*

$$V := \{(u_0, \dot{u}_0, p) \in E_4; \|(u_0 - u_0^{ref}, \dot{u}_0 - \dot{u}_0^{ref}, p)\|_{E_4} < \delta^*\},$$

the map $d\Phi_T(u_0, \dot{u}_0, p)$ has a right inverse

$$d\Phi_T(u_0, \dot{u}_0, p)^{-1} : F_7 \to E_6$$

such that, for every $(U_0, \dot{U}_0, U_T, \dot{U}_T) \in F_7$,

$$\|d\Phi_T(u_0,\dot{u}_0,p)^{-1}.(U_0,\dot{U}_0,U_T,\dot{U}_T)\|_{E_2} \leqslant C\|(U_0,\dot{U}_0,U_T,\dot{U}_T)\|_{E_3},$$
$$\|d\Phi_T(u_0,\dot{u}_0,p)^{-1}.(U_0,\dot{U}_0,U_T,\dot{U}_T)\|_{E_6} \leqslant C[\|(U_0,\dot{U}_0,U_T,\dot{U}_T)\|_{E_7} + \|(u_0-u_0^{ref},\dot{u}_0-\dot{u}_0^{ref},p)\|_{E_8}\|(U_0,\dot{U}_0,U_T,\dot{U}_T)\|_{E_3}]$$

- *and the map*

$$\begin{array}{ccccc} E_8 \cap V & \times & F_7 & \to & E_6 \\ ((u_0,\dot{u}_0,p) & , & (U_0,\dot{U}_0,U_T,\dot{U}_T)) & \mapsto & d\Phi_T(u_0,\dot{u}_0,p)^{-1}.(U_0,\dot{U}_0,U_T,\dot{U}_T) \end{array}$$

is continuous.

Remark 5. A natural strategy for proving Proposition 17 consists in deducing it from Proposition 11 together with an argument of close linear maps. Indeed, if we prove that, for $(u_0,\dot{u}_0,p) \in E_8 \cap V$, $d\Phi_T(u_0,\dot{u}_0,p)$ is a continuous linear map from E_6 to F_7 such that

$$\left\| d\Phi_T(u_0,\dot{u}_0,p) - d\Phi_T(u_0^{ref},\dot{u}_0^{ref},0)\right\|_{\mathcal{L}(E_6,F_7)} < 1, \tag{9.1}$$

then we know that $d\Phi_T(u_0,\dot{u}_0,p)$ has a right inverse $d\Phi_T(u_0,\dot{u}_0,p)^{-1} : F_7 \to E_6$. This strategy is more or less the one used in this section. However, we chose to use the closeness between moment problems (associated to the controllability of the linearized systems) instead of the closeness between $d\Phi_T(u_0,\dot{u}_0,p)$ and $d\Phi_T(u_0^{ref},\dot{u}_0^{ref},0)$. Note that the results of section 2 are not sufficient to prove (9.1). Indeed, Proposition 3 ensures only that $d\Phi_T(u_0,\dot{u}_0,p)$ is a continuous linear map from E_6 to F_6 (and not F_7) and provides bounds only for

$$\left\| d\Phi_T(u_0,\dot{u}_0,p) - d\Phi_T(u_0^{ref},\dot{u}_0^{ref},0)\right\|_{\mathcal{L}(E_6,F_6)}. \tag{9.2}$$

Therefore, in order to get (9.1), we need to work more.

In subsection 9.1, we detail the strategy of the proof. We explain that the proof of Proposition 17 is the consequence of two other results stated in Proposition 20s and 21. The proof of these propositions needs preliminary work done in subsection 9.2. Finally, in subsection 9.3, we prove Proposition 20, and, in subsection 9.4, we prove Proposition 21.

9.1. Strategy. In order to prove Proposition 17, we will transform the controllability of the linearized system around a trajectory (u, u_t, p) into the solvability of a "generalized moment problem." Then we will use the closeness between the generalized moment problem associated to the linearized system around (u, u_t, p) and the moment problem associated to the linearized system around $(u^{ref}, u_t^{ref}, 0)$ to deduce the solvability of the first one from the solvability of the second one (which was proved in the previous section).

First, let us write the controllability of the linearized system around a trajectory (u, u_t, p) into a generalized moment problem. In order to do that, we need some new notations. For $\gamma \in \mathbb{R}$, we introduce the operator A_γ defined by

$$D(A_\gamma) := H^4 \cap H_0^2((0,1),\mathbb{C}), \quad A_\gamma u := u_{xxxx} + \gamma u_{xx}.$$

It is a symmetric unbounded operator on $L^2((0,1),\mathbb{R})$. Let $(\lambda_{k,\gamma})_{k\in\mathbb{N}^*}$ be the nondecreasing sequence of the eigenvalues of A_γ and $(\varphi_{k,\gamma})_{k\in\mathbb{N}^*}$ associated eigenvectors, which form an orthonormal basis of $L^2((0,1),\mathbb{R})$. The maps $\gamma \mapsto \lambda_{k,\gamma}$ and $\gamma \mapsto \varphi_{k,\gamma}$ are analytic, which gives a sense to the notations

$$\left.\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_{\gamma_1} \quad \text{and} \quad \frac{d}{d\gamma}\left[\frac{1}{\sqrt{\lambda_{k,\gamma}}}\varphi_{k,\gamma}\right]_{\gamma_1},$$

which are, respectively, the derivative of the map $\gamma \mapsto \varphi_{k,\gamma}$ considered at the point $\gamma = \gamma_1$ and the derivative of the map $\gamma \mapsto \varphi_{k,\gamma}/\sqrt{\lambda_{k,\gamma}}$ considered at the point $\gamma = \gamma_1$.

Now we transform the controllability of the linearized system around a trajectory (u, u_t, p) into a generalized moment problem. Let $(u_0, \dot{u}_0, p) \in E_8$ and $u \in C^0([0,T], H^8((0,1),\mathbb{R}))$ be the solution of

$$\begin{cases} u_{tt} + u_{xxxx} + p(t)u_{xx} = 0, x \in (0,1), t \in (0,T), \\ u = u_x = 0 \text{ at } x = 0,1, \\ u(0) = u_0, u_t(0) = \dot{u}_0. \end{cases}$$

Let $(U_0, \dot{U}_0, P) \in E_6$. We have

$$d\Phi_T(u_0, \dot{u}_0, P).(U_0, \dot{U}_0, P) = (U_0, \dot{U}_0, U(T), U_t(T)),$$

where $U \in C^0([0,T], H^6((0,1),\mathbb{R}))$ is the solution of

$$\begin{cases} U_{tt} + U_{xxxx} + p(t)U_{xx} + P(t)u_{xx} = 0, x \in (0,1), t \in (0,T), \\ U = U_x = 0 \text{ at } x = 0,1, \\ U(0) = U_0, U_t(0) = \dot{U}_0. \end{cases}$$

We have, for every $t \in [0,T]$,

$$\text{(9.3)} \qquad \begin{pmatrix} U(t) \\ U_t(t) \end{pmatrix} = \sum_{k=1}^{\infty} 2\Re\left(x_k(t) X_{k,p(t)}\right),$$

where, for every $k \in \mathbb{N}^*$,

$$\text{(9.4)} \qquad x_k(t) := \frac{1}{2}\int_0^1 \left[U(t,x)\varphi_{k,p(t)}(x) - \dot{U}(t,x)\frac{1}{i\sqrt{\lambda_{k,p(t)}}}\varphi_{k,p(t)}(x)\right] dx,$$

$$\text{(9.5)} \qquad X_{k,\gamma} := \begin{pmatrix} \varphi_{k,\gamma} \\ -i\sqrt{\lambda_{k,\gamma}}\varphi_{k,\gamma} \end{pmatrix}.$$

By using the partial differential equation solved by U, we get

$$\begin{aligned} \dot{x}_k &= \frac{1}{2}\int_0^1 \left(U_t\varphi_{k,p} + \left[U_{xxxx} + p(t)U_{xx} + P(t)u_{xx}\right] \frac{\varphi_{k,p}}{i\sqrt{\lambda_{k,p}}} \right) dx \\ &\quad + \frac{\dot{p}}{2}\int_0^1 \left(U \left[\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_{p(t)} - U_t \frac{d}{d\gamma}\left[\frac{\varphi_{k,\gamma}}{i\sqrt{\lambda_{k,\gamma}}}\right]_{p(t)} \right) dx \\ &= -i\sqrt{\lambda_{k,p(t)}}x_k(t) - \frac{i}{2}P(t)\left\langle u_{xx}(t), \frac{\varphi_{k,p(t)}}{\sqrt{\lambda_{k,p(t)}}} \right\rangle \\ &\quad + \frac{1}{2}\dot{p}(t)\left(\left\langle U(t), \left.\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_{p(t)} \right\rangle + i\left\langle \dot{U}(t), \frac{d}{d\gamma}\left[\frac{\varphi_{k,\gamma}}{\sqrt{\lambda_{k,\gamma}}}\right]_{p(t)} \right\rangle \right). \end{aligned}$$

This first order ordinary differential equation can be solved explicitly. Then the equality

$$d\Phi_T(u_0,\dot{u}_0,p).(U_0,\dot{U}_0,P)=(U_0,\dot{U}_0,U_T,\dot{U}_T)$$

is equivalent to $(U(T),\dot{U}(T))=(U_T,\dot{U}_T)$, which is equivalent to the generalized moment problem

$$\Upsilon_{(u_0,\dot{u}_0,p)}(P)=d(U_0,\dot{U}_0,U_T,\dot{U}_T),$$

where $\Upsilon_{(u_0,\dot{u}_0,p)}(P) := (\Upsilon_{(u_0,\dot{u}_0,p)}(P)_k)_{k\in\mathbb{N}^*}$, $d(U_0,\dot{U}_0,U_T,\dot{U}_T) := (d(U_0,\dot{U}_0,U_T,\dot{U}_T)_k)_{k\in\mathbb{N}^*}$, and, for every $k\in\mathbb{N}^*$,

(9.6)

$$\begin{aligned}\Upsilon_{(u_0,\dot{u}_0,p)}(P)_k := \int_0^T \Bigg\{ & \frac{-i}{2}P(t)\left\langle u_{xx}(t),\frac{\varphi_{k,p(t)}}{\sqrt{\lambda_{k,p(t)}}}\right\rangle+\frac{1}{2}\dot{p}(t)\left\langle U(t),\left.\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_{p(t)}\right\rangle \\ & +\frac{i}{2}\dot{p}(t)\left\langle \dot{U}(t),\frac{d}{d\gamma}\left[\frac{\varphi_{k,\gamma}}{\sqrt{\lambda_{k,\gamma}}}\right]_{p(t)}\right\rangle\Bigg\} e^{i\int_0^t\sqrt{\lambda_{k,p(s)}}ds}dt,\\ d(U_0,\dot{U}_0,U_T,\dot{U}_T)_k := {} & \frac{1}{2}e^{i\int_0^T\sqrt{\lambda_{k,p(s)}}ds}\left[\langle U_T,\varphi_k\rangle-\frac{1}{i\sqrt{\lambda_k}}\langle\dot{U}_T,\varphi_k\rangle\right]\\ & -\frac{1}{2}\left[\langle U_0,\varphi_k\rangle-\frac{1}{i\sqrt{\lambda_k}}\langle\dot{U}_0,\varphi_k\rangle\right].\end{aligned}$$

Notice that the right-hand side $d(U_0,\dot{U}_0,U_T,\dot{U}_T)$ belongs to $h^7(\mathbb{N}^*,\mathbb{C})$ when $(U_0,\dot{U}_0,U_T,\dot{U}_T)\in F_7$. Thus, Proposition 17 is equivalent to the following proposition.

PROPOSITION 18. *There exist $\delta^*>0$, $C>0$ such that*

- *for every $(u_0,\dot{u}_0,p)\in E_8\cap V$, where*

$$V:=\{(u_0,\dot{u}_0,p)\in E_4;\|(u_0-u_0^{ref},\dot{u}_0-\dot{u}_0^{ref},p)\|_{E_4}<\delta^*\},$$

the map $\Upsilon_{(u_0,\dot{u}_0,p)}$ has a right inverse

$$\Upsilon_{(u_0,\dot{u}_0,p)^{-1}}:h^7(\mathbb{N}^*,\mathbb{C})\to H^2\cap H_0^1((0,T),\mathbb{R})$$

such that, for every $d\in h^7(\mathbb{N}^*,\mathbb{C})$,

$$\|\Upsilon^{-1}_{(u_0,\dot{u}_0,p)}.d\|_{L^2((0,T),\mathbb{R})}\leqslant C\|d\|_{h^3},$$

$$\|\Upsilon^{-1}_{(u_0,\dot{u}_0,p)}.d\|_{H^2}\leqslant C[\|d\|_{h^7}+\|(u_0-u_0^{ref},\dot{u}_0-\dot{u}_0^{ref},p)\|_{E_8}\|d\|_{h^3}],$$

- *the map*

$$\begin{array}{cccccc} E_8\cap V & \times & h^7(\mathbb{N}^*,\mathbb{C}) & \to & H^2\cap H_0^1((0,T),\mathbb{R}) \\ ((u_0,\dot{u}_0,p) & , & d) & \mapsto & \Upsilon^{-1}_{(u_0,\dot{u}_0,p)}.d \end{array}$$

is continuous.

We will get Proposition 18 by applying the following proposition with $\widetilde{\mathcal{M}}$ replaced by $\Upsilon_{(u_0^{ref},\dot{u}_0^{ref},0)}$, $\mathcal{M}$ replaced by $\Upsilon_{(u_0,\dot{u}_0,p)}$, Δ_4 replaced by $C\|(u_0-u_0^{ref},\dot{u}_0-\dot{u}_0^{ref},p)\|_{E_4}$, and Δ_8 replaced by $C\|(u_0-u_0^{ref},\dot{u}_0-\dot{u}_0^{ref},p)\|_{E_8}$.

PROPOSITION 19. *Let $\widetilde{\mathcal{M}}$ be a continuous linear map $L^2((0,T),\mathbb{R}) \to h^3(\mathbb{N}^*,\mathbb{C})$ and $H_0^2((0,T),\mathbb{R}) \to h^7(\mathbb{N}^*,\mathbb{C})$ that has a continuous right inverse $\widetilde{\mathcal{M}}^{-1} : h^7(\mathbb{N}^*,\mathbb{C}) \to H_0^2((0,T),\mathbb{R})$ satisfying, for every $d \in h^7(\mathbb{N}^*,\mathbb{C})$,*

$$\|\widetilde{\mathcal{M}}^{-1}(d)\|_{L^2} \leqslant C_0\|d\|_{h^3},$$
$$\|\widetilde{\mathcal{M}}^{-1}(d)\|_{H_0^2} \leqslant C_0\|d\|_{h^7},$$

with some positive constant C_0.

(1) *Every linear map $\mathcal{M}$ for which there exists $\Delta_4, \Delta_8 > 0$, with $C_0\Delta_4 < 1$ such that, for every $P \in H_0^2((0,T),\mathbb{R})$,*

$$\begin{aligned} &\|(\widetilde{\mathcal{M}} - \mathcal{M})(P)\|_{h^3} \leqslant \Delta_4\|P\|_{L^2}, \\ &\|(\widetilde{\mathcal{M}} - \mathcal{M})(P)\|_{h^7} \leqslant \Delta_4\|P\|_{H_0^2} + \Delta_8\|P\|_{L^2}, \end{aligned} \tag{9.7}$$

has a right inverse $\mathcal{M}^{-1} : h^7(\mathbb{N}^*,\mathbb{C}) \to H_0^2((0,T),\mathbb{R})$ such that, for every $d \in h^7(\mathbb{N}^*,\mathbb{C})$,

$$\begin{aligned} &\|\mathcal{M}^{-1}(d)\|_{L^2} \leqslant \frac{C_0}{1 - C_0\Delta_4}\|d\|_{h^3}, \\ &\|\mathcal{M}^{-1}(d)\|_{H_0^2} \leqslant \frac{C_0}{1 - C_0\Delta_4}\|d\|_{h^7} + \left(\frac{C_0}{1 - C_0\Delta_4} + \frac{C_0\Delta_8}{(1 - C_0\Delta_4)^2}\right)\|d\|_{h^3}. \end{aligned} \tag{9.8}$$

(2) *Let $(\mathcal{M}_\epsilon)_{\epsilon>0}$, $\mathcal{M}$ be linear maps that satisfy the assumptions of statement* (1) *with constants Δ_4^ϵ, Δ_8^ϵ, Δ_4, Δ_8. Let $(d_\epsilon)_{\epsilon>0}$, $d \in h^7(\mathbb{N}^*,\mathbb{C})$. We assume the following:*

(a) *$\mathcal{M}_\epsilon \to \mathcal{M}$ weakly in $\mathcal{L}(H_0^2((0,T),\mathbb{R}), h^7(\mathbb{N}^*,\mathbb{C}))$, i.e., for every $P \in H_0^2((0,T),\mathbb{R})$, $\mathcal{M}_\epsilon(P) \to \mathcal{M}(P)$ in $h^7(\mathbb{N}^*,\mathbb{C}))$ when $\epsilon \to 0$;*

(b) *$d_\epsilon \to d$ in $h^7(\mathbb{N}^*,\mathbb{C})$ when $\epsilon \to 0$;*

(c) *there exists $\Delta_4^*, \Delta_8^* > 0$ such that $C_0\Delta_4^* < 1$, and for every $\epsilon > 0$, $\Delta_4^\epsilon \leqslant \Delta_4^*$ and $\Delta_8^\epsilon \leqslant \Delta_8^*$.*

Then $\mathcal{M}_\epsilon^{-1}(d_\epsilon) \to \mathcal{M}^{-1}(d)$ in $H_0^2((0,T),\mathbb{R})$.

Proof of Proposition 19. First, we prove statement (1). Let $d \in h^7(\mathbb{N}^*,\mathbb{C}))$. We define a sequence $(P_n)_{n\in\mathbb{N}^*} \subset H_0^2((0,T),\mathbb{R})$ by

$$\begin{cases} P_0 := \widetilde{\mathcal{M}}^{-1}(P_0), \\ P_{n+1} := \widetilde{\mathcal{M}}^{-1}[(\widetilde{\mathcal{M}} - \mathcal{M})(P_n)] \ \forall n \in \mathbb{N}. \end{cases}$$

Then, for every $n \in \mathbb{N}$, we have

$$\mathcal{M}\left(\sum_{k=0}^{n} P_k\right) = d + (\mathcal{M} - \widetilde{\mathcal{M}})(P_n). \tag{9.9}$$

Thanks to (9.7), we have, for every $n \in \mathbb{N}$,

$$\begin{aligned} &\|P_n\|_{L^2} \leqslant C_0(C_0\Delta_4)^n\|d\|_{h^3}, \\ &\|P_n\|_{H_0^2} \leqslant C_0(C_0\Delta_4)^n\|d\|_{h^7} + y_n\|d\|_{h^3}, \end{aligned} \tag{9.10}$$

where $(y_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ is defined by

$$\begin{cases} y_0 = C_0, \\ y_{n+1} = C_0\Delta_4 y_n + C_0\Delta_8(C_0\Delta_4)^{n+1} \ \forall n \in \mathbb{N}. \end{cases} \tag{9.11}$$

Since $C_0\Delta_4 < 1$, then $\sum P_n$ converges in $H^2_0((0,T),\mathbb{R})$ to $P := \sum_{n=0}^{\infty} P_n$. By using (9.9), the convergence of P_n to zero in $H^2_0((0,T),\mathbb{R})$, and the continuity of $\mathcal{M} - \widetilde{\mathcal{M}} : H^2_0 \to h^7$, we get $\mathcal{M}(P) = d$. Moreover, we have

$$\|P\|_{L^2} \leqslant \sum_{n=0}^{\infty} \|P_n\|_{L^2} \leqslant \sum_{n=0}^{\infty} C_0(C_0\Delta_4)^n \|d\|_{h^3} \leqslant \frac{C_0}{1-C_0\Delta_4}\|d\|_{h^3},$$
$$\|P\|_{H^2_0} \leqslant \sum_{n=0}^{\infty} \|P_n\|_{H^2_0} \leqslant \frac{C_0}{1-C_0\Delta_4}\|d\|_{h^7} + \left(\sum_{n=0}^{\infty} y_n\right)\|d\|_{h^3}.$$

By using (9.11), we get

$$\left(\sum_{n=0}^{\infty} y_n\right) - C_0 = C_0\Delta_4\left(\sum_{n=0}^{\infty} y_n\right) + \frac{C_0^2\Delta_4\Delta_8}{1-C_0\Delta_4},$$

which gives (9.8).

Now we prove statement (2). Let $P^\epsilon := \mathcal{M}_\epsilon^{-1}(d_\epsilon)$, $P := \mathcal{M}^{-1}(d) \in H^2_0((0,T),\mathbb{R})$ built with the previous construction, i.e., $P^\epsilon = \sum_{n=0}^{\infty} P_n^\epsilon$ and $P = \sum_{n=0}^{\infty} P_n$. We want to prove that $P^\epsilon \to P$ in $H^2_0((0,T),\mathbb{R})$ when $\epsilon \to 0$.

First we prove by induction on $n \in \mathbb{N}$ that, for every $n \in \mathbb{N}$,

$$\mathbf{H}(n) : P_n^\epsilon \to P_n \text{ in } H^2_0((0,T),\mathbb{R}) \text{ when } \epsilon \to 0.$$

It is clear that $P_0^\epsilon := \widetilde{\mathcal{M}}^{-1}(d_\epsilon)$ converges to $P_0 = \widetilde{\mathcal{M}}^{-1}(d)$ in $H^2_0((0,T),\mathbb{R})$. Indeed, $\widetilde{\mathcal{M}}^{-1} : h^7 \to H^2_0$ is continuous and $d_\epsilon \to d$ in h^7. This proves $\mathbf{H}(0)$.

Let $n \in \mathbb{N}$. We assume $\mathbf{H}(n)$. Let us recall that

$$P_{n+1}^\epsilon = \widetilde{\mathcal{M}}^{-1}[(\widetilde{\mathcal{M}} - \mathcal{M}_\epsilon)(P_n^\epsilon)].$$

Since

$$\widetilde{\mathcal{M}} - \mathcal{M}_\epsilon \to \widetilde{\mathcal{M}} - \mathcal{M} \text{ weakly in } \mathcal{L}(H^2_0, h^7) \text{ when } \epsilon \to 0,$$

and $P_n^\epsilon \to P_n$ strongly in H^2_0, then

$$(\widetilde{\mathcal{M}} - \mathcal{M}_\epsilon)(P_n^\epsilon) \to (\widetilde{\mathcal{M}} - \mathcal{M})(P_n) \text{ in } h^7 \text{ when } \epsilon \to 0.$$

Thanks to the continuity of $\widetilde{\mathcal{M}}^{-1} : h^7 \to H^2_0$, we deduce that $\mathbf{H}(n+1)$ holds. This ends the proof by induction.

Now, let us prove statement (2) thanks to the dominated convergence theorem. For every $\epsilon > 0$, for every $n \in \mathbb{N}^*$, we know that

$$\|P_n^\epsilon\|_{H^2_0} \leqslant C_0(C_0\Delta_4^\epsilon)^n\|d\|_{h^7} + y_n^\epsilon\|d\|_{h^3},$$

where $(y_n^\epsilon)_{\epsilon>0} \subset \mathbb{R}$ is defined by

$$\begin{cases} y_0^\epsilon = C_0, \\ y_{n+1}^\epsilon = C_0\Delta_4^\epsilon y_n^\epsilon + C_0\Delta_8^\epsilon(C_0\Delta_4^\epsilon)^n \ \forall n \in \mathbb{N}. \end{cases}$$

Let $(y_n^*)_{\epsilon>0} \subset \mathbb{R}$ be defined by

$$\begin{cases} y_0^* = C_0, \\ y_{n+1}^* = C_0\Delta_4^* y_n^* + C_0\Delta_8^*(C_0\Delta_4^*)^n. \end{cases}$$

Since $C_0\Delta_4^* < 1$, then $(y_n^*)_{n\in\mathbb{N}} \in l^1(\mathbb{N},\mathbb{C})$. Moreover, for every $\epsilon > 0$, for every $n \in \mathbb{N}$, $y_n^\epsilon \leqslant y_n^*$. Therefore, we have

- $P_n^\epsilon \to P_n$ in $H_0^2((0,T),\mathbb{R})$ when $\epsilon \to 0$,
- and for every $\epsilon > 0$ and for every $n \in \mathbb{N}$,

$$\|P_n^\epsilon\|_{H_0^2} \leqslant C_0(C_0\Delta_4^*)^n\|d\|_{h^7} + C_0\Delta_8^* y_n^* \|d\|_{h^3}.$$

The right-hand side of the previous inequality defines a sequence in $l^1(\mathbb{N},\mathbb{C})$, and thus the dominated convergence theorem allows one to conclude that

$$P^\epsilon = \sum_{n=0}^\infty P_n^\epsilon \to \sum_{n=0}^\infty P_n = P \text{ in } H_0^2((0,T),\mathbb{R}) \text{ when } \epsilon \to 0.$$

This ends the proof of Proposition 19. ◻

Now let us explain how we apply Proposition 19 with $\widetilde{\mathcal{M}} = \Upsilon_{(u_0^{ref},\dot{u}_0^{ref},0)}$, $\mathcal{M} = \Upsilon_{(u_0,\dot{u}_0,p)}$, $\mathcal{M}_\epsilon = \Upsilon_{(u_0^\epsilon,\dot{u}_0^\epsilon,p^\epsilon)}$, where $(u_0^\epsilon,\dot{u}_0^\epsilon,p^\epsilon) \to (u_0,\dot{u}_0,p)$ in E_8. First, note that the first assumption of Proposition 19, with $\widetilde{\mathcal{M}} = \Upsilon_{(u_0^{ref},\dot{u}_0^{ref},0)}$, holds thanks to Proposition 12. In order to prove the estimate (9.7), we use the following decomposition:

$$\begin{aligned}&(\Upsilon_{(u_0,\dot{u}_0,p)} - \Upsilon_{(u^{ref}(0),\dot{u}^{ref}(0),0)})(P)\\ &\quad = \frac{-i}{2}\Upsilon^1_{(u_0,\dot{u}_0,p)}(P) + \frac{1}{2}\Upsilon^2_{(u_0,\dot{u}_0,p)}(P) + \frac{i}{2}\Upsilon^3_{(u_0,\dot{u}_0,p)}(P),\end{aligned}$$

where, for every $k \in \mathbb{N}^*$,

$$\begin{aligned}&\Upsilon^1_{(u_0,\dot{u}_0,p)}(P)_k\\ &\quad := \int_0^T P(t)\left(\left\langle u_{xx}(t), \frac{\varphi_{k,p(t)}}{\sqrt{\lambda_{k,p(t)}}}\right\rangle e^{i\int_0^t \sqrt{\lambda_{k,p(s)}}ds} - \left\langle u_{xx}^{ref}(t), \frac{\varphi_k}{\sqrt{\lambda_k}}\right\rangle e^{i\sqrt{\lambda_k}t}\right)dt,\end{aligned}$$

$$\Upsilon^2_{(u_0,\dot{u}_0,p)}(P)_k := \int_0^T \dot{p}(t)\left\langle U(t), \left.\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_{p(t)}\right\rangle e^{i\int_0^t\sqrt{\lambda_{k,p(s)}}ds}dt,$$

$$\Upsilon^3_{(u_0,\dot{u}_0,p)}(P)_k := \int_0^T \dot{p}(t)\left\langle \dot{U}(t), \frac{d}{d\gamma}\left[\frac{\varphi_{k,\gamma}}{\sqrt{\lambda_{k,\gamma}}}\right]_{p(t)}\right\rangle e^{i\int_0^t\sqrt{\lambda_{k,p(s)}}ds}dt.$$

In sections 9.3 and 9.4, we prove the following proposition that allows us to deduce Proposition 18 from Proposition 19.

Proposition 20. *There exist $\delta^* > 0$, $C > 0$ such that, for every $j \in \{1,2,3\}$, for every $(u_0,\dot{u}_0,p) \in E_8$, and for every $P \in H_0^2((0,T),\mathbb{R})$,*

$$\|\Upsilon^j_{(u_0,\dot{u}_0,p)}(P)\|_{h^3(\mathbb{N}^*,\mathbb{C})} \leqslant C\|(u_0 - u_0^{ref}, \dot{u}_0 - \dot{u}_0^{ref}, p)\|_{E_4}\|P\|_{L^2},$$

$$\begin{aligned}&\|\Upsilon^j_{(u_0,\dot{u}_0,p)}(P)\|_{h^7(\mathbb{N}^*,\mathbb{C})}\\ &\quad \leqslant C[\|(u_0 - u_0^{ref}, \dot{u}_0 - \dot{u}_0^{ref}, p)\|_{E_8}\|P\|_{L^2} + \|(u_0 - u_0^{ref}, \dot{u}_0 - \dot{u}_0^{ref}, p)\|_{E_4}\|P\|_{H_0^2}].\end{aligned}$$

Proposition 21. *Let $((u_0^\epsilon,\dot{u}_0^\epsilon,p^\epsilon))_{\epsilon>0}$, $(u_0,\dot{u}_0,p)$ in E_8 such that $(u_0^\epsilon,\dot{u}_0^\epsilon,p^\epsilon) \to (u_0,\dot{u}_0,p)$ in E_8 when $\epsilon \to 0$. Then $\Upsilon_{(u_0^\epsilon,\dot{u}_0^\epsilon,p^\epsilon)} \to \Upsilon_{(u_0,\dot{u}_0,p)}$ weakly in $\mathcal{L}(H_0^2,h^7)$, when $\epsilon \to 0$.*

The proofs of Propositions 20 and 21 need preliminary work that is developed in the next subsection.

9.2. Preliminaries. In this subsection, we prove technical results which are useful for the proof of Propositions 20 and 21.

9.2.1. Technical results about $\lambda_{k,\gamma}$ and $\varphi_{k,\gamma}$. In this section, we state some useful results on the eigenvalues $(\lambda_{k,\gamma})_{k\in\mathbb{N}^*}$ and eigenfunctions $(\varphi_{k,\gamma})_{k\in\mathbb{N}^*}$ introduced in section 9.1. When $\gamma=0$, we write λ_k and φ_k instead of $\lambda_{k,0}$ and $\varphi_{k,0}$. In this case, explicit expressions and asymptotic behaviors are given in Proposition 6. It is well known that $\varphi_{k,\gamma}$ and $\lambda_{k,\gamma}$ are analytic functions of the parameter γ:

$$
\begin{aligned}
\varphi_{k,\gamma} &= \varphi_k + \gamma\varphi_k^{(1)} + \gamma^2\varphi_k^{(2)} + \gamma^3\varphi_k^{(3)} + \cdots, \\
\lambda_{k,\gamma} &= \lambda_k + \gamma\lambda_k^{(1)} + \gamma^2\lambda_k^{(2)} + \gamma^3\lambda_k^{(3)} + \cdots.
\end{aligned} \tag{9.12}
$$

LEMMA 2. *There exists $c\in(0,1)$ such that, for every $j,k\in\mathbb{N}^*$ with the same parity*

$$
(1-c)\pi|k-j| \leqslant |\nu_k-\nu_j| \leqslant (1+c)\pi|k-j|, \tag{9.13}
$$

(9.14)

$$
(1-c)\left(\frac{\pi}{2}\right)^4|(2k+1)^4-(2j+1)^4| \leqslant |\lambda_k-\lambda_j| \leqslant (1+c)\left(\frac{\pi^2}{2}\right)^4|(2k+1)^4-(2j+1)^4|.
$$

Proof of Lemma 2. When $k,j\in\mathbb{N}^*$, $k\neq j$, and k,j have the same parity, thanks to (3.3), we have

$$
\frac{\pi}{2}|k-j| \leqslant |\nu_k-\nu_j| \leqslant 2\pi|k-j|
$$

and

$$
\begin{aligned}
&\left|(\lambda_k-\lambda_j)-\left(\frac{\pi}{2}\right)^4[(2k+1)^4-(2j+1)^4]\right| \\
&\qquad \leqslant \max\left\{\left|\lambda_k-\left(\frac{\pi}{2}(2k+1)\right)^4\right|,\left|\lambda_j-\left(\frac{\pi}{2}(2j+1)\right)^4\right|\right\}.
\end{aligned}
$$

Moreover, by using (3.4), we get

$$
\begin{aligned}
\left|\lambda_k-\left(\frac{\pi}{2}(2k+1)\right)^4\right| &\leqslant 15\left(\frac{\pi}{2}(2k+1)\right)^3 x_k \text{ because } 0<x_k<1 \\
&\leqslant \left(\frac{\pi}{2}\right)^4 m\frac{15}{2320}\frac{(2k+1)^3}{\cosh(k\pi)},
\end{aligned}
$$

where $m:=\inf\{|(2l+1)^4-(2i+1)^4|; l\neq i\in\mathbb{N}^* \text{ with the same parity}\}=2320$. Thus,

$$
\begin{aligned}
\left|\lambda_k-\left(\frac{\pi}{2}(2k+1)\right)^4\right| &\leqslant \left(\frac{\pi}{2}\right)^4 mc_1 \\
&\leqslant \left(\frac{\pi}{2}\right)^4\left|(2k+1)^4-(2j+1)^4\right|c_1,
\end{aligned}
$$

where

$$
c_1 := \sup\left\{\frac{15}{2320}\frac{(2k+1)^3}{\cosh(\pi)}; k\in\mathbb{N}^*\right\} = \frac{81}{464\cosh(\pi)} < 1,
$$

which ends the proof. ∎

PROPOSITION 22. *For every $k \in \mathbb{N}^*$, we have*

$$\frac{d^4\varphi_k^{(1)}}{dx^4} + \frac{d^2\varphi_k}{dx^2} = \lambda_k\varphi_k^{(1)} + \lambda_k^{(1)}\varphi_k, \tag{9.15}$$

$$\lambda_k^{(1)} = -\|\varphi_k'\|^2_{L^2((0,1),\mathbb{R})}, \tag{9.16}$$

$$\varphi_k^{(1)} = \sum_{j\in\mathbb{N}^*, j\neq k, P(j)=P(k)} x_{k,j}\varphi_j, \text{ where } x_{k,j} := \frac{\langle\varphi_k'', \varphi_j\rangle}{(\lambda_k - \lambda_j)}, \tag{9.17}$$

where the sum is taken over all of the integers j different from k, with the same parity as k. There exists a constant $C > 0$ such that, for every $k \in \mathbb{N}^*$,

$$|\lambda_k^{(1)}| \leqslant Ck^2, \tag{9.18}$$

$$\|\varphi_k^{(1)}\|_{L^2((0,1),\mathbb{R})} \leqslant \frac{C}{k}. \tag{9.19}$$

Proof of Proposition 22. Equation (9.15) corresponds to the term of first order with respect to γ, in the equality $A_\gamma\varphi_{k,\gamma} = \lambda_{k,\gamma}\varphi_{k,\gamma}$ developed thanks to (9.12). Notice that the equality

$$\|\varphi_{k,\gamma}\|_{L^2((0,1),\mathbb{R})} = 1$$

implies $\langle\varphi_k^{(1)}, \varphi_k\rangle >= 0$. Then we get (9.16) by taking the L^2-scalar product of (9.15) with the vector φ_k. The expression (9.17) comes from (9.15), and the parity of the functions φ_j justify that half of the components vanish.

Thanks to (9.16), the convexity of the H^s-norms, the behavior (3.3), and the equalities

$$\|\varphi_k\|_{L^2((0,1),\mathbb{R})} = 1, \quad \left\|\frac{d^4\varphi_k}{dx^4}\right\|_{L^2((0,1),\mathbb{R})} = \lambda_k,$$

we get (9.18). By using (9.14) and $\|\varphi_k''\|_{L^2} = \sqrt{\lambda_k} \leqslant Ck^2$, we get

$$\begin{aligned}\|\varphi_k^{(1)}\|_{L^2((0,1),\mathbb{R})} &= \left[\sum_{j\in\mathbb{N}^*, j\neq k, P(j)=P(k)} \left(\frac{\langle\varphi_k'', \varphi_j\rangle >}{\lambda_k - \lambda_j}\right)^2\right]^{1/2} \\ &\leqslant Ck^2\left[\sum_{j\in\mathbb{N}^*, j\neq k, P(j)=P(k)} \frac{1}{[(2k+1)^4 - (2j+1)^4]^2}\right]^{1/2}.\end{aligned}$$

Thanks to the explicit expression for $x > 0$, $x \neq K$,

$$\begin{aligned}\int \frac{dx}{(x^4 - K^4)^2} = &\frac{3}{16K^7}\ln\left(\frac{x+K}{|x-K|}\right) \\ &- \frac{1}{16K^6}\frac{2x}{x^2 - K^2} + \frac{3}{8K^7}\arctan\left(\frac{x}{K}\right) + \frac{1}{8K^6}\frac{x}{x^2+K^2}\end{aligned}$$

we get

$$\sum_{j\in\mathbb{N}^*, j\neq k, P(j)=P(k)} \frac{1}{[(2k+1)^4 - (2j+1)^4]^2} \leqslant \frac{C}{k^6}, \tag{9.20}$$

which gives the conclusion. ☐

COROLLARY 1. *There exist $\gamma^* > 0$ and $C > 0$ such that, for every $\gamma_1 \in (-\gamma^*, \gamma^*)$, for every $k \in \mathbb{N}^*$, we have*

$$\|\varphi_{k,\gamma_1} - \varphi_k\|_{H^s((0,1),\mathbb{R})} \leqslant C|\gamma_1|k^{s-1} \text{ for every integer } s \in [0,4], \tag{9.21}$$

$$|\lambda_{k,\gamma_1} - \lambda_k| \leqslant C|\gamma_1|k^2, \tag{9.22}$$

$$|\sqrt{\lambda_{k,\gamma_1}} - \sqrt{\lambda_k}| \leqslant C|\gamma_1|. \tag{9.23}$$

Proof of Corollary 1. The inequalities (9.22) and (9.23) are consequences of (9.18). The inequality (9.21) for $s = 0$ is a consequence of (9.19). By using the equation

$$\frac{d^4}{dx^4}[\varphi_{k,\gamma_1} - \varphi_k] + \gamma_1 \varphi''_{k,\gamma_1} = \lambda_{k,\gamma_1}(\varphi_{k,\gamma_1} - \varphi_k) + (\lambda_{k,\gamma_1} - \lambda_k)\varphi_k,$$

we get (9.21) for $s = 4$. Indeed, we have (9.22) and

$$\|\varphi''_{k,\gamma_1}\|_{L^2((0,1),\mathbb{R})} = \sqrt{\lambda_{k,\gamma_1}} \leqslant Ck^2.$$

Then (9.21) for $s = 2, 3$ comes from the logarithmic convexity of the H^s-norm. ☐

PROPOSITION 23. *For every $k \in \mathbb{N}^*$, we have*

$$\frac{d^4\varphi_k^{(2)}}{dx^4} + \frac{d^2\varphi_k^{(1)}}{dx^2} = \lambda_k\varphi_k^{(2)} + \lambda_k^{(1)}\varphi_k^{(1)} + \lambda_k^{(2)}\varphi_k, \tag{9.24}$$

$$\lambda_k^{(2)} = \left\langle \frac{d^2\varphi_k^{(1)}}{dx^2}, \varphi_k \right\rangle, \tag{9.25}$$

$$\varphi_k^{(2)} = -\frac{1}{2}\|\varphi_k^{(1)}\|_{L^2}^2\varphi_k + \sum_{j\in\mathbb{N}^*, j\neq k, P(j)=P(k)} \frac{\langle \frac{d^2\varphi_k^{(1)}}{dx^2}, \varphi_j\rangle - \lambda_k^{(1)}x_{k,j}}{(\lambda_k - \lambda_j)}\varphi_j. \tag{9.26}$$

There exists a constant $C > 0$ such that, for every $k \in \mathbb{N}^*$,

$$\|\varphi_k^{(2)}\|_{L^2((0,1),\mathbb{R})} \leqslant \frac{C}{k^2}, \tag{9.27}$$

$$|\lambda_k^{(2)}| \leqslant Ck. \tag{9.28}$$

Proof of Proposition 23. The proof of this proposition is very similar to the one of Proposition 22; thus, here, we justify only the bound (9.27). By using (9.14), we get

$$\|\varphi_k^{(2)}\|_{L^2} \leqslant \frac{1}{2}\|\varphi_k^{(1)}\|_{L^2}^2 + C\left(\left\|\frac{d^2\varphi_k^{(1)}}{dx^2}\right\|_{L^2} + |\lambda_k^{(1)}|\|\varphi_k^{(1)}\|_{L^2}\right) \left[\sum_{j=1, j\neq k, P(j)=P(k)}^{\infty} \frac{1}{[(2k+1)^4 - (2j+1)^4]^2}\right]^{1/2}.$$

Equation (9.24) gives

$$\left\|\frac{d^4\varphi_k^{(1)}}{dx^4}\right\|_{L^2} \leqslant Ck^3,$$

which, with (9.19) and the convexity of the norms, leads to

$$\left\|\frac{d^2\varphi_k^{(1)}}{dx^2}\right\|_{L^2} \leqslant Ck.$$

We conclude by using also (9.18), (9.19), and (9.20). ☐

The vectors φ_k and the real numbers $\lambda_{k,\gamma}$ are analytic functions of the parameter γ, so we can consider their derivatives with respect to γ. We introduce the notations

$$\left.\frac{d^j\varphi_{k,\gamma}}{d\gamma^j}\right]_{\gamma_1}$$

for the jth derivative of the function $\gamma \mapsto \varphi_{k,\gamma}$ evaluated at the point $\gamma = \gamma_1$ and

$$\lambda'_{k,\gamma_1}, \lambda''_{k,\gamma_1}$$

for the first and the second derivative, respectively, of the function $\gamma \mapsto \lambda_{k,\gamma}$ evaluated at the point $\gamma = \gamma_1$.

COROLLARY 2. *There exist $\gamma^* > 0$, $C > 0$, and $l \in \mathbb{R}$ such that, for every $\gamma_1 \in (-\gamma^*, \gamma^*)$, for every $k \in \mathbb{N}^*$, we have*

$$\left\| \left.\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_{\gamma_1} - \left.\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_{0} \right\|_{H^s((0,1),\mathbb{R})} \leqslant C|\gamma_1|k^{s-2} \text{ for every integer } s \in [0,4], \tag{9.29}$$

$$\left\| \left.\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_{\gamma_1} \right\|_{H^s((0,1),\mathbb{R})} \leqslant Ck^{s-1}, \tag{9.30}$$

$$\|\varphi_{k,\gamma_1} - \varphi_k - \gamma_1\varphi_k^{(1)}\|_{L^2((0,1),\mathbb{R})} \leqslant C\frac{|\gamma_1|^2}{k^2}, \tag{9.31}$$

$$|\lambda'_{k,\gamma_1} - \lambda'_k| \leqslant C|\gamma_1|k \text{ and } |\lambda'_{k,\gamma_1}| \leqslant Ck^2, \tag{9.32}$$

$$\left|\sqrt{\lambda_{k,\gamma_1}} - \sqrt{\lambda_k} - \gamma_1 l\right| \leqslant C\frac{|\gamma_1|}{k}. \tag{9.33}$$

Proof of Corollary 2. The proof of Corollary 2 is similar to the one of Corollary 1; thus we justify only (9.33). We deduce from (9.28) that there exist $C > 0$ and $\gamma^* > 0$ such that, for every $\gamma \in (-\gamma^*, \gamma^*)$, for every $k \in \mathbb{N}^*$,

$$\left|\sqrt{\lambda_{k,\gamma}} - \sqrt{\lambda_k} - \gamma\frac{\lambda'_k}{2\sqrt{\lambda_k}}\right| \leqslant C\frac{\gamma^2}{k}.$$

Thus, we need only to prove the existence of constants $C > 0$ and $l \in \mathbb{R}$ such that, for every $k \in \mathbb{N}^*$,

$$\left|\frac{\lambda'_k}{2\sqrt{\lambda_k}} - l\right| \leqslant \frac{C}{k}. \tag{9.34}$$

By using (9.16), (3.7), and (3.14), we get

$$\frac{\lambda'_k}{\sqrt{\lambda_k}} = -\frac{1}{\nu_k\|v_k\|_{L^2}^2}\left(\xi_k^2 I_1(\nu_k) + \zeta_k I_2(\nu_k) + 2\xi_k\zeta_k I_3(\nu_k)\right),$$

where

$$\begin{aligned} I_1(x) &:= \int_0^x (\sin(y)+\sinh(y))^2 dy \\ &= \frac{\sinh(2x)}{4} + \sin(x)\cosh(x) - \cos(x)\sinh(x) - \frac{\sin(2x)}{4}, \\ I_2(x) &:= \int_0^x (\cos(y)-\cosh(y))^2 dy \\ &= \frac{\sinh(2x)}{4} - \cos(x)\sinh(x) - \sin(x)\cosh(x) + x + \frac{\sin(2x)}{4}, \\ I_3(x) &:= \int_0^x (\sin(y)+\sinh(y))(-\cos(y)+\cosh(y)) dy \\ &= \frac{\cosh(2x)}{4} - \cos(x)\cosh(x) + \frac{1}{4} + \frac{\cos(x)^2}{2}. \end{aligned}$$

We get (9.34) thanks to (3.8) and the asymptotic behaviors (3.13). ∎

In the same way as we proved the two previous propositions, we can get the next one.

PROPOSITION 24. *There exists $C>0$ such that, for every $k \in \mathbb{N}^*$,*

$$\|\varphi_k^{(3)}\|_{L^2} \leqslant \frac{C}{k^2}.$$

Thus, there exist $\gamma^*>0$ and $C>0$ such that, for every $\gamma \in (-\gamma^*, \gamma^*)$, for every $k \in \mathbb{N}^*$, we have

$$\left\| \left. \frac{d^2 \varphi_{k,\gamma}}{d\gamma^2} \right]_{\gamma_1} \right\|_{L^2} \leqslant \frac{C}{k^2}. \tag{9.35}$$

9.2.2. Technical results about sequences with the same form as $\boldsymbol{\Upsilon_{(u_0,\dot{u}_0,p)}(P)}$. In this section, we prove bounds for the h^1- or h^3-norm of sequences $S=(S_k)_{k\in\mathbb{N}^*}$ that have a form similar to the one of $\Upsilon_{(u_0,\dot{u}_0,p)}(P)$. For example,

$$S_k := \int_0^T w(t) \left\langle f(t), \frac{\varphi_{k,p(t)}}{\sqrt{\lambda_{k,p(t)}}} \right\rangle e^{i\int_0^t \sqrt{\lambda_{k,p(s)}}ds},$$

where $p, w : [0,T] \to \mathbb{R}$ and $f : [0,T] \to L^2((0,1),\mathbb{R})$.

LEMMA 3. *Let γ^* be as in Corollaries 1 and 2. There exists $C>0$ such that, for every $\gamma_1 \in (-\gamma^*, \gamma^*)$, for every $f \in L^2((0,1),\mathbb{R})$,*

$$\sum_{k=1}^{\infty} \left| k \left\langle f, \left. \frac{d\varphi_{k,\gamma}}{d\gamma} \right]_{\gamma_1} \right\rangle \right|^2 \leqslant C \|f\|^2_{L^2((0,1),\mathbb{R})}, \tag{9.36}$$

$$\sum_{k=1}^{\infty} \left| k \langle f, \varphi_{k,\gamma_1} - \varphi_k \rangle \right|^2 \leqslant C\gamma_1^2 \|f\|^2_{L^2((0,1),\mathbb{R})}. \tag{9.37}$$

Proof of Lemma 3. Note that, in order to get (9.36), it is sufficient to prove it with $\gamma_1 = 0$. Indeed, by using Corollary 2, we have, for $\gamma_1 \in (-\gamma^*, \gamma^*)$ and $f \in L^2((0,1),\mathbb{R})$,

$$\left| k \left\langle f, \left. \frac{d\varphi_{k,\gamma}}{d\gamma} \right]_{\gamma_1} - \left. \frac{d\varphi_{k,\gamma}}{d\gamma} \right]_0 \right\rangle \right| \leqslant \|f\|_{L^2((0,1),\mathbb{R})} \frac{C|\gamma_1|}{k}.$$

For $f \in L^2((0,1),\mathbb{R})$, we have

$$\sum_{k=1}^{\infty} \left| k \left\langle f, \frac{d\varphi_{k,\gamma}}{d\gamma}\right]_0 \right\rangle \right|^2 = \sum_{k=1}^{\infty} \left| \sum_{j=1, j\neq k, P(j)=P(k)}^{\infty} a_{k,j} \langle f, \varphi_j \rangle \right|^2,$$

where the second sum is taken over all $j \in \mathbb{N}^*$ having the same parity as k such that $j \neq k$, $a_{k,j} := k x_{k,j}$, $x_{k,j}$ is defined in (9.17) when $P(k) = P(j)$, and $a_{k,j} = 0$ when $P(k) \neq P(j)$.

If we prove that there exists $C > 0$ such that

(9.38)
$$\forall j \in \mathbb{N}^* \sum_{k \in \mathbb{N}^*, k \neq j, P(k)=P(j)} |a_{k,j}| \leqslant C \text{ and } \forall k \in \mathbb{N}^* \sum_{j \in \mathbb{N}^*, j \neq k, P(j)=P(k)} |a_{k,j}| \leqslant C,$$

then Cauchy–Schwarz inequality provides

$$\forall (x_j)_{j \in \mathbb{N}^*} \in l^2(\mathbb{N}^*, \mathbb{C}), \sum_{k=1}^{\infty} \left| \sum_{j=1}^{\infty} a_{j,k} x_j \right| \leqslant C^2 \sum_{j=1}^{\infty} |x_j|^2,$$

which gives the conclusion. Let us prove (9.38).

By using (3.14), (3.10), (3.17), and (9.14), we get

$$|a_{k,j}| \leqslant C \frac{k^3 j^2 \max\{k,j\}}{[(2k+1)^4 - (2j+1)^4]^2} \text{ when } P(k) \neq P(j).$$

Thus, we have, for every $k \in \mathbb{N}^*$,

$$\begin{aligned} \sum_{j=1}^{\infty} |a_{k,j}| &\leqslant C k^4 \sum_{j<k, P(j)=P(k)}^{\infty} \frac{j^2}{[(2k+1)^4 - (2j+1)^4]^2} \\ &\quad + C k^3 \sum_{j>k, P(j)=P(k)}^{\infty} \frac{j^3}{[(2j+1)^4 - (2k+1)^4]^2} \\ &\leqslant C k^4 \int_3^{2k} \frac{x^2}{[(2k+1)^4 - x^4]^2} dx + C k^3 \int_{2k+2}^{\infty} \frac{x^3}{[(2k+1)^4 - x^4]^2} dx \leqslant C. \end{aligned}$$

For every $j \in \mathbb{N}^*$, we have

$$\begin{aligned} \sum_{k=1}^{\infty} |a_{k,j}| &\leqslant C j^3 \sum_{k<j, P(j)=P(k)}^{\infty} \frac{k^3}{[(2j+1)^4 - (2k+1)^4]^2} \\ &\quad + C j^2 \sum_{k>j, P(j)=P(k)}^{\infty} \frac{k^4}{[(2j+1)^4 - (2k+1)^4]^2} \\ &\leqslant C j^3 \int_3^{2j} \frac{x^3}{[(2j+1)^4 - x^4]^2} dx + C j^2 \int_{2j+2}^{\infty} \frac{x^4}{[(2j+1)^4 - x^4]^2} dx \leqslant C. \end{aligned}$$

Now let us prove (9.37). We use the decomposition

$$\varphi_{k,\gamma_1} - \varphi_k = \left(\varphi_{k,\gamma_1} - \varphi_k - \gamma_1 \frac{d\varphi_{k,\gamma}}{d\gamma}\right]_0 \right) + \gamma_1 \frac{d\varphi_{k,\gamma}}{d\gamma}\right]_0.$$

For the first term, we use (9.31), and, for the second one, we apply (9.36). ☐

LEMMA 4. *Let γ^* be as in Corollaries 1 and 2. There exists $C > 0$ such that, for every $\gamma_1 \in (-\gamma^*, \gamma^*)$, for every $f \in H_0^2((0,1),\mathbb{R})$,*

$$\sum_{k=1}^{\infty} \left| k^3 \left\langle f, \left. \frac{d\varphi_{k,\gamma}}{d\gamma} \right]_{\gamma_1} \right\rangle \right|^2 \leqslant C \|f\|^2_{H_0^2((0,1),\mathbb{R})}. \tag{9.39}$$

Proof of Lemma 4. First, we prove Lemma 4 for $\gamma_1 = 0$ with the same argument as for the previous lemma. For $f \in H_0^2((0,1),\mathbb{R})$, thanks to integrations by parts, we get

$$\sum_{k=1}^{\infty} \left| k^3 \left\langle f, \left. \frac{d\varphi_{k,\gamma}}{d\gamma} \right]_{0} \right\rangle \right|^2 = \sum_{k=1}^{\infty} \left| \sum_{j=1}^{\infty} b_{k,j} \left\langle f'', \frac{\varphi_j''}{\sqrt{\lambda_j}} \right\rangle \right|^2,$$

where

$$b_{k,j} := \frac{k^3}{\sqrt{\lambda_j}} x_{k,j} \text{ when } P(k) = P(j), k \neq j,$$

$b_{k,j} = 0$ in the other cases, and $x_{k,j}$ is defined in (9.17). Note that $(\varphi_j''/\sqrt{\lambda_j})_{j\in\mathbb{N}^*}$ is an orthonormal family of $L^2((0,1),\mathbb{R})$, and thus

$$\sum_{j=1}^{\infty} \left| \left\langle f'', \frac{\varphi_j''}{\sqrt{\lambda_j}} \right\rangle \right|^2 \leqslant \|f\|_{H_0^2((0,1),\mathbb{R})}.$$

We have

$$|b_{k,j}| \leqslant C \frac{k^5 \max\{k,j\}}{[(2k+1)^4 - (2j+1)^4]^2} \text{ when } P(k) \neq P(j).$$

We have, for every $k \in \mathbb{N}^*$,

$$\begin{aligned} \sum_{j=1}^{\infty} |b_{k,j}| &\leqslant Ck^6 \sum_{j<k, P(j)=P(k)}^{\infty} \frac{1}{[(2k+1)^4 - (2j+1)^4]^2} \\ &\quad + Ck^5 \sum_{j>k, P(j)=P(k)}^{\infty} \frac{j}{[(2j+1)^4 - (2k+1)^4]^2} \\ &\leqslant Ck^6 \int_3^{2k} \frac{1}{[(2k+1)^4 - x^4]^2} dx + Ck^5 \int_{2k+2}^{\infty} \frac{x}{[(2k+1)^4 - x^4]^2} dx \leqslant C. \end{aligned}$$

For every $j \in \mathbb{N}^*$, we have

$$\begin{aligned} \sum_{k=1}^{\infty} |b_{k,j}| &\leqslant Cj \sum_{k<j, P(j)=P(k)}^{\infty} \frac{k^5}{[(2j+1)^4 - (2k+1)^4]^2} \\ &\quad + C \sum_{k>j, P(j)=P(k)}^{\infty} \frac{k^6}{[(2j+1)^4 - (2k+1)^4]^2} \\ &\leqslant Cj \int_3^{2j} \frac{x^5}{[(2j+1)^4 - x^4]^2} dx + C \int_{2j+2}^{\infty} \frac{x^6}{[(2j+1)^4 - x^4]^2} dx \leqslant C. \end{aligned}$$

This gives the conclusion for $\gamma_1 = 0$.

Now let us prove that, for $\gamma_1 \in (-\gamma^*, \gamma^*)$ and $f \in H_0^2((0,1),\mathbb{C})$, we have

$$\left| \left\langle f, \left.\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_{\gamma_1} - \left.\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_0 \right\rangle \right| \leqslant \frac{C|\gamma_1|}{k^4} \|f\|_{H_0^2((0,1),\mathbb{R})}, \tag{9.40}$$

which gives the conclusion. Thanks to the equation

$$A_{\gamma_1} \left.\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_{\gamma_1} + \varphi''_{k,\gamma_1} = \lambda_{k,\gamma_1} \left.\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_{\gamma_1} + \lambda'_{k,\gamma_1}\varphi_{k,\gamma_1},$$

we have

$$\begin{aligned}
\left\langle f, \left.\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_{\gamma_1} - \left.\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_0 \right\rangle = {} & \frac{1}{\lambda_{k,\gamma_1}} \left\langle f'', \left(\left.\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_{\gamma_1} - \left.\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_0 \right)'' \right\rangle \\
& + \frac{\gamma_1}{\lambda_{k,\gamma_1}} \left\langle f'', \left.\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_{\gamma_1} \right\rangle \\
& + \frac{1}{\lambda_{k,\gamma_1}} \langle f'', \varphi_{k,\gamma_1} - \varphi_k \rangle - \frac{\lambda'_{k,\gamma_1}}{\lambda_{k,\gamma_1}} \langle f, \varphi_{k,\gamma_1} - \varphi_k \rangle \\
& + \left(\frac{1}{\lambda_{k,\gamma_1}} - \frac{1}{\lambda_k} \right) \left(\left\langle f'', \left.\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_0'' \right\rangle + \langle f'', \varphi_k \rangle \right) \\
& - \left(\frac{\lambda'_{k,\gamma_1}}{\lambda_{k,\gamma_1}} - \frac{\lambda'_k}{\lambda_k} \right) \langle f, \varphi_k \rangle.
\end{aligned}$$

We get (9.40) by using (9.29), (9.30), (9.21), (9.22), and (9.32). □

PROPOSITION 25. *There exists a constant $C > 0$ such that, for every $w \in L^2((0,T),\mathbb{R})$ and for every $f \in C^0([0,T], H^1((0,1),\mathbb{R}))$, the $h^1(\mathbb{N}^*,\mathbb{C})$-norm of the sequence $(S_k)_{k\in\mathbb{N}^*}$ defined by*

$$S_k := \int_0^T w(t)\langle f(t), \varphi_k\rangle e^{-i\sqrt{\lambda_k}t} dt$$

is bounded by

$$C\|w\|_{L^2((0,T),\mathbb{R})}\|f\|_{C^0((0,T),H^1((0,T),\mathbb{R}))}.$$

Proof of Proposition 25. We introduce the function $g \in C^0([0,T], H_0^1((0,T),\mathbb{R}))$ defined by

$$g(t,x) := f(t,x) - f(t,0)(1-x) - f(t,1)x.$$

We have

$$\begin{aligned}
S_k = {} & \int_0^T w(t)\langle g(t), \varphi_k\rangle e^{-i\sqrt{\lambda_k}t} dt \\
& + \langle (1-x), \varphi_k\rangle \int_0^T w(t) f(t,0) e^{-i\sqrt{\lambda_k}t} + \langle x, \varphi_k\rangle \int_0^T w(t) f(t,1) e^{-i\sqrt{\lambda_k}t}.
\end{aligned} \tag{9.41}$$

Thanks to Cauchy–Schwarz inequality in $L^2((0,T),\mathbb{R})$, we get the following bound for the h^1-norm of the first term of the right-hand side of (9.41):

$$\|w\|_{L^2}\left(\int_0^T\sum_{k=1}^{\infty}|k\langle g(t),\varphi_k\rangle|^2dt\right)^{1/2}\leqslant C\|w\|_{L^2}\|g\|_{C^0([0,T],H_0^1((0,1),\mathbb{R}))}.$$

Thanks to (3.7), (3.8), and (3.10) we get

$$(-1)^k\langle(1-x),\varphi_k\rangle=\langle x,\varphi_k\rangle\sim\frac{C}{\nu_k},$$

which gives the conclusion. □

PROPOSITION 26. *Let γ^* be as in Corollaries 1 and 2. There exists a constant $C>0$ such that, for every $p\in H^1((0,T),\mathbb{R})$, with $\|p\|_{L^\infty((0,T),\mathbb{R})}<\gamma^*$, for every $w\in L^2((0,T),\mathbb{R})$, and for every $f\in C^0([0,T],H^1((0,1),\mathbb{R}))$, the $h^3(\mathbb{N}^*,\mathbb{C})$-norm of the sequences $(S_k)_{k\in\mathbb{N}^*}$, $(\widetilde{S}_k)_{k\in\mathbb{N}^*}$ defined by*

$$S_k:=\int_0^T w(t)\left\langle f(t),\frac{\varphi_{k,p(t)}}{\sqrt{\lambda_{k,p(t)}}}\right\rangle e^{-i\sqrt{\lambda_k}t}dt,$$
$$\widetilde{S}_k:=\int_0^T w(t)\left\langle f(t),\frac{\varphi_{k,p(t)}}{\sqrt{\lambda_{k,p(t)}}}\right\rangle e^{-i\int_0^t\sqrt{\lambda_{k,p(s)}}ds}dt$$

are bounded by

$$C\|w\|_{L^2((0,T),\mathbb{R})}\|f\|_{C^0([0,T],H^1((0,T),\mathbb{R})},$$

and, moreover,

$$\|S-\widetilde{S}\|_{h^3}\leqslant C\|p\|_{L^\infty}\|w\|_{L^2}\|f\|_{C^0([0,T],L^2)}.$$

Proof of Proposition 26. Let us consider the decomposition $S_k=S^1+S^2+S^3$, where, for every $k\in\mathbb{N}^*$,

$$\begin{aligned}
S_k^1&:=\int_0^T w(t)\left(\frac{1}{\sqrt{\lambda_{k,p(t)}}}-\frac{1}{\sqrt{\lambda_k}}\right)\langle f(t),\varphi_{k,p(t)}\rangle e^{i\sqrt{\lambda_k}t}dt,\\
S_k^2&:=\frac{1}{\sqrt{\lambda_k}}\int_0^T w(t)\langle f(t),\varphi_{k,p(t)}-\varphi_k\rangle e^{i\sqrt{\lambda_k}t}dt,\\
S_k^3&:=\frac{1}{\sqrt{\lambda_k}}\int_0^T w(t)\langle f(t),\varphi_k\rangle e^{i\sqrt{\lambda_k}t}dt.
\end{aligned}\tag{9.42}$$

Thanks to (9.23) we have

$$\left|\frac{1}{\sqrt{\lambda_{k,p(t)}}}-\frac{1}{\sqrt{\lambda_k}}\right|\leqslant\frac{C|p(t)|}{k^4}.$$

Thus, thanks to the Cauchy–Schwarz inequality, the $h^3(\mathbb{N}^*,\mathbb{C})$-norm of S^1 is bounded by

$$C\|w\|_{L^2}\|p\|_{L^\infty}\|f\|_{C^0([0,T],L^2)}\leqslant C\|w\|_{L^2}\|f\|_{C^0([0,T],L^2)}.$$

Thanks to Lemma 3, the $h^3(\mathbb{N}^*,\mathbb{C})$-norm of S^2 is bounded by

$$C\|w\|_{L^2}\|p\|_{L^\infty}\|f\|_{C^0([0,T],L^2)} \leqslant C\|w\|_{L^2}\|f\|_{C^0([0,T],L^2)}.$$

We conclude the study of S by applying Proposition 25 to S^3.

By using (see Corollary 1)

$$\sqrt{\lambda_{k,p(s)}} = \sqrt{\lambda_k} + p(s)l + \epsilon_k(s), \text{ where } l \in \mathbb{R} \text{ and } |\epsilon_k(s)| \leqslant C\frac{\|p\|_{L^\infty}}{k}, \tag{9.43}$$

we get $\widetilde{S} - S = \delta S^1 + \delta S^2$, where, for every $k \in \mathbb{N}^*$,

$$\begin{aligned} \delta S^1_k &:= \int_0^T w(t)(e^{il\int_0^t p(s)ds} - 1)\left\langle f(t), \frac{\varphi_{k,p(t)}}{\sqrt{\lambda_{k,p(t)}}}\right\rangle e^{i\sqrt{\lambda_k}t}dt \\ \delta S^2_k &:= \int_0^T w(t)e^{il\int_0^t p(s)ds}\left\langle f(t), \frac{\varphi_{k,p(t)}}{\sqrt{\lambda_{k,p(t)}}}\right\rangle e^{i\sqrt{\lambda_k}t}[e^{i\int_0^t \epsilon_k(s)ds} - 1]dt. \end{aligned} \tag{9.44}$$

The first part of Proposition 26 gives the following bound for the $h^3(\mathbb{N}^*,\mathbb{C})$-norm of δS^1:

$$C\|w\|_{L^2}\|p\|_{L^\infty}\|f\|_{C^0([0,T],H^1)}.$$

Thanks to Cauchy–Schwarz inequality and the bound on ϵ_k given in (9.43), we get the following bound for the $h^3(\mathbb{N}^*,\mathbb{C})$-norm of δS^2:

$$C\|w\|_{L^2}\|p\|_{L^\infty}\|f\|_{C^0([0,T],L^2)}. \qquad \square$$

9.3. Proof of Proposition 20. In all of this section, $(u_0, \dot{u}_0, p) \in E_8$ is fixed, and we use the notations

$$\begin{aligned} \delta_4 &:= \|(u_0 - u_0^{ref}, \dot{u}_0 - \dot{u}_0^{ref}, p)\|_{E_4}, \\ \delta_6 &:= \|(u_0 - u_0^{ref}, \dot{u}_0 - \dot{u}_0^{ref}, p)\|_{E_6}, \\ \delta_8 &:= \|(u_0 - u_0^{ref}, \dot{u}_0 - \dot{u}_0^{ref}, p)\|_{E_8}. \end{aligned} \tag{9.45}$$

We assume that $\delta_4 \in [0,1]$ and that δ_4 is small enough so that $\|p\|_{L^\infty((0,T),\mathbb{R})} < \gamma^*$, where γ^* is given in Corollaries 1 and 2. We write Υ^j instead of $\Upsilon^j_{(u_0,\dot{u}_0,p)}(P)$.

Proof of Proposition 20 *for* $j = 1$. For the study of Υ^1 in $h^3(\mathbb{N}^*,\mathbb{C})$, we consider the decomposition

$$\begin{aligned} \Upsilon^1(P)_k = &\int_0^T P(t)\left\langle u_{xx}(t), \frac{\varphi_{k,p(t)}}{\sqrt{\lambda_{k,p(t)}}}\right\rangle [e^{i\int_0^t \sqrt{\lambda_{k,p(s)}}ds} - e^{i\sqrt{\lambda_k}t}]dt \\ &+ \int_0^T P(t)\left(\frac{1}{\sqrt{\lambda_{k,p(t)}}} - \frac{1}{\sqrt{\lambda_k}}\right)\langle u_{xx}(t), \varphi_{k,p(t)}\rangle e^{i\sqrt{\lambda_k}t}dt \\ &+ \frac{1}{\sqrt{\lambda_k}}\int_0^T P(t)\langle u_{xx}(t), \varphi_{k,p(t)} - \varphi_k\rangle e^{i\sqrt{\lambda_k}t}dt \\ &+ \int_0^T P(t)\left\langle (u - u^{ref})_{xx}(t), \frac{\varphi_k}{\sqrt{\lambda_k}}\right\rangle e^{i\lambda_k t}dt. \end{aligned} \tag{9.46}$$

Thanks to Proposition 26, we have the following bound for the $h^3(\mathbb{N}^*, \mathbb{C})$-norm of the first term of the right-hand side of (9.46):

$$C\|P\|_{L^2}\|p\|_{L^\infty}\|u\|_{C^0([0,T],H^2)} \leqslant C\delta_4(1+\delta_4)\|P\|_{L^2} \leqslant C\delta_4\|P\|_{L^2}.$$

Thanks to (9.23) and Proposition 1, the $h^3(\mathbb{N}^*, \mathbb{C})$-norm of the second term of the right-hand side of (9.46) is bounded by

$$C\|P\|_{L^2}\|p\|_{L^\infty}\|u\|_{C^0([0,T],H^2)} \leqslant C\delta_4(1+\delta_4)\|P\|_{L^2} \leqslant C\delta_4\|P\|_{L^2}.$$

Thanks to Lemma 3 and Proposition 1, we have the following bound for the $h^3(\mathbb{N}^*, \mathbb{C})$-norm of the third term of the right-hand side of (9.46):

$$C\|P\|_{L^2}\|p\|_{L^\infty}\|u\|_{C^0([0,T],H^2)} \leqslant C\delta_4(1+\delta_4)\|P\|_{L^2} \leqslant C\delta_4\|P\|_{L^2}.$$

Thanks to Propositions 25 and 2, we have the following bound for the $h^3(\mathbb{N}^*, \mathbb{C})$-norm of the last term of the right-hand side of (9.46):

$$C\|P\|_{L^2}\|(u-u^{ref})_{xx}\|_{C^0([0,T],H^1)} \leqslant C\|P\|_{L^2}\delta_4.$$

In conclusion, we have proved that

$$\|\Upsilon^1\|_{h^3} \leqslant C\delta_4\|P\|_{L^2}.$$

In order to study Υ^1 in h^7, first we study it in $h^5(\mathbb{N}^*, \mathbb{C})$. By using an integration by parts, one gets

(9.47)
$$\begin{aligned}
-\Upsilon^1_k = & \frac{1}{i\sqrt{\lambda_k}}\int_0^T \dot{P}(t)\left(\left\langle u_{xx}(t), \frac{\varphi_{k,p(t)}}{\sqrt{\lambda_{k,p(t)}}}\right\rangle e^{i\int_0^t \sqrt{\lambda_{k,p(s)}}ds} - \left\langle u_{xx}^{ref}(t), \varphi_k\right\rangle e^{i\lambda_k t}\right) dt \\
& + \int_0^T \left(\frac{1}{i\sqrt{\lambda_{k,p(t)}}} - \frac{1}{i\sqrt{\lambda_k}}\right)\dot{P}(t)\left\langle u_{xx}(t), \frac{\varphi_{k,p(t)}}{\sqrt{\lambda_{k,p(t)}}}\right\rangle e^{i\int_0^t \sqrt{\lambda_{k,p(s)}}ds} dt \\
& + \frac{1}{i\sqrt{\lambda_k}}\int_0^T P(t)\left(\left\langle \dot{u}_{xx}(t), \frac{\varphi_{k,p(t)}}{\sqrt{\lambda_{k,p(t)}}}\right\rangle e^{i\int_0^t \sqrt{\lambda_{k,p(s)}}ds} - \left\langle \dot{u}_{xx}^{ref}(t), \frac{\varphi_k}{\sqrt{\lambda_k}}\right\rangle e^{i\lambda_k t}\right) dt \\
& + \int_0^T \left(\frac{1}{i\sqrt{\lambda_{k,p(t)}}} - \frac{1}{i\sqrt{\lambda_k}}\right) P(t)\left\langle \dot{u}_{xx}(t), \frac{\varphi_{k,p(t)}}{\sqrt{\lambda_{k,p(t)}}}\right\rangle e^{i\int_0^t \sqrt{\lambda_{k,p(s)}}ds} dt \\
& + i\int_0^T P(t)\dot{p}(t)\frac{\lambda'_{k,p(t)}}{\lambda^2_{k,p(t)}}\langle u_{xx}(t), \varphi_{k,p(t)}\rangle e^{i\int_0^t \sqrt{\lambda_{k,p(s)}}ds} dt \\
& + \int_0^T P(t)\dot{p}(t)\frac{1}{i\lambda_{k,p(t)}}\left\langle u_{xx}(t), \left.\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_{p(t)}\right\rangle e^{i\int_0^t \sqrt{\lambda_{k,p(s)}}ds} dt.
\end{aligned}$$

Thanks to the study of Υ^1 in $h^3(\mathbb{N}^*, \mathbb{C})$, we have the following bound for the $h^5(\mathbb{N}^*, \mathbb{C})$-norm of the first (resp., third) term of the right-hand side of (9.47):

$$C\|P\|_{H^1_0}\delta_4 \text{ (resp., } C\|P\|_{L^2}\delta_6).$$

Thanks to (9.23), we get the following bound for the $h^5(\mathbb{N}^*, \mathbb{C})$-norm of the second (resp., fourth) term of the right-hand side of (9.47):

$$C\|P\|_{H^1_0}\delta_4 \text{ (resp., } C\|P\|_{L^2}\delta_4).$$

Thanks to (9.32), we get the following bound for the $h^5(\mathbb{N}^*,\mathbb{C})$-norm of the fifth term of the right-hand side of (9.47):

$$C\|P\|_{L^2}\|p\|_{H^2}\|u\|_{C^0([0,T],H^2)} \leqslant C\|P\|_{L^2}\delta_6.$$

Thanks to Lemma 3, we get the following bound for the $h^5(\mathbb{N}^*,\mathbb{C})$-norm of the last term of the right-hand side of (9.47):

$$C\|P\|_{L^2}\delta_4.$$

In conclusion, we have proved that

$$\|\Upsilon^1\|_{h^5(\mathbb{N}^*,\mathbb{C})} \leqslant C[\|P\|_{L^2}\delta_6 + \|P\|_{H^1_0}\delta_4].$$

Finally, let us study Υ^1 in $h^7(\mathbb{N}^*,\mathbb{C})$. Thanks to the study of Υ^1 in $h^5(\mathbb{N}^*,\mathbb{C})$, we have the following bound for the $h^7(\mathbb{N}^*,\mathbb{C})$-norm of the first (resp., third) term of the right-hand side of (9.47):

$$C[\|P\|_{H^1_0}\delta_6 + \|P\|_{H^2_0}\delta_4] \text{ (resp., } C[\|P\|_{L^2}\delta_8 + \|P\|_{H^1_0}\delta_6]).$$

The study of the second and fourth terms of the right-hand side of (9.47) can be done in the same way. Let us work on the first one. We use (9.43). We have

(9.48)
$$\begin{aligned}
&\int_0^T \frac{lp(t)+\epsilon_k(t)}{\sqrt{\lambda_{k,p(t)}\lambda_k}}\dot{P}(t)\left\langle u_{xx}(t), \frac{\varphi_{k,p(t)}}{\sqrt{\lambda_{k,p(t)}}}\right\rangle e^{i\int_0^t \sqrt{\lambda_{k,p(s)}}ds}dt = \\
&\frac{l}{\lambda_k}\int_0^T \dot{P}(t)p(t)\left\langle u_{xx}(t), \frac{\varphi_{k,p(t)}}{\sqrt{\lambda_{k,p(t)}}}\right\rangle e^{i\int_0^t \sqrt{\lambda_{k,p(s)}}ds}dt \\
&+\frac{l}{\sqrt{\lambda_k}}\int_0^T \left(\frac{1}{\sqrt{\lambda_{k,p(t)}}} - \frac{1}{\sqrt{\lambda_k}}\right)\dot{P}(t)p(t)\left\langle u_{xx}(t), \frac{\varphi_{k,p(t)}}{\sqrt{\lambda_{k,p(t)}}}\right\rangle e^{i\int_0^t \sqrt{\lambda_{k,p(s)}}ds}dt \\
&+\int_0^T \frac{\epsilon_k(s)}{\sqrt{\lambda_k}\sqrt{\lambda_{k,p(t)}}}\dot{P}(t)\left\langle u_{xx}(t), \frac{\varphi_{k,p(t)}}{\sqrt{\lambda_{k,p(t)}}}\right\rangle e^{i\int_0^t \sqrt{\lambda_{k,p(s)}}ds}dt.
\end{aligned}$$

Thanks to Proposition (26), we get the following bound for the $h^7(\mathbb{N}^*,\mathbb{C})$-norm of the first term of the right-hand side of (9.48):

$$C\|P\|_{H^1_0}\|p\|_{L^\infty}\|u\|_{C^0([0,T],H^3)} \leqslant C\delta_4(1+\delta_4)\|P\|_{H^1_0} \leqslant C\delta_4\|P\|_{H^1_0}.$$

By using (9.23), we get the following bound for the $h^7(\mathbb{N}^*,\mathbb{C})$-norm of the second term of the right-hand side of (9.48):

$$C\|P\|_{H^1_0}\|p\|_{L^\infty}\|u\|_{C^0([0,T],H^2)} \leqslant C\delta_4\|P\|_{H^1_0}.$$

Thanks to the asymptotic behavior of ϵ_k, we get the same bound for the the $h^7(\mathbb{N}^*,\mathbb{C})$-norm of the last term of the right-hand side of (9.48).

Finally, we get the following bound for the $h^7(\mathbb{N}^*,\mathbb{C})$-norm of the second (resp., fourth) term of the right-hand side of (9.47):

$$C\|P\|_{H^1_0}\delta_4 \text{ (resp., } C\|P\|_{L^2}\delta_6).$$

For the study of the fifth term of the right-hand side of (9.47), we consider the decomposition

$$\int_0^T P(t)\dot{p}(t)\frac{\lambda'_{k,p(t)}}{\lambda^2_{k,p(t)}}\langle u_{xx}(t),\varphi_{k,p(t)}\rangle e^{i\int_0^t\sqrt{\lambda_{k,p(s)}}ds}dt$$
$$= \frac{\lambda'_k}{\lambda_k^{3/2}}\int_0^T P(t)\dot{p}(t)\left\langle u_{xx}(t),\frac{\varphi_{k,p(t)}}{\sqrt{\lambda_{k,p(t)}}}\right\rangle e^{i\int_0^t\sqrt{\lambda_{k,p(s)}}ds}dt$$
$$+\int_0^T P(t)\dot{p}(t)\left(\frac{\lambda'_{k,p(t)}}{\lambda^{3/2}_{k,p(t)}}-\frac{\lambda'_k}{\lambda_k^{3/2}}\right)\left\langle u_{xx}(t),\frac{\varphi_{k,p(t)}}{\sqrt{\lambda_{k,p(t)}}}\right\rangle e^{i\int_0^t\sqrt{\lambda_{k,p(s)}}ds}dt.$$

We study the first term of this decomposition with Proposition 26 and (9.32) and the second one with (9.32); then we get the following bound for the $h^7(\mathbb{N}^*,\mathbb{C})$-norm of the fifth term of the right-hand side of (9.47):

$$C\|P\|_{L^2}\delta_6.$$

For the last term of the right-hand side of (9.47), we cannot apply Lemma 4 because $u_{xx}\notin H^2_0((0,1),\mathbb{R})$; thus, we perform another integration by parts:

$$\begin{aligned}
&-i\int_0^T P(t)\dot{p}(t)\frac{1}{\lambda_{k,p(t)}}\left\langle u_{xx}(t),\left.\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_{p(t)}\right\rangle e^{i\sqrt{\lambda_{k,p(s)}}ds}dt\\
&=\int_0^T [\dot{P}(t)\dot{p}(t)+P(t)\ddot{p}(t)]\frac{1}{(\lambda_{k,p(t)})^{3/2}}\left\langle u_{xx}(t),\left.\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_{p(t)}\right\rangle e^{i\sqrt{\lambda_{k,p(s)}}ds}dt\\
&+\int_0^T P(t)\dot{p}(t)\frac{1}{(\lambda_{k,p(t)})^{3/2}}\left\langle \dot{u}_{xx}(t),\left.\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_{p(t)}\right\rangle e^{i\sqrt{\lambda_{k,p(s)}}ds}dt\\
&-\int_0^T P(t)\dot{p}(t)^2\frac{3\lambda'_{k,p(t)}}{2(\lambda_{k,p(t)})^{5/2}}\left\langle u_{xx}(t),\left.\frac{d\varphi_{k,\gamma}}{d\gamma}\right]_{p(t)}\right\rangle e^{i\sqrt{\lambda_{k,p(s)}}ds}dt\\
&+\int_0^T P(t)\dot{p}(t)^2\frac{1}{(\lambda_{k,p(t)})^{3/2}}\left\langle u_{xx}(t),\left.\frac{d^2\varphi_{k,\gamma}}{d\gamma^2}\right]_{p(t)}\right\rangle e^{i\sqrt{\lambda_{k,p(s)}}ds}dt.
\end{aligned}\tag{9.49}$$

We study the first two terms of the right-hand side of (9.49) thanks to Lemma 3 and the last two thanks to (9.32) and (9.35). In conclusion, we have proved that

$$\|\Upsilon^1\|_{h^7(\mathbb{N}^*,\mathbb{C})}\leqslant C[\|P\|_{L^2}\delta_8+\|P\|_{H^1_0}\delta_6+\|P\|_{H^2_0}\delta_4],$$

and the logarithmic convexity of the norms justifies that

$$\|\Upsilon^1\|_{h^7(\mathbb{N}^*,\mathbb{C})}\leqslant C[\|P\|_{L^2}\delta_8+\|P\|_{H^2_0}\delta_4]. \qquad \square$$

Proof of Proposition 20 *for* $j=2,3$. Lemma 4 gives the bound in $h^3(\mathbb{N}^*,\mathbb{C})$ for Υ^2. As in the previous proof, one can study Υ^2 in h^5 thanks to an integration by parts. Then we deduce the bound in $h^7(\mathbb{N}^*,\mathbb{C})$. The study of Υ^3 can be done in the same way. $\square$

9.4. Proof of Proposition 21. The goal of this section is to sketch the proof of Proposition 21. Let $((u_0^\epsilon,\dot{u}_0^\epsilon,p^\epsilon))_{\epsilon>0}$, $(u_0,\dot{u}_0,p)\in E_8$ such that $(u_0^\epsilon,\dot{u}_0^\epsilon,p^\epsilon)\to(u_0,\dot{u}_0,p)$ in E_8 and $P\in H^2_0((0,T),\mathbb{R})$. By doing again for

$$\Upsilon_{(u_0^\epsilon,\dot{u}_0^\epsilon,p^\epsilon)}(P)-\Upsilon_{(u_0,\dot{u}_0,p)}(P)$$

the same analysis we did for

$$\Upsilon_{(u_0,\dot{u}_0,p)}(P) - \Upsilon_{(u_0^{ref},\dot{u}_0^{ref},0)}(P)$$

in the previous subsection, one can prove that

$$\Upsilon_{(u_0^\epsilon,\dot{u}_0^\epsilon,p^\epsilon)}(P) \to \Upsilon_{(u_0,\dot{u}_0,p)}(P) \text{ in } h^7(\mathbb{N}^*,\mathbb{C}) \text{ when } \epsilon \to 0.$$

10. Remarks, conjectures, prospects. In Theorem 1, the regularity assumption $H^{5+\epsilon}_{(0)} \times H^{3+\epsilon}_{(0)}((0,1),\mathbb{R})$, with $\epsilon > 0$, is technical and related to the use of the Nash–Moser theorem. We conjecture that (Σ) is controllable

- in $H^3_{(0)} \times H^1_0((0,1),\mathbb{R})$ with control functions in $L^2_{\text{loc}}(\mathbb{R},\mathbb{R})$,
- in $H^5_{(0)} \times H^3_{(0)}((0,1),\mathbb{R})$ with control functions in $H^1_{\text{loc}}(\mathbb{R},\mathbb{R})$,
- in $H^7_{(0)} \times H^5_{(0)}((0,1),\mathbb{R})$ with control functions in $H^2_{\text{loc}}(\mathbb{R},\mathbb{R})$, etc.,

because it is the case for the linearized system studied in section 3.

Theorem 1 provides the local controllability in time $T := 8/\pi$. This choice ($T := 8/\pi$) is also technical. The existence of a minimal time for the controllability of this system is an open problem.

Most probably, the proof presented in this article also works for the proof of the local controllability of the same system around

$$\varphi_k(x)\left\{\begin{array}{c}\cos(\sqrt{\lambda_k}t)\\ \text{or}\\ \sin(\sqrt{\lambda_k}t)\end{array}\right\} + \varphi_j(x)\left\{\begin{array}{c}\cos(\sqrt{\lambda_j}t)\\ \text{or}\\ \sin(\sqrt{\lambda_j}t)\end{array}\right\}$$

when k and j are integers with different parities. The same argument should also be adaptable to the beam equation with follower loads

$$\begin{cases} u_{tt} + u_{xxxx} + p(t)u_{xx} = 0, (t,x) \in \mathbb{R}_+ \times (0,1),\\ u = u_x = 0 \text{ at } x = 0,\\ u_{xx} = u_{xxx} = 0 \text{ at } x = 1, \end{cases}$$

which is also proved to be not controllable in $H^2 \times L^2((0,1),\mathbb{R})$ with $L^r_{\text{loc}}(\mathbb{R},\mathbb{R})$-control functions, in [1].

Since the coefficients $\langle \varphi_1'', \varphi_k \rangle$ vanish when k is even, the linearized system of (Σ) around the trajectory $(u(t,x) = \varphi_1(x)\sin(\sqrt{\lambda_1}t), p \equiv 0)$ is not controllable. In order to prove the local controllability of (Σ) around this trajectory, the return method would probably work, as in [3]. This method was introduced by Coron in [5] in order to solve a stabilization problem. It has been used in order to get controllability results for partial differential equations by Coron in [8], [6], [7], by Coron and Fursikov in [9], by Fursikov and Imanuvilov in [11], by Glass in [12], [14], [16], [13], [15], [17], [18], and by Horsin in [21]; see also the book [10] by Coron.

The strategy developed in [4] could be used in order to prove steady-state controllability results of the type: For every $k, l \in \mathbb{N}^*$, there exists $T > 0$ and $p \in H^1_0((0,T),\mathbb{R})$ such that the solution of (Σ) with $u(0) = \varphi_k$ and control p satisfies $u(T) = \varphi_l$.

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