WEAK TRUNCATION ERROR ESTIMATES FOR ELLIPTIC PDES WITH LOGNORMAL COEFFICIENTS

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Abstract. In this work, we are interested in the weak error committed on the solution of an elliptic partial differential equation with a lognormal coefficient, resulting from the approximation of the lognormal coefficient through a Karhunen-Loève expansion. We wish to improve results of a previous work, in which L^p-estimates of the weak error are provided. Only small enough values of p (the corresponding values of p depend on the space dimension) could be considered and such bounds are not sufficient to be applied to practical cases. Moreover, the optimality of this weak order (which turns out to be twice the strong order) has not been studied numerically. Therefore, the aim of this paper is double. First we improve drastically the weak error estimate by providing a bound of the $\mathcal{C}^1$-norm of the weak error. This requires regularity results in Hölder spaces, with explicit bounds for the constants. We also consider much more general test functions in the definition of the weak error. Finally, we show the optimality of the weak order and illustrate this weak convergence with numerical results.

Key words. uncertainty quantification, elliptic PDE with random coefficients, Karhunen-Loève expansion, weak error estimate, lognormal distribution.

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1. Introduction. In the context of uncertainty quantification, uncertainties on some physical properties of a media can be modelled by using random fields, leading to partial differential equations with random coefficients. The aim is then to compute the law of the solution. In this paper, we are concerned with the case of an elliptic partial differential equation with a lognormal coefficient. Such models are in particular widely used in hydrogeology to model flow in porous media with uncertainty, see e.g. [8, 14]. Elliptic PDEs with lognormal coefficients have been studied e.g. in [3, 5, 9, 13], where questions of existence and uniqueness, regularity, spatial discretization of the solution have been addressed among other questions more specific to each of these papers. We note that more general models could be easily considered to model uncertainty on the coefficients, considering functions of gaussian fields which are not necessarily the exponential one. Under adequate assumptions, such models would provide similar results using the same techniques. However, for the sake of simplicity we focus on the case of lognormal coefficients, which is also of particular interest in view of applications.

More precisely, we focus here on the question of the approximation of the lognormal coefficients by a finite number of random variables. Such an approximation is used in many numerical methods for PDE with random coefficients. It is in particular the first and fundamental step of the so called stochastic Galerkin methods, see e.g. [2, 7, 10, 11, 16, 17, 20, 25] and stochastic collocation methods, see e.g. [1, 16, 22, 26], which have been extensively studied lately. The numerical cost of such methods tends to increase very fast when the number of random variables used to approximate the coefficients gets larger. Methods are developed to reduce this increase of the cost with respect to the stochastic dimension but it is still a major difficulty in concrete applications. Thus it is important to have sharp error estimates on the error result-

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ing on the solution from the approximation of the coefficient in a finite dimensional stochastic space. Such sharp error estimates can also be used in the framework of multilevel Monte-Carlo methods, when the coefficients are approximated through a truncated Karhunen-Loève expansion, see e.g. [15], or in the improvement proposed in [21], where Karhunen-Loève expansions truncated at different order for each grid are used. Approximation of the random coefficient in a finite dimensional space is also used in the context of quasi Monte-Carlo methods, see for example [6]. Here, we consider the case where a truncated Karhunen-Loève expansion of the logarithm of the coefficient is used to achieve such an approximation of the lognormal coefficient. Our aim is then to estimate the resulting error on the solution of the PDE, more precisely we are here interested in the weak error, i.e. the error made on the law. Such an estimate is indeed an important element to get complete numerical analysis of the above mentioned numerical methods.

The question of estimating the error made on the law of the solution has already been addressed in [3], where the first author provided bounds for the L^p norm of the weak error, for p small enough (the corresponding values of p depend on the spatial dimension). The bounds depend on the eigenpairs of the Karhunen-Loève expansion and a weak order equal to twice the strong order was obtained. However such an estimate of the weak error, is not sufficient to provide error estimates for most of the output functionals considered in practical cases.

More precisely, for f in $L^2(D)$ and a a lognormal coefficient, we consider the equation:

$$\begin{cases} -\mathrm{div}(a(\omega,x)\nabla u(\omega,x)) & = f(x) \quad \text{on } D, \\ u & = 0 \quad \text{on } \partial D, \end{cases}$$

and its approximation when a is replaced by a^N another lognormal random field obtained by a truncation of the Karhunen-Loève expansion of the logarithm of a.

Natural quantities that one wants to approximate are: mean point values of the pressure $x \mapsto \mathbb{E}[u(\omega,x)]$, variance of point values of the pressure $x \mapsto \mathbb{E}[(u(\omega,x) - \mathbb{E}[u(\omega,x)])^2]$, mean value of the L^p norm (to the power p) of the pressure $\mathbb{E}[\|u(\omega,x)\|^p_{L^p_x}]$, mean value of the outflow through a part Γ of the boundary $\mathbb{E}[\int_\Gamma -a(\omega,x)\nabla u(\omega,x).d\vec{\nu}]$. For most of these output functionals, the results of [3] do not apply. More precisely they only give weak error bounds for the computation of $x \mapsto \mathbb{E}[\varphi(u(\omega,x))]$, for some functionals $\varphi : \mathbb{R} \to \mathbb{R}$. They provide bounds in H^1-norm if φ is supposed to be linear, and bounds in L^p norm for some values of p if φ is quadratic or has bounded derivatives. Therefore it does not enable in particular to treat neither the computation of point values of the pressure, nor the computation of the second moment of the pressure in dimension 3, nor the computation of outflow through a part of the boundary. Comments on possible output functionals and corresponding analysis of the multilevel Monte Carlo method can be found in [21].

In the present work, the main improvement with respect to [3] is the use a regularity result in Hölder spaces for elliptic PDEs, which makes explicit how the Hölder norm of the solution depends on the coefficients, instead of the basic estimates in Sobolev spaces used in [3]. The latter only provide estimates of the weak error in L^p norm for small values of p. We are then able to bound the weak error (corresponding to the computation of $x \mapsto \mathbb{E}[\varphi(u(\omega,x))]$) in C^1 norm. We note that we get spatial pointwise estimates of the weak error. Moreover, we also consider more general test functions than in [3], we only suppose the test functions to have derivatives with at most only polynomial growth, and do not suppose them to have bounded derivatives or to be linear or quadratic. It enables in particular to treat the case of polynomial functional, which is of course of particular interest for applications. To complete this result, we also treat the case of test functions defined on $\mathcal{C}^{1,\alpha}$: we consider the computation of $\mathbb{E}[\psi(u)]$ where $\psi : \mathcal{C}^{1,\alpha} \to \mathbb{R}$, and even the more general case where ψ depends on u and also on a, leading to a slightly different definition of the weak error. Using the bounds obtained for these two types of weak error, we can apply them to a large range of output functionals which are natural for applications, including all the examples given above. In particular, it completes the results of [21], where general output functionals are considered in the numerical analysis of the multilevel Monte Carlo method. Corresponding optimal finite element error bounds are established, whereas strong error estimates are used to bound the truncation error, leading to a non optimal order. We indeed show that we get a weak order equal to twice the strong order for the truncation error. The second novelty of this work is to study numerically the weak convergence. We focus on the case of a 1D lognormal coefficient with exponential covariance, which is a frequently used model in hydrogeology, and corresponds to a case where the convergence of the Karhunen-Loève expansion is quite slow.

The outline of the paper is as follows. In the first section, we recall the settings, some useful lemma given in [3] and state the two main results of this paper. The three subsequent sections are devoted to the proofs of these results. We first give in the second section a regularity result for elliptic PDEs, in Hölder spaces, which gives an explicit dependance of the constant on the coefficient a. We note that this result is quite general, and could be useful for other applications. For example it may also be used to get L^∞ and $W^{1,\infty}$ finite element error estimates, see e.g. [23] for this topic. Using this crucial preliminary result, we give in sections three and four the proofs of the weak error bounds for the two types of weak error given in the first section. Finally we propose numerical results in section 5. We show the optimality of the convergence rate for the weak errors obtained in section three and four in the case of a one dimensional lognormal field with exponential covariance, considering different output functionals. We also study the influence of the parameters appearing in the covariance, namely the covariance length, which physically represents the characteristic size of the covariance of uncertainty, and the coefficient which describes the amplitude of the uncertainty.

2. Preliminaries and main results. In this section, we recall the framework, some main results and useful preliminary results of [3] before improving and generalising one of its main results. We first define the homogeneous lognormal random field a and make some regularity assumptions on its covariance kernel, then we define a linear elliptic PDE with a random coefficient, namely a, and finally recall an existence and uniqueness of the solution of this equation.

Let D be an open bounded domain in $\mathbb{R}^d$ with $\mathcal{C}^2$ boundary and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We consider a function $k \in \mathcal{C}^{0,1}(\mathbb{R}^+, \mathbb{R})$, and $g : \Omega \times \bar{D} \to \mathbb{R}$ a mean-free gaussian field with covariance function $cov[g](x,y) = k(\|x-y\|)$. Under these assumptions, g and hence a admit a version whose trajectories belong to $\mathcal{C}^{0,\alpha}(\bar{D})$ a.s. for $\alpha < 1/2$. In what follows g and a will denote these versions. Therefore we can define for almost every ω, $a^{min}(\omega) = \min_{x \in \bar{D}} a(\omega, x)$ and $a^{max}(\omega) = \max_{x \in \bar{D}} a(\omega, x)$. We define the lognormal homogeneous random field $a : \Omega \times \bar{D} \to \mathbb{R}$ as $a(\omega, x) = e^{g(\omega, x)}$.

PROPOSITION 2.1. *Let f in $L^2(D)$, then the equation:*

$$\begin{cases} -\operatorname{div}(a(\omega,x)\nabla u(\omega,x)) &= f(x) \quad \text{on } D, \\ u &= 0 \quad \text{on } \partial D, \end{cases} \tag{2.1}$$

admits a unique solution u, which belongs to $L^q(\Omega, H_0^1(D))$, for any $q \geq 1$.

REMARK 2.2. *All the following results hold in the case where the forcing term f is stochastic and g is not mean-free, under adequate assumptions.*

Next we define the approximated random field a_N through a Karhunen-Loève expansion truncated at order N, before estimating the corresponding error on the solution. Such an approximation of coefficients in a finite dimensional stochastic space is the first and fundamental step of several numerical methods, in particular the so-called spectral stochastic methods (stochastic Galerkin methods, see e.g. [2, 7, 10, 11, 16, 17, 20, 25] and stochastic Collocation methods, see e.g [1, 16, 22, 24, 26]). Moreover, the numerical cost of these spectral stochastic methods and even the possibility of using these methods strongly depend on the stochastic dimension, i.e. N. Therefore the question of the approximation of coefficients in a finite dimensional stochastic space of small dimension is crucial.

Let $(\lambda_n, b_n)_{n\geq 0}$ denote the sequence of eigenpairs associated with the compact self-adjoint operator that maps

$$f \in L^2(D) \mapsto \int_D \operatorname{cov}[g](x,.)f(x)dx \ \in L^2(D),$$

where $\lambda_1 \geq \lambda_2 \geq ... \geq 0$, and the eigenfunctions are orthonormal. Then the truncated Karhunen-Loève expansion (see [18, 19] for more details) g_N of the stochastic gaussian process g and its exponential a_N are defined by

$$g_N(\omega,x) = \sum_{n=1}^{N} \sqrt{\lambda_n} b_n(x) Y_n(\omega), \quad a_N(\omega,x) = e^{\sum_{n=1}^{N} \sqrt{\lambda_n} b_n(x) Y_n(\omega)},$$

where the real random variables $(Y_n)_{n\geq 1}$ are uniquely determined by

$$Y_n(\omega) = \frac{1}{\sqrt{\lambda_n}} \int_D g(\omega,x) b_n(x) dx.$$

They are independent gaussian random variables with mean zero and unit variance. We also define for later use the more general quantity

$$a_{\gamma,N}(\omega,x) = e^{\sum_{n=1}^{N} \sqrt{\lambda_n}\gamma_n b_n(x) Y_n(\omega)},$$

for any $\gamma \in [0,1]^N$. We recall that one of the main goals of [3] was to estimate both the strong and the weak error resulting on the solution of the PDE (2.1) from this truncation. In all this paper, we will use the following notation for a function v Hölder continuous with exponent α :

$$|v|_\alpha = \sup_{x,\hat{x}\in\Omega, x\neq\hat{x}} \frac{|v(x)-v(\hat{x})|}{|x-\hat{x}|^\alpha},$$

so that

$$\|v\|_{C^{0,\alpha}(\bar{D})} = \|v\|_{C^0(\bar{D})} + |v|_\alpha.$$

From now on, we make the following assumption in order to get estimates of the truncation error:

ASSUMPTION 2.3.

The eigenfunctions b_n are Hölder continuous with exponent $1/2$ and the series $\sum_{n\geq 1}\lambda_n\|b_n\|_{C^{0,1/2}(\bar{D})}$ is convergent.

REMARK 2.4. *Such assumptions are fulfilled in the case of an exponential (for the $\|\cdot\|_1$-norm) covariance on a rectangular domain, and in the case of a gaussian covariance, and more generally in the case of an analytic covariance function, for more details see [3].*

The minimum and maximum of the continuous function $a_{\gamma,N}(\omega,\cdot)$ will be naturally denoted by $a^{\max}_{\gamma,N}(\omega)$ and $a^{\min}_{\gamma,N}(\omega)$.

DEFINITION 2.5. *In the case where assumption 2.3 is fulfilled, we define for $0\leq\alpha\leq 1/2$ and $N\in\mathbb{N}$*

$$R_N^\alpha=\sum_{n>N}\lambda_n\|b_n\|_{C^{0,\alpha}(\bar{D})}.$$

This quantity will be used to express the truncation error on the solution of the PDE 2.1.

We now recall two preliminary results which will be useful later. These propositions are slight modifications of results which can be found in [3].

PROPOSITION 2.6. *For any $0\leq\beta<1/2$ and $q\geq 1$, $\|a\|_{C^{0,\beta}(\bar{D})}$, a^{max} and $\frac{1}{a^{min}}$ belong to $L^q(\Omega)$. Moreover for any $N\in\mathbb{N}$, and $\gamma\in[0,1]^N$, $\|a_{\gamma,N}\|_{C^{0,\beta}(\bar{D})}$, $a^{max}_{\gamma,N}$ and $\frac{1}{a^{min}_{\gamma,N}}$ also belong to $L^q(\Omega)$ and there exists constants $D_{\beta,q}$, D_q such that for any $N\in\mathbb{N}$, and $\gamma\in[0,1]^N$*

$$\|a_{\gamma,N}\|_{L^q(\Omega,C^{0,\beta}(\bar{D}))}\leq D_{\beta,q},\quad \left\|\frac{1}{a^{min}_{\gamma,N}}\right\|_{L^q(\Omega)}\leq D_q,\ \text{and}\quad \|a^{max}_{\gamma,N}\|_{L^q(\Omega)}\leq D_q.$$

This is proved in [3] (Propositions 2.3 and 3.10) for $\beta=0$. These proofs can be easily generalised to the case where $0<\beta<1/2$, since it is proved in [3] that g belongs to $L^q(\Omega,\mathcal{C}^{0,\beta}(\bar{D}))$ and $g_{\gamma,N}$ is bounded in $L^q(\Omega,\mathcal{C}^{0,\beta}(\bar{D}))$. It remains then to replace the estimates in $\mathcal{C}^0(\bar{D})$ norm by estimates in $\mathcal{C}^{0,\beta}(\bar{D})$ norm by applying the Fernique theorem in the separable Banach space $\mathcal{C}^{0,\beta}_0(\bar{D})$ instead of $\mathcal{C}^0(\bar{D})$. Indeed

$$\mathcal{C}^{0,\beta}_0(\bar{D})=\left\{f\in\mathcal{C}^{0,\beta}(\bar{D})|\forall\varepsilon>0\quad\exists\eta>0\text{ such that }|x-y|\leq\eta\Rightarrow\frac{|f(x)-f(y)|}{\|x-y\|^\beta}\leq\varepsilon\right\}$$

is a separable subspace of $\mathcal{C}^{0,\beta}(\bar{D})$ and we can always find $\varepsilon>0$ such that $\beta+\varepsilon<1/2$ and hence $\mathcal{C}^{0,\beta+\varepsilon}(\bar{D})\subset\mathcal{C}^{0,\beta}_0(\bar{D})$.

PROPOSITION 2.7. *For any α,β with $0\leq\beta<\alpha<1/2$ and $q\geq 1$, there exists a constant $A_{\alpha,\beta,p}$ such that for all N in $\mathbb{N}$:*

$$\|a_N-a\|_{L^q(\Omega,\mathcal{C}^{0,\beta}(\bar{D}))}\leq E_{\alpha,\beta,q}(R^\alpha_N)^{\frac{1}{2}}.$$

This has been proved in [3](proposition 3.11) for $\beta=0$. The proof can be easily generalised to the case where $0<\beta<\alpha<1/2$, by noticing that for almost all ω we have

$$\begin{aligned}\|(e^g-e^{g_N})(\omega)\|_{\mathcal{C}^{0,\beta}(\bar{D})}\leq&\|(a+a_N)(\omega)\|_{\mathcal{C}^0(\bar{D})}\|(g-g_N)(\omega)\|_{\mathcal{C}^{0,\beta}(\bar{D})}\\&+\|g-g_N\|_\infty(\|g\|_{\mathcal{C}^{0,\beta}(\bar{D})}+\|g_N\|_{\mathcal{C}^{0,\beta}(\bar{D})}).\end{aligned}$$

It remains to apply Hölder inequality, Proposition 3.4 of [3] and Proposition 2.6.

We now introduce the solutions of the corresponding PDEs. Since for all $N \in \mathbb{N}$ and $\gamma \in [0,1]^N$, the random variables $a_{\gamma,N}^{max}$ and $\frac{1}{a_{\gamma,N}^{min}}$ belong to $L^q(\Omega)$ for all $q \geq 1$, the equation

$$\left\{ \begin{array}{rll} -\mathrm{div}(a_{\gamma,N}(\omega,x)\nabla u_{\gamma,N}(\omega,x)) & = f(x), & \text{on } D, \\ u_{\gamma,N}(\omega,x) & = 0, & \text{on } \partial D, \end{array} \right. \tag{2.2}$$

admits therefore a unique solution $u_{\gamma,N} \in L^q(\Omega, H_0^1(D))$ for all $q \geq 1$. In particular, for $\gamma = (1,1,...,1)$, we have $a_{\gamma,N} = a_N$ and we denote by u_N the solution. Let us set for $(y_1,...,y_N) \in \mathbb{R}^N$ and $x \in D$, $\tilde{a}_{\gamma,N}(y_1,...,y_N,x) = e^{\sum_{n=1}^N \sqrt{\lambda_n}\gamma_n b_n(x) y_n}$ and $\tilde{u}_{\gamma,N}(y_1,...,y_N,\cdot)$ be the solution of

$$\left\{ \begin{array}{rll} -\mathrm{div}(\tilde{a}_N(y,x)\nabla \tilde{u}_N(y,x)) & = f(x), & \text{on } D, \\ \tilde{u}_N(y,x) & = 0, & \text{on } \partial D. \end{array} \right. \tag{2.3}$$

It is classical that $\tilde{u}_N$ is a $\mathcal{C}^\infty$ function of $y_1,...,y_N$. When we need to emphasise the dependance of $\tilde{u}_N$ on $y_1,...,y_N$, we write $\tilde{u}_N(y_1,...,y_N)$. We have then $\tilde{u}_N \in \mathcal{C}^\infty(\mathbb{R}^N, H_0^1(D))$. We notice that a.s. $a_{\gamma,N}(\omega,x) = \tilde{a}_{\gamma,N}(Y_1(\omega),...,Y_N(\omega),x)$, and hence it is easy to see that we have $u_{\gamma,N}(\omega,x) = \tilde{u}_{\gamma,N}(Y_1(\omega),...,Y_N(\omega),x)$.

We now state the two main results of this paper, which establish bounds for two kinds of weak truncation errors. More precisely we estimate the error committed by approximating the law of the solution u of (2.1) by the law of the solution u_N of the PDE (2.2) with coefficient a_N. First we consider regular functionals $\varphi : \mathbb{R} \to \mathbb{R}$, whose derivatives have at most polynomial growth, and bound $x \mapsto \mathbb{E}_\omega[\varphi(u_N(\omega,x)) - \varphi(u(\omega,x))]$ in $\mathcal{C}^{1,\beta}$ norm. We recall that such quantities were bounded in [3] in L^q norm, for some finite values of q depending on the spatial dimension d, in the cases where the functionals φ are linear, quadratic or have bounded derivatives.

THEOREM 2.8. *For any $f \in L^p(D)$ with $p > d$ and any β such that $\frac{1}{2} \wedge \left(1 - \frac{d}{p}\right) > \beta > 0$, with $\varphi \in \mathcal{C}^6(\mathbb{R},\mathbb{R})$ such that φ and its derivatives have at most polynomial growth, there exists a constant $C_{w1}(\beta,f,p,\varphi)$ such that for all $N \in \mathbb{N}$, , we have the following weak error bound:*

$$\|\mathbb{E}_\omega[\varphi(u_N) - \varphi(u)]\|_{\mathcal{C}^{1,\beta}(D)} \leq C_{w1}(\beta,f,p,\varphi) R_N^\beta.$$

REMARK 2.9. *In the proof of this theorem (see equation (4.4)) we will show that the strong error $u - u_N$ can be bounded for any $q \geq 1$ in $L^q(\Omega, \mathcal{C}^{1,\beta}(\bar{D}))$ by $\sqrt{R_N^\alpha}$ (up to a constant) for any $\beta < \alpha < \frac{1}{2}$, which in particular improves the strong error estimate of [3] in H_0^1 norm. As stated in the Theorem above, the weak error at order N is bounded by R_N^β for any β such that $\frac{1}{2} \wedge \left(1 - \frac{d}{p}\right) > \beta > 0$. Therefore we have that the weak order is twice the strong order, as seen in [3] where we had almost the same orders.*

REMARK 2.10. *Compared to the weak (and also to the strong) error estimate of [3], we have a slightly stronger assumption on f since it is required to be in $L^p(D)$ with $p > d$ but the topology is much stronger.*
Moreover we suppose here the test function φ to be six times continuously differentiable, indeed in the proof we need to derive four times with respect to the parameter

y and then to take the $\mathcal{C}^{1,\beta}$ norm with respect to the spatial variable x, which mean we derive (almost) two times more.

In order to cover more quantities of interest, and in particular some of the output functionals given in the introduction, we now consider regular functionals $\psi : \mathcal{C}^{0,\beta}(\bar{D}) \times \mathcal{C}^{1,\beta}(\bar{D}) \to \mathbb{R}$ and the weak error committed on the computation of $\mathbb{E}[\psi(a,u)]$.

THEOREM 2.11. *For any $f \in L^p(D)$ with $p > d$, β such that $\frac{1}{2} \wedge \left(1 - \frac{d}{p}\right) > \beta > 0$, with $\psi \in \mathcal{C}^4(\mathcal{C}^{0,\beta}(\bar{D}) \times \mathcal{C}^{1,\beta}(\bar{D}), \mathbb{R})$ whose derivatives have at most polynomial growth, i.e. such that for any $u \in \mathcal{C}^{0,\beta}(\bar{D})$, $a \in \mathcal{C}^{1,\beta}(D)$ and $k = 0, \dots 4$ we have*

$$\|D^k \psi(a,u)\|_{\mathcal{L}((\mathcal{C}^{0,\beta}(\bar{D}) \times \mathcal{C}^{1,\beta}(\bar{D}))^k, \mathbb{R})} \leq C_\psi (1 + \|u\|^{2d_\psi}_{\mathcal{C}^{1,\beta}(\bar{D})} + \|a\|^{2d_\psi}_{\mathcal{C}^{0,\beta}(\bar{D})}),$$

there exists a constant $C_{w2}(\beta, f, p, \varphi)$ such that for all $N \in \mathbb{N}$ we have the following weak error bound:

$$|\mathbb{E}[\psi(a,u) - \psi(a_N, u_N)]| \leq C_{w2}(\beta, f, p, \varphi) R_N^\beta.$$

REMARK 2.12. *We note that all the cases of output functionals given in the introduction can be treated with these two results.*

REMARK 2.13. *The constants $C_{w1}(\beta, f, p, \varphi)$ and $C_{w2}(\beta, f, p, \varphi)$ also depend on a (i.e. on $\operatorname{cov}[g]$), which is supposed to be fixed here.*

3. A regularity result. In this section, we prove a polynomial dependance with respect to the coefficient of the $C^{1,\alpha}$ norm of the solution of an elliptic equation in divergence form, which will be crucial in the proofs of Theorems 2.8 and 2.11. Such a regularity result is classical (see for instance [12]) but we have not found any reference where the growth of the coefficients is made explicit.

THEOREM 3.1. *Let D be a $\mathcal{C}^2$ bounded domain of $\mathbb{R}^d$, $f \in L^p(D)$, with $p > d$ and $p \geq 2$, $g = (g_1, \dots, g_d) \in C^{0,\alpha}(\bar{D})$and $a \in \mathcal{C}^{0,\alpha}(\bar{D})$, with $1 - \frac{d}{p} > \alpha > 0$, such that for any $x \in D$, we have $a_{min} \leq a(x) \leq a_{max}$ and $\|a\|_{C^{0,\alpha}}(\bar{D}) \leq a_\alpha$ (with $a_\alpha \geq a_{max}$). We consider then the following elliptic partial differential equation:*

$$\begin{cases} -\operatorname{div}(a(x)\nabla u(x)) = f(x) + \operatorname{div} g(x), & \text{on } D, \\ u(x) = 0, & \text{on } \partial D. \end{cases} \tag{3.1}$$

It has a unique solution in $u \in H_0^1(D)$. Moreover $u \in C^{1,\alpha}(\bar{D})$ and

$$\|u\|_{C^{1,\alpha}(\bar{D})} \leq C_r(p,\alpha,D)\, P\left(\frac{1}{a_{\min}}, a_\alpha\right) \left(\|f\|_{L^p(D)} + \|g\|_{C^{0,\alpha}(\bar{D})}\right).$$

Where $P(x,y) = x + x(1 + x + xy)\left(x^{\frac{1+\alpha}{\alpha}} y^{\frac{1+\alpha}{\alpha}} + x^{\frac{p}{2\alpha}} y^{\frac{p}{2}\left(1+\frac{1}{\alpha}\right)} x^{\frac{1+\alpha^2}{\alpha(1-\alpha)}} y^{\frac{1+2\alpha}{\alpha(1-\alpha)}}\right)$ and the constant $C_r(p,\alpha,D)$ depends only on the arguments.

Proof. First, let u be a solution of the previous equation in $\mathcal{C}^{1,\alpha}(\bar{D})$. From [12], Theorem 8.15, the following estimate holds:

$$\|u\|_{C^0(\bar{D})} \leq C(\|u\|_{L^2(D)} + a_{min}^{-1}(\|f\|_{L^{p/2}(D)} + \|g\|_{L^p(D)}))$$

where here and below, C denotes a constant which may depend only on p, α, d or D and whose value may change from one line to the other. Since

$$\|u\|_{L^2(D)} \leq C a_{min}^{-1}(\|f\|_{L^2(D)} + \|g\|_{L^2(D)})$$

we deduce

$$\|u\|_{C^0(\bar{D})} \leq C\, a_{min}^{-1}(\|f\|_{L^p(D)} + \|g\|_{C^{0,\alpha}(\bar{D})}). \tag{3.2}$$

We now estimate the gradient. We take $1 \geq r > 0$ to be chosen below, cover $\overline{D}$ by balls $(B(x_i, r/4))_{i=1,\ldots,N}$ and take

$$\varphi_i \in C_0^\infty(B(x_i, r)),\ \text{such that } 0 \leq \varphi_i \leq 1,\ \text{and } \varphi_i \equiv 1 \text{ on } B(x_i, r/2).$$

It is possible to choose these functions such that we have $\|\nabla\varphi_i\|_{C^0(\bar{D})} \leq Cr^{-1}$ and $\|\nabla\varphi_i\|_{C^{0,\alpha}(\bar{D})} \leq Cr^{-1-\alpha}$. We set $a_i = a(x_i)$, $u_i = \varphi_i u$. The function u_i is then the solution of the following PDE:

$$\begin{cases} -a_i\Delta u_i &= \operatorname{div}[(a-a_i)\nabla u_i] + \varphi_i f + \operatorname{div}(\varphi_i g) - g\nabla\varphi_i - a\nabla u\nabla\varphi_i - \operatorname{div}[au\nabla\varphi_i] \\ u_i &= 0 \text{ on } \partial D. \end{cases}$$

Let C_0 be a constant depending on p, α and D such that the solution of

$$-\Delta w = g_0 + \sum_{i=1}^d \partial_{x_i} h_i,\ \text{on } D\ \ w = 0 \text{ on } \partial D,$$

satisfies:

$$\|\nabla w\|_{C^{0,\alpha}(\bar{D})} \leq C_0[\|g_0\|_{L^p(D)} + \sum_{i=1}^d \|h_i\|_{C^{0,\alpha}(\bar{D})}]. \tag{3.3}$$

In the case where $g_0 = 0$, the existence of this constant follows from [12], Theorem 8.33 and Theorem 8.15. In the case where $h_i = 0$ for any i, it follows from [12], Theorem 9.9 together with the fact that the semi-norm $\|D^2u\|_{L^p(D)}$ is equivalent to the $W^{2,p}$ norm on $W_0^{2,p}(D)$ and the Sobolev embedding $W^{2,p}(D) \subset C^{1,\alpha}(\bar{D})$.

We apply for any i this bound to the solution u_i of (3.3) and get:

$$\begin{aligned} a_i\|\nabla u_i\|_{C^{0,\alpha}(\bar{D})} \leq C_0\Big[&\|(a-a_i)\nabla u_i\|_{C^{0,\alpha}(\bar{D})} + \|f\|_{L^p(D)} + \|\varphi_i\|_{C^{0,\alpha}(\bar{D})}\|g\|_{C^{0,\alpha}(\bar{D})} \\ &+\|\nabla\varphi_i\|_{C^0(\bar{D})}\|g\|_{L^p(D)} + \|a\|_{C^0(\bar{D})}\|\nabla\varphi_i\|_{C^0(\bar{D})}\|\nabla u\|_{L^p(D)} \\ &+\|a\|_{C^{0,\alpha}(\bar{D})}\|u\|_{C^{0,\alpha}(\bar{D})}\|\nabla\varphi_i\|_{C^{0,\alpha}(\bar{D})}\Big]. \end{aligned} \tag{3.4}$$

By interpolation, there exists $C > 0$ such that we have for any $\epsilon > 0$:

$$\|\nabla u_i\|_{C^0(\bar{D})} \leq \epsilon\|\nabla u_i\|_{C^{0,\alpha}(\bar{D})} + \frac{C}{\epsilon^{1/\alpha}}\|u_i\|_{C^0(\bar{D})}.$$

The proof of this interpolation result is similar to the proof of [12], Lemma 6.32.

Choosing $0 < \epsilon = r^\alpha \le 1$, we obtain:

$$\|(a-a_i)\nabla u_i\|_{C^{0,\alpha}(\bar D)} \le \|a-a_i\|_{C^0(\bar B(x_i,r))}\|\nabla u_i\|_{C^0(\bar D)} \tag{3.5}$$
$$+ \|a-a_i\|_{C^0(\bar B(x_i,r))}|\nabla u_i|_\alpha + |a|_\alpha \|\nabla u_i\|_{C^0(\bar D)}$$
$$\le |a|_\alpha r^\alpha \|\nabla u_i\|_{C^0(\bar D)} + |a|_\alpha r^\alpha |\nabla u_i|_\alpha + |a|_\alpha \|\nabla u_i\|_{C^0(\bar D)}$$
$$\le |a|_\alpha (2r^\alpha \|\nabla u_i\|_{C^{0,\alpha}(\bar D)} + \frac{C}{r}\|u_i\|_{C^0(\bar D)}). \tag{3.6}$$

We choose $r = \left(\frac{a_{min}}{4C_0 a_\alpha}\right)^{1/\alpha} \wedge 1$ (where C_0 is the constant defined in equation **??**), then $2C_0 a_\alpha r^\alpha \le \frac{a_{min}}{2}$ and $r \le 1$ and we deduce from (3.2) applied to u_i, (3.4) and (3.3) :

$$a_i\|\nabla u_i\|_{C^{0,\alpha}} \le C\Bigg[a_\alpha a_{min}^{-1} r^{-1}(\|f\|_{L^p(D)} + \|g\|_{C^{0,\alpha}(\bar D)}) + \|f\|_{L^p(D)} + r^{-1}\|g\|_{C^{0,\alpha}(\bar D)}$$
$$+ r^{-1}\|a\|_{C^0(\bar D)}\|\nabla u\|_{L^p(D)} + r^{-1-\alpha}\|a\|_{C^{0,\alpha}(\bar D)}\|u\|_{C^{0,\alpha}(\bar D)}\Bigg]$$
$$\le C\left[r^{-1-\alpha}(\|f\|_{L^p(D)} + \|g\|_{C^{0,\alpha}(\bar D)}) + r^{-1}\|a\|_{C^0(\bar D)}\|\nabla u\|_{L^p(D)}\right. \tag{3.7}$$
$$+ r^{-1-\alpha}\|a\|_{C^{0,\alpha}(\bar D)}\|u\|_{C^{0,\alpha}(\bar D)}.$$

(3.9)

Since $p \ge 2$, we also have the elementary interpolation inequality

$$\|\nabla u\|^p_{L^p(D)} \le \|\nabla u\|^{p-2}_{C^0(\bar D)} |\nabla u|^2_{L^2(D)}.$$

We deduce

$$\|\nabla u\|^p_{L^p(D)} \le C\|\nabla u\|^{p-2}_{C^0(\bar D)} a_{min}^{-2}(\|f\|_{L^2(D)} + \|g\|_{L^2(D)})^2$$
$$\le C\|\nabla u\|^{p-2}_{C^0(\bar D)} a_{min}^{-2}(\|f\|_{L^p(D)} + \|g\|_{C^{0,\alpha}(\bar D)})^2.$$

Therefore, for $\eta > 0$ to be fixed later, Young's inequality yields

$$r^{-1}\|a\|_{C^0(\bar D)}\|\nabla u\|_{L^p(D)} \le C a_{max} a_{min}^{-\frac{2}{p}} r^{-1}\|\nabla u\|^{\frac{p-2}{p}}_{C^0(\bar D)}(\|f\|_{L^p(D)} + \|g\|_{C^{0,\alpha}(\bar D)})^{\frac{2}{p}}$$
$$\le \frac{C}{\eta} a_{max}^{\frac{p}{2}} a_{min}^{-(1+\frac{p}{2\alpha})} a_\alpha^{\frac{p}{2\alpha}}(\|f\|_{L^p(D)} + \|g\|_{C^{0,\alpha}(\bar D)}) + \eta\|\nabla u\|_{C^0(\bar D)}, \tag{3.10}$$

which enables to bound the second term of (3.6). For the last term of (3.6), we again use the same interpolation argument:

$$r^{-1-\alpha}\|a\|_{C^{0,\alpha}(\bar D)}\|u\|_{C^{0,\alpha}(\bar D)} \le C\, r^{-1-\alpha} a_\alpha \|u\|^{1-\alpha}_{C^0(\bar D)}\|\nabla u\|^\alpha_{C^0(\bar D)}$$
$$\le \frac{C}{\eta}\, a_\alpha^{\frac{1}{1-\alpha}} r^{-\frac{1+\alpha}{1-\alpha}}\|u\|_{C^0(\bar D)} + \eta\|\nabla u\|_{C^0(\bar D)}.$$

Therefore by (3.2) and the definition of r, we get:

$$r^{-1-\alpha}\|a\|_{C^{0,\alpha}(\bar D)}\|u\|_{C^{0,\alpha}(\bar D)} \le \frac{C}{\eta} a_{min}^{-k_\alpha} a_\alpha^{\tilde k_\alpha}(\|f\|_{L^p(D)} + \|g\|_{C^{0,\alpha}(\bar D)}) + \eta\|\nabla u\|_{C^0(\bar D)}, \tag{3.11}$$

with $k_\alpha = \frac{1+\alpha}{\alpha(1-\alpha)} + 1$, $\tilde{k}_\alpha = \frac{1+\alpha}{\alpha(1-\alpha)} + \frac{1}{1-\alpha}$. We deduce, thanks to (3.6), (3.7) and (3.8),

$$\begin{aligned}\|\nabla u_i\|_{C^{0,\alpha}(\bar{D})} \leq C\, a_{min}^{-1}\Big[a_{min}^{-\frac{1+\alpha}{\alpha}} a_\alpha^{\frac{1+\alpha}{\alpha}} + \frac{C}{\eta} a_{max}^{\frac{p}{2}} a_{min}^{-(1+\frac{p}{2\alpha})} a_\alpha^{\frac{p}{2\alpha}} + \frac{C}{\eta} a_{min}^{-k_\alpha} a_\alpha^{\tilde{k}_\alpha}\Big] \quad (3.12)\\ \times (\|f\|_{L^p(D)} + \|g\|_{C^{0,\alpha}(\bar{D})}) + 2\eta a_{min}^{-1} \|\nabla u\|_{C^0(\bar{D})}.\end{aligned}$$

Let $x_0 \in \bar{D}$ such that $\nabla u(x_0) = \|\nabla u\|_{C^0(\bar{D})}$, and take i_0 such that $x_0 \in B(x_{i_0}, r/4)$. Then $\nabla u(x_0) = \nabla u_{i_0}(x_0)$. Therefore

$$\|\nabla u\|_{C^0(\bar{D})} \leq \|\nabla u_{i_0}\|_{C^0(\bar{D})} \leq \|\nabla u_{i_0}\|_{C^{0,\alpha}(\bar{D})}$$

and thanks to (3.9)

$$\begin{aligned}\|\nabla u\|_{C^0(\bar{D})} \quad &\leq \quad C\, a_{min}^{-1}\Big[a_{min}^{-\frac{1+\alpha}{\alpha}} a_\alpha^{\frac{1+\alpha}{\alpha}} + \frac{C}{\eta} a_{max}^{\frac{p}{2}} a_{min}^{-(1+\frac{p}{2\alpha})} a_\alpha^{\frac{p}{2\alpha}} + \frac{C}{\eta} a_{min}^{-k_\alpha} a_\alpha^{\tilde{k}_\alpha}\Big]\\ &\times \quad (\|f\|_{L^p(D)} + \|g\|_{C^{0,\alpha}(\bar{D})}) + 2\eta\, a_{min}^{-1} \|\nabla u\|_{C^0(\bar{D})}.\end{aligned}$$

Taking η such that $2a_{min}^{-1}\eta = \frac{1}{2}$, we deduce the following bound on $\|\nabla u\|_{C^0(\bar{D})}$

$$\begin{aligned}\|\nabla u\|_{C^0(\bar{D})} \leq 2C\, a_{min}^{-1}\Big[a_{min}^{-\frac{1+\alpha}{\alpha}} a_\alpha^{\frac{1+\alpha}{\alpha}} + \frac{C}{\eta} a_{max}^{\frac{p}{2}} a_{min}^{-(1+\frac{p}{2\alpha})} a_\alpha^{\frac{p}{2\alpha}} + \frac{C}{\eta} a_{min}^{-k_\alpha} a_\alpha^{\tilde{k}_\alpha}\Big]\\ \times (\|f\|_{L^p(D)} + \|g\|_{C^{0,\alpha}(\bar{D})}),\end{aligned}$$

which in turns, thanks to (3.9), implies the following bound for each $\|\nabla u_i\|_{C^{0,\alpha}(\bar{D})}$:

$$\begin{aligned}\|\nabla u_i\|_{C^{0,\alpha}(\bar{D})} \leq \quad \frac{2C}{a^{\min}}\Big[a_{min}^{-\frac{1+\alpha}{\alpha}} a_\alpha^{\frac{1+\alpha}{\alpha}} + \frac{C}{\eta} a_{max}^{\frac{p}{2}} a_{min}^{-(1+\frac{p}{2\alpha})} a_\alpha^{\frac{p}{2\alpha}} + \frac{C}{\eta} a_{min}^{-k_\alpha} a_\alpha^{\tilde{k}_\alpha}\Big]\\ \times (\|f\|_{L^p(D)} + \|g\|_{C^{0,\alpha}(\bar{D})}).\end{aligned}$$

The estimate on $\|\nabla u\|_{C^{0,\alpha}(\bar{D})}$ follows from the inequality

$$\|\nabla u\|_{C^{0,\alpha}(\bar{D})} \leq \max\{\max_i \|\nabla u_i\|_{C^{0,\alpha}(\bar{D})}, 2^{1+2\alpha} r^{-\alpha} \|\nabla u\|_{C^0(\bar{D})}\}$$

which is easily obtained by considering the points where the maximum in the definition of $|\nabla u|_\alpha$ is reached. Indeed, if x and y are such points, $|\nabla u|_\alpha$ is bounded by $\frac{2^{1+2\alpha}\|\nabla u\|_{\mathcal{C}^0(\bar{D})}}{r^\alpha}$ if $\|x-y\| \geq \frac{r}{4}$ and by $\|\nabla u_i\|_{\mathcal{C}^{0,\alpha}(\bar{D})}$ otherwise, where i is such that $x, y \in \mathcal{B}(x_i, \frac{r}{2})$. Moreover we have the obvious inequality $\|\nabla u\|_{\mathcal{C}^0(\bar{D})} \leq \max_i \|\nabla u_i\|_{\mathcal{C}^0(\bar{D})}$.

It remains to prove that the unique solution u in $H_0^1(D)$ indeed belongs to $\mathcal{C}^{1,\alpha}$. In the case where $f = 0$, it is Theorem 8.34 of [12]. In the case where $g = 0$, it is a slight modification of Theorem 9.15 for divergence form equations. More precisely, we can use an approximation argument. We approximate f using a sequence $(f_k)_{k\geq 0}$ of L^∞ functions, for each f_k the corresponding PDE admits a unique solution u_k in $\mathcal{C}^{1,\alpha}(\bar{D})$ thanks to theorem 8.34 of [12]. Thanks to the above estimate, $(u_k)_{k\geq 0}$ is bounded in $\mathcal{C}^{1,\alpha}(\bar{D})$. Using the Theorem of Ascoli, we get that $(u_k)_{k\geq 0}$ is relatively compact in $\mathcal{C}^1(\bar{D})$. The sequence $(u_k)_{k\geq 0}$ converges then to u in $\mathcal{C}^1(\bar{D})$, because of the uniqueness of the solution of (3.1) in $H_0^1(D)$. Therefore u belongs to $\mathcal{C}^{1,\alpha}(\bar{D})$. □

4. Proof of Theorem 2.8. In this section we prove Theorem 2.8. In order to bound the weak error $x \mapsto \mathbb{E}[\varphi(u_N) - \varphi(u)]$ in $\mathcal{C}^{1,\beta}$ norm, for some regular function φ, we need estimates on the growth of the derivatives of $\varphi(u_N)$ with respect to the y_i, which follow from the estimates below on the derivatives of u_N with respect to the y_i. We use the estimate in $C^{1,\beta}$ norm on the solution of an elliptic equation in divergence form given in Theorem 3.1 to derive the bound of Theorem 2.8. Hence, in this section and in the following section, we assume that $f \in L^p(D)$ with $p > d$.

PROPOSITION 4.1. *For any β such that $\frac{1}{2} \wedge \left(1 - \frac{d}{p}\right) > \beta > 0$, and any integer k, there exists a constant $C_1(\beta, k, p, D)$ such that for any $N \in \mathbb{N}$, for any multi-index $\delta \in \mathbb{N}^N$ with length k, we have the following estimate on the growth of the derivatives of u_N with respect to y : for any $y \in \mathbb{R}^N$*

$$\left\| \frac{\partial^\delta \tilde{u}_N}{\partial y^\delta}(y) \right\|_{C^{1,\beta}(D)} \leq C_1(\beta,k,p,D)(1+\tilde{\kappa}_{N,\beta}(y))^{k+1}(1+\|\tilde{a}_N(y)\|_{C^{0,\beta}(\bar{D})})^k \|f\|_{L^p(D)} \times \prod_{i\in\mathbb{N}} \sqrt{\lambda_i^{\delta_i}} \|b_i\|_{C^{0,\beta}(D)}^{\delta_i},$$

where $\tilde{\kappa}_{N,\beta}(y) = P\left(\frac{1}{a_N^{min}(y)}, \|\tilde{a}_N(y)\|_{\mathcal{C}^{0,\beta}}\right)$, P given in Theorem 3.1.

Proof. For $k = 0$, this is exactly Theorem 3.1 with $g = 0$. For $k = 1$, we write for $1 \leq i \leq N$

$$-\mathrm{div}\left(\tilde{a}_N \nabla \frac{\partial \tilde{u}_N}{\partial y_i}\right) = \mathrm{div}\left(\frac{\partial \tilde{a}_N}{\partial y_i} \nabla \tilde{u}_N\right)$$

and $\frac{\partial \tilde{u}_N}{\partial y_i} = 0$ on ∂D. By Theorem 3.1, we deduce

$$\begin{aligned}
\left\| \frac{\partial \tilde{u}_N}{\partial y_i} \right\|_{C^{1,\beta}(\bar{D})} &\leq C\tilde{\kappa}_{N,\beta} \left\| \frac{\partial \tilde{a}_N}{\partial y_i} \nabla \tilde{u}_N \right\|_{C^{0,\beta}(\bar{D})} \\
&\leq C\tilde{\kappa}_{N,\beta} \left\| \frac{\partial \tilde{a}_N}{\partial y_i} \right\|_{C^{0,\beta}(\bar{D})} \|\nabla \tilde{u}_N\|_{C^{0,\beta}(\bar{D})} \\
&\leq C\,\tilde{\kappa}_{N,\beta}^2 \left\| \frac{\partial \tilde{a}_N}{\partial y_i} \right\|_{C^{0,\beta}(\bar{D})} \|f\|_{L^p(D)}
\end{aligned}$$

where C is a constant which may depend on β, p, D and change from one line to the other.

We compute the derivatives of $\tilde{a}_N$ with respect to the y_i. For all $1 \leq i \leq N$:

$$\frac{\partial \tilde{a}_N}{\partial y_i}(y,x) = \sqrt{\lambda_i} b_i(x) \tilde{a}_N(y,x).$$

It follows

$$\left\| \frac{\partial \tilde{u}_N}{\partial y_i} \right\|_{C^{1,\beta}(\bar{D})} \leq C\,\tilde{\kappa}_{N,\beta}^2 \sqrt{\lambda_i} \|b_i\|_{C^{0,\beta}(\bar{D})} \|\tilde{a}_N\|_{C^{0,\beta}(\bar{D})} \|f\|_{L^p(D)}.$$

The estimate for $k = 2$ is similar, we write for $1 \leq i, j \leq N$:

$$-\mathrm{div}\left(\tilde{a}_N \nabla \frac{\partial^2 \tilde{u}_N}{\partial y_i \partial y_j}\right) = \mathrm{div}\left(\frac{\partial \tilde{a}_N}{\partial y_i} \frac{\partial \nabla \tilde{u}_N}{\partial y_j}\right) + \mathrm{div}\left(\frac{\partial \tilde{a}_N}{\partial y_j} \frac{\partial \nabla \tilde{u}_N}{\partial y_i}\right) + \mathrm{div}\left(\frac{\partial^2 \tilde{a}_N}{\partial y_i \partial y_j} \nabla \tilde{u}_N\right)$$

and use Theorem 3.1:

$$\left\|\frac{\partial^2\tilde u_N}{\partial y_i\partial y_j}\right\|_{C^{1,\beta}} \le C\tilde\kappa_{N,\beta}\left(\left\|\frac{\partial\tilde a_N}{\partial y_i}\frac{\partial\nabla\tilde u_N}{\partial y_j}\right\|_{C^{0,\beta}}+\left\|\frac{\partial\tilde a_N}{\partial y_j}\frac{\partial\nabla\tilde u_N}{\partial y_i}\right\|_{C^{0,\beta}}+\left\|\frac{\partial^2\tilde a_N}{\partial y_i\partial y_j}\nabla\tilde u_N\right\|_{C^{0,\beta}}\right)$$
$$\le C\tilde\kappa_{N,\beta}\left(\left\|\frac{\partial\tilde a_N}{\partial y_i}\right\|_{C^{0,\beta}}\left\|\frac{\partial\nabla\tilde u_N}{\partial y_j}\right\|_{C^{0,\beta}}+\left\|\frac{\partial\tilde a_N}{\partial y_j}\right\|_{C^{0,\beta}}\left\|\frac{\partial\nabla\tilde u_N}{\partial y_i}\right\|_{C^{0,\beta}}+\left\|\frac{\partial^2\tilde a_N}{\partial y_i\partial y_j}\right\|_{C^{0,\beta}}\|\nabla\tilde u_N\|_{C^{0,\beta}}\right)$$
$$\le C\left(1+\tilde\kappa_{N,\beta}\right)^3\sqrt{\lambda_i}\sqrt{\lambda_j}\|b_i\|_{C^{0,\beta}}\|b_j\|_{C^{0,\beta}}(1+\|\tilde a_N\|_{C^{0,\beta}})^2\|f\|_{L^p(D)}$$

The result follows for $|\delta|\ge 3$ by induction. ▯

PROPOSITION 4.2. *Let $\varphi\in\mathcal{C}^6(\mathbb{R},\mathbb{R})$, whose derivatives have at most polynomial growth, then for any β such that $\frac{1}{2}\wedge\left(1-\frac{d}{p}\right)>\beta>0$, there exists a constant $C_2(\beta,\varphi)$ such that for any $n\in\mathbb{N}$ and any multi-index $\delta\in\mathbb{N}^N$ with $|\delta|\le 4$, we have the following estimate on the growth of the derivatives of $\varphi\circ\tilde u_N$ with respect to y: for any $y\in\mathbb{R}^N$*

$$\left\|\frac{\partial^\delta\varphi\circ\tilde u_N}{\partial y^\delta}(y)\right\|_{C^{1,\beta}(\bar D)} \le C_2(\beta,\varphi)(1+\tilde\kappa_{N,\beta}(y))^{2|\delta|+2(1+d_\varphi)}\left(1+\|\tilde a_N(y)\|_{C^{0,\beta}(\bar D)}\right)^{|\delta|}$$
$$\times(1+\|f\|_{L^p(D)})^{|\delta|+2(1+d_\varphi)}\prod_{i\in\mathbb{N}}\sqrt{\lambda_i^{\delta_i}}\|b_i\|^{\delta_i}_{C^{0,\beta}(\bar D)}.$$

where for $k=0,...,6$ we have for some integer d_φ and some constant C_φ

$$|\varphi^{(k)}(x)|\le C_\varphi(1+|x|^{2d_\varphi}) \tag{4.1}$$

Proof. First we recall some basic inequalities, which will be used in the following proof. Let $g,g_1,g_2\in\mathcal{C}^{1,\beta}$, $\varphi\in\mathcal{C}^6(\mathbb{R},\mathbb{R})$ such that (4.1) holds and $1\le k\le 6$, then we have

$$\begin{aligned}\|g_1g_2\|_{\mathcal{C}^{0,\beta}(D)} &\le \|g_1\|_{\mathcal{C}^0(D)}\|g_2\|_{\mathcal{C}^{0,\beta}(D)}+\|g_2\|_{\mathcal{C}^0(D)}\|g_1\|_{\mathcal{C}^{0,\beta}(D)}\\ &\le 2\|g_1\|_{\mathcal{C}^{0,\beta}(D)}\|g_2\|_{\mathcal{C}^{0,\beta}(D)}\\ \|g_1g_2\|_{\mathcal{C}^{1,\beta}(D)} &\le 6\|g_1\|_{\mathcal{C}^{1,\beta}(D)}\|g_2\|_{\mathcal{C}^{1,\beta}(D)}\\ \|\varphi^{(k)}\circ g\|_{\mathcal{C}^{0,\beta}} &\le C_\varphi(1+\|g\|^{2d_\varphi}_{\mathcal{C}^0(D)})\|g\|_{\mathcal{C}^{0,\beta}}.\end{aligned}$$

For $|\delta|=0$, we have

$$\begin{aligned}\|\varphi\circ\tilde u_N\|_{\mathcal{C}^{1,\beta}(\bar D)} &\le \|\varphi\circ\tilde u_N\|_{\mathcal{C}^0(\bar D)}+\|\nabla(\varphi\circ\tilde u_N)\|_{\mathcal{C}^{0,\beta}(\bar D)}\\ &\le C_\varphi(1+\|\tilde u_N\|^{2d_\varphi}_{\mathcal{C}^0(\bar D)})+2\|\nabla\tilde u_N\|_{\mathcal{C}^{0,\beta}(\bar D)}\|\varphi'\circ\tilde u_N\|_{\mathcal{C}^{0,\beta}(\bar D)}\\ &\le 2C_\varphi(1+\|\tilde u_N\|^2_{\mathcal{C}^{1,\beta}(\bar D)})(1+\|\tilde u_N\|^{2d_\varphi}_{\mathcal{C}^{1,\beta}(\bar D)})\\ &\le 2C_2(\beta,\varphi)(1+\tilde\kappa_{N,\beta}(y))^{2(1+d_\varphi)}(1+\|f\|_{L^p(D)})^{2(1+d_\varphi)},\end{aligned}$$

with $C_2(\beta,\varphi)=2C_\varphi(1+C_1(\beta,0,p,D))^{2(1+d_\varphi)}$.

For $|\delta|=1$, we have:

$$\frac{\partial\varphi\circ\tilde u_N}{\partial y_i}(y)=\varphi'\circ\tilde u_N(y)\frac{\partial\tilde u_N}{\partial y_i}(y).$$

Using the second preliminary inequality, we have

$$\left\| \frac{\partial \varphi \circ \tilde{u}_N}{\partial y_i}(y) \right\|_{C^{1,\beta}(\bar{D})} \leq 6 \left\| \varphi' \circ \tilde{u}_N(y) \right\|_{C^{1,\beta}(\bar{D})} \left\| \frac{\partial \tilde{u}_N}{\partial y_i}(y) \right\|_{C^{1,\beta}(\bar{D})}.$$

And we have

$$\|\varphi' \circ \tilde{u}_N(y)\|_{C^{1,\beta}(\bar{D})} \leq 2C_\varphi(1 + \|\tilde{u}_N(y)\|^2_{\mathcal{C}^{1,\beta}})(1 + \|\tilde{u}_N(y)\|^{2d_\varphi}_{\mathcal{C}^{1,\beta}}). \tag{4.2}$$

similarly to what precedes. Therefore, by Proposition 4.1 we get

$$\begin{aligned}\left\| \frac{\partial \varphi \circ \tilde{u}_N}{\partial y_i}(y) \right\|_{C^{1,\beta}(\bar{D})} \leq & C_2(\beta,\varphi)(1 + \tilde{\kappa}_{N,\beta}(y))^{4+2d_\varphi} \left(1 + \|\tilde{a}_N(y)\|_{C^{0,\beta}(\bar{D})}\right) \\ & (1 + \|f\|_{L^p(D)})^{3+2d_\varphi} \lambda_i \|b_i\|_{C^{0,\beta}(\bar{D})},\end{aligned}$$

with $C_2(\beta,\varphi) = 12C_\varphi(1 + C_1(\beta,1,p,D))(1 + C_1(\beta,0,p,D))^{2(1+d_\varphi)}$.

Similarly, for $|\delta| = 2$, using (4.2) again, a similar estimate for $\varphi'' \circ \tilde{u}_N$ and Proposition 4.1 we get:

$$\frac{\partial^2 \varphi \circ \tilde{u}_N}{\partial y_i \partial y_j}(y) = \varphi' \circ \tilde{u}_N(y) \frac{\partial^2 \tilde{u}_N}{\partial y_i \partial y_j}(y) + \varphi'' \circ \tilde{u}_N(y) \frac{\partial \tilde{u}_N}{\partial y_i}(y) \frac{\partial \tilde{u}_N}{\partial y_j}(y).$$

$$\begin{aligned}\left\| \frac{\partial^2 \varphi \circ \tilde{u}_N}{\partial y_i \partial y_j}(y) \right\|_{C^{1,\beta}(\bar{D})} &\leq 6 \left\| \varphi' \circ \tilde{u}_N(y) \right\|_{C^{1,\beta}(\bar{D})} \left\| \frac{\partial^2 \tilde{u}_N}{\partial y_i \partial y_j}(y) \right\|_{C^{1,\beta}(\bar{D})} \\ &+ 6 \left\| \varphi'' \circ \tilde{u}_N(y) \right\|_{C^{1,\beta}(\bar{D})} \left\| \frac{\partial \tilde{u}_N}{\partial y_i}(y) \right\|_{C^{1,\beta}(\bar{D})} \left\| \frac{\partial \tilde{u}_N}{\partial y_j}(y) \right\|_{C^{1,\beta}(\bar{D})} \\ &\leq 12C_\varphi(1 + \|\tilde{u}_N(y)\|^2_{\mathcal{C}^{1,\beta}})(1 + \|\tilde{u}_N(y)\|^{2d_\varphi}_{\mathcal{C}^{1,\beta}}) \\ &\quad \left(\left\| \frac{\partial^2 \tilde{u}_N}{\partial y_i \partial y_j}(y) \right\|_{C^{1,\beta}(\bar{D})} + \left\| \frac{\partial \tilde{u}_N}{\partial y_i}(y) \right\|_{C^{1,\beta}(\bar{D})} \left\| \frac{\partial \tilde{u}_N}{\partial y_j}(y) \right\|_{C^{1,\beta}(\bar{D})} \right) \\ &\leq C_2(\beta,\varphi)(1 + \tilde{\kappa}_{N,\beta}(y))^{6+2d_\varphi} \left(1 + \|\tilde{a}_N(y)\|_{C^{0,\beta}(\bar{D})}\right)^2 \\ &\quad (1 + \|f\|_{L^p(D)})^{4+2d_\varphi} \sqrt{\lambda_i \lambda_j} \|b_i\|_{C^{0,\beta}(\bar{D})} \|b_j\|_{C^{0,\beta}(\bar{D})},\end{aligned}$$

with $C_2(\beta,\varphi) = 12C_\varphi(1+C_1(\beta,2,p,D))(1+C_1(\beta,1,p,D))^2(1+C_1(\beta,0,p,D))^{2(1+d_\varphi)}$. The cases $|\delta| = 3, 4$ are treated similarly. It remains just to notice that the constants $C_2(\beta,\varphi)$ can be chosen increasingly, so that the final value holds for $|\delta| = 0, 1, 2, 3, 4$. ∎

We are now ready to estimate the weak error, i.e. the quantity $\mathbb{E}[\varphi(u_N) - \varphi(u)]$ in $C^{1,\beta}$-norm. Before proving Theorem 2.8, we recall the basic idea of the proof, which is similar to the one of the estimate in L^p norm in [3]. To estimate the weak error, we consider the Taylor expansion at order 2 of $\varphi(u_N) - \varphi(u)$ and remark that first order terms and second order terms such that $i \neq j$ are mean-free. In the case where φ is the identity, formally the second order development is:

$$\begin{aligned}u(\omega,x) - u_N(\omega,x) &= \tilde{u}(Y_1(\omega),...,Y_N(\omega),Y_{N+1}(\omega),...,x) - \tilde{u}(Y_1(\omega),...,Y_N(\omega),0,...,x) \\ &= \sum_{i>N} \frac{\partial \tilde{u}}{\partial y_i}(Y_1(\omega),...,Y_N(\omega),0,...,x) Y_i(\omega) \\ &+ \frac{1}{2} \sum_{i,j>N} \frac{\partial^2 \tilde{u}}{\partial y_i \partial y_j}(Y_1(\omega),...,Y_N(\omega),0,...,x) Y_i(\omega) Y_j(\omega) + ...\end{aligned}$$

Combining the independence of the Y_i with the fact that the Y_i are mean-free yields that the following terms are mean-free:

$$\mathbb{E}\left[\frac{\partial \tilde{u}}{\partial y_i}(Y_1(\omega),...,Y_N(\omega),0,...,x)Y_i(\omega)\right] = \mathbb{E}\left[\frac{\partial \tilde{u}}{\partial y_i}(Y_1(\omega),...,Y_N(\omega),0,...,x)\right]\mathbb{E}[Y_i(\omega)] \\ = 0.$$

Analogously, for $i \neq j$,

$$\begin{aligned} &\mathbb{E}\left[\frac{\partial^2 \tilde{u}}{\partial y_i \partial y_j}(Y_1(\omega),...,Y_N(\omega),0,...,x)Y_i(\omega)Y_j(\omega)\right] \\ =&\mathbb{E}\left[\frac{\partial^2 \tilde{u}}{\partial y_i \partial y_j}(Y_1(\omega),...,Y_N(\omega),0,...,x)\right]\mathbb{E}[Y_i(\omega)]\mathbb{E}[Y_j(\omega)] \\ =&0. \end{aligned}$$

The proof below shows that indeed the dominant in the error on the expected value is

$$\sum_{i>N}\mathbb{E}\left[\frac{\partial^2 \tilde{u}}{\partial y_i^2}(Y_1(\omega),...,Y_N(\omega),0,...,x)\right].$$

We now give a detailed proof of Theorem 2.8.

Let $M > N$, and $x \in D$, the first order Taylor theorem with integral remainder gives:

$$\begin{aligned} &\mathbb{E}_\omega[(\varphi(u_M)-\varphi(u_N))(\omega,x)] \\ =&\mathbb{E}_\omega[\varphi(\tilde{u}_M)(Y_1(\omega),...Y_M(\omega),x)-\varphi(\tilde{u}_M)(Y_1(\omega),...Y_N(\omega),0,...0,x)] \\ =&\mathbb{E}_\omega[D_y(\varphi\circ\tilde{u}_M)(Y_1(\omega),...Y_N(\omega),0,...0,x).(0,...0,Y_{N+1}(\omega),...,Y_M(\omega))] \\ &+\mathbb{E}_\omega\left[\int_0^1(1-t)D_y^2(\varphi\circ\tilde{u}_M)(Y_1,...Y_N,tY_{N+1},...,tY_M,x).(0,...0,Y_{N+1},...,Y_M)^2dt\right]. \end{aligned}$$

Since the random variables Y_i are independent, with mean zero and unit variance, the first order term is mean-free:

$$\begin{aligned} &\mathbb{E}_\omega\left[D_y(\varphi\circ\tilde{u}_M)(Y_1(\omega),...Y_N(\omega),0,...0,x).(0,...0,Y_{N+1}(\omega),...,Y_M(\omega))\right] \\ &=\sum_{i=N+1}^{M}\mathbb{E}_\omega\left[\frac{\partial(\varphi\circ\tilde{u}_M)}{\partial y_i}(Y_1(\omega),...Y_N(\omega),0,...0,x)\right]\mathbb{E}_\omega[Y_i(\omega)] \\ &=0. \end{aligned}$$

We now bound the integral remainder term, to begin with we split it into two terms:

$$\begin{aligned} &\mathbb{E}_\omega\left[(\varphi(u_M)-\varphi(u_N))(\omega,x)\right] \\ =&\sum_{N+1\leq i\leq M}\int_0^1(1-t)\mathbb{E}_\omega\left[\frac{\partial^2(\varphi\circ\tilde{u}_M)}{\partial y_i^2}(Y_1,...Y_N,tY_{N+1},...,tY_M,x)Y_i^2\right]dt \\ &+\sum_{N+1\leq i\neq j\leq M}\int_0^1(1-t)\mathbb{E}_\omega\left[\frac{\partial^2(\varphi\circ\tilde{u}_M)}{\partial y_i\partial y_j}(Y_1,...Y_N,tY_{N+1},...,tY_M,x)Y_iY_j\right]dt. \end{aligned}$$

First we give a bound for the first error contribution. Using the bound of the derivatives of $\varphi \circ \tilde{u}_M$ given in Proposition 4.2, we get for $N+1 \le i \le M$:

$$\begin{aligned}
&\left\| \int_0^1 (1-t)\mathbb{E}_\omega \left[\frac{\partial^2(\varphi \circ \tilde{u}_M)}{\partial y_i^2}(Y_1, ...Y_N, tY_{N+1}, ..., tY_M, x) Y_i^2 \right] dt \right\|_{C^{1,\beta}(\bar{D})} \\
&\le \int_0^1 (1-t)\mathbb{E}_\omega \left[\left\| \frac{\partial^2(\varphi \circ \tilde{u}_M)}{\partial y_i^2}(Y_1, ...Y_N, tY_{N+1}, ..., tY_M, x) \right\|_{C^{1,\beta}(\bar{D})} Y_i^2 \right] dt \\
&\le C_2(\beta,\varphi) \|b_i\|^2_{C^{0,\beta}(\bar{D})} \lambda_i (1 + \|f\|_{L^p(D)})^{4+2d_\varphi} \\
&\quad \times \int_0^1 \mathbb{E}_\omega \left[\left((1+\tilde{\kappa}_{N,\beta})^{6+2d_\varphi} \left(1 + \|\tilde{a}_N\|_{C^{0,\beta}(\bar{D})}\right)^2 \right) (Y_1, ..., Y_N, tY_{N+1}, ..., tY_M) Y_i^2 \right] dt.
\end{aligned}$$

We define, for $t \in [0,1]$, $\gamma_t \in [0,1]^M$ by $\gamma_t(i) = 1$ for $i \le N$ and $\gamma_t(i) = t$ for $i > N$, then, recalling that Y_i is normal, there exist a constant $c(\beta,\varphi)$ depending only on β and φ (and also on $\mathrm{cov}[g]$ which is supposed to be fixed here) such that :

$$\begin{aligned}
&\mathbb{E}_\omega \left[\left((1+\tilde{\kappa}_{N,\beta})^{6+2d_\varphi} \left(1 + \|\tilde{a}_N\|_{C^{0,\beta}(\bar{D})}\right)^2 \right) (Y_1, ..., Y_N, tY_{N+1}, ..., tY_M) Y_i^2 \right] \\
&= \mathbb{E}_\omega \left[(1+\kappa_{N,\beta,\gamma_t})^{6+2d_\varphi} \left(1 + \|a_{\gamma_t,N}\|_{C^{0,\beta}(\bar{D})}\right)^2 Y_i^2 \right] \\
&\le (1 + \|\kappa_{N,\beta,\gamma_t}\|_{L^{18+6d_\varphi}(\Omega)})^{6+2d_\varphi} (1 + \|a_{\gamma_t,N}\|_{L^6(\Omega, C^{0,\beta}(\bar{D}))})^2 \|Y_i\|^2_{L^6(\Omega)} \\
&\le c(\beta,\varphi),
\end{aligned}$$

where for any $\gamma \in [0,1]^N$, $\tilde{\kappa}_{N,\beta,\gamma}$ and $\kappa_{N,\beta,\gamma}$ are naturally defined by $\tilde{\kappa}_{N,\beta,\gamma}(y) = P\left(\frac{1}{\tilde{a}^{\min}_{\gamma,N}(y)}, \|\tilde{a}_{\gamma,N}(y)\|_{C^{1,\beta}(\bar{D})}\right)$ and $\kappa_{N,\beta,\gamma}(\omega) = \tilde{\kappa}_{N,\beta,\gamma}(Y_1(\omega), ...Y_N(\omega))$. By the Hölder inequality and Proposition 2.6, we deduce that for any β and $p \ge 1$ $(\kappa_{N,\beta,\gamma})_{N\ge 0}$ is bounded in $L^p(\Omega)$, with a bound independant of γ, which proves the existence of $c(\beta,\varphi)$, together with Proposition 2.6. We finally obtain the following bound for the first term of the error contribution.

$$\begin{aligned}
&\left\| \sum_{N+1 \le i \le M} \int_0^1 (1-t)\mathbb{E}_\omega \left[\frac{\partial^2(\varphi \circ \tilde{u}_M)}{\partial y_i^2}(Y_1, ..., Y_N, tY_{N+1}, ..., tY_M, x) Y_i^2 \right] dt \right\|_{C^{1,\beta}(\bar{D})} \\
&\le C_2(\beta,\varphi) c(\beta,\varphi) (1 + \|f\|_{L^p(D)})^{4+2d_\varphi} \sum_{N+1 \le i \le M} \lambda_i \|b_i\|^2_{C^{0,\beta}(\bar{D})} \\
&\le k_1(\beta, f, \varphi) R_N^\beta.
\end{aligned}$$

Where $k_1(\beta, f, \varphi) = C_2(\beta,\varphi) c(\beta,\varphi) (1 + \|f\|_{L^p(D)})^{4+2d_\varphi}$. Next we give an estimate for the second term of the error contribution, by using once again the independence of the random variables Y_i, for $N+1 \le i < j \le M$ we get:

$$\begin{aligned}
&\int_0^1 (1-t)\mathbb{E}_\omega\left[\frac{\partial^2(\varphi\circ\tilde{u}_M)}{\partial y_i\partial y_j}(X_{i,j}^{t,1,1},x)Y_iY_j\right]dt\\
&=\int_0^1 (1-t)\mathbb{E}_\omega\left[\frac{\partial^2(\varphi\circ\tilde{u}_M)}{\partial y_i\partial y_j}(X_{i,j}^{t,1,1},x)Y_iY_j\right]dt\\
&-\int_0^1 (1-t)\mathbb{E}_\omega\left[\frac{\partial^2(\varphi\circ\tilde{u}_M)}{\partial y_i\partial y_j}(X_{i,j}^{t,0,1},x)Y_iY_j\right]dt\\
&=\mathbb{E}_\omega\left[\iint_{[0,1]^2}(1-t)t\frac{\partial^3(\varphi\circ\tilde{u}_M)}{\partial y_i^2\partial y_j}(X_{i,j}^{t,u,1},x)Y_i^2Y_j dtdu\right]\\
&=\mathbb{E}_\omega\left[\iiint_{[0,1]^3}(1-t)t^2\frac{\partial^4(\varphi\circ u_M)}{\partial y_i^2\partial y_j^2}(X_{i,j}^{t,u,s},x)Y_i^2Y_j^2 dtduds\right].
\end{aligned}$$

Where the random variables $X_{i,j}^{t,r,s}(\omega)$ are defined by

$$X_{i,j}^{t,r,s}(\omega)=(Y_1,...,Y_N,tY_{N+1},...,trY_i,...,tsY_j,...,tY_M)(\omega).$$

By Proposition 4.2, we have then:

$$\begin{aligned}
&\left\|\mathbb{E}_\omega\left[\iiint_{[0,1]^3}(1-t)t^2\frac{\partial^4(\varphi\circ\tilde{u}_M)}{\partial y_i^2\partial y_j^2}(X_{i,j}^{t,s,u},x)Y_i^2Y_j^2 dtduds\right]\right\|_{C^{1,\beta}(\bar{D})}\\
&\le\iiint_{[0,1]^3}(1-t)t^2\mathbb{E}_\omega\left[\left\|\frac{\partial^4(\varphi\circ\tilde{u}_M)}{\partial y_i^2\partial y_j^2}(X_{i,j}^{t,s,u},x)\right\|_{C^{1,\beta}(\bar{D})}Y_i^2Y_j^2\right]dtduds\\
&\le C_2(\beta,\varphi)(1+\|f\|_{L^p(D)})^{6+2d_\varphi}\|b_i\|^2_{C^{0,\beta}(\bar{D})}\|b_j\|^2_{C^{0,\beta}(\bar{D})}\lambda_i\lambda_j\\
&\quad\times\iiint_{[0,1]^3}\mathbb{E}_\omega\left[(1+\tilde{\kappa}_{N,\beta})^{10+2d_\varphi}\left(1+\|\tilde{a}_N\|_{C^{0,\beta}(\bar{D})}\right)^4(X_{i,j}^{t,s,u})Y_i^2Y_j^2\right]dtduds.
\end{aligned}$$

We define, for $t,s,u\in[0,1]$, $\gamma_{t,s,u}\in[0,1]^M$ by $\gamma_{t,s,u}(n)=1$ for $n\le N$, $\gamma_{t,s,u}(n)=t$ for $n>N$ such that $n\ne i, n\ne j$, $\gamma_{t,s,u}(i)=tu$, and $\gamma_{t,s,u}(j)=ts$.
Then, Hölder inequality combined with Proposition 2.6 yields the existence of $c'(\beta,\varphi)$ such that:

$$\begin{aligned}
&\mathbb{E}_\omega\left[(1+\tilde{\kappa}_{N,\beta})^{10}\left(1+\|\tilde{a}_N\|_{C^{0,\beta}(\bar{D})}\right)^4(X_{i,j}^{t,s,u})Y_i^2Y_j^2\right]\\
&=\mathbb{E}_\omega\left[(1+\kappa_{N,\beta,\gamma_{s,t,u}})^{10}\left(1+\|a_{\gamma_{s,t,u},N}\|_{C^{0,\beta}(\bar{D})}\right)^4\right]\\
&\le c'(\beta,\varphi).
\end{aligned}$$

We have finally the following estimate for the second term of the error contribution:

$$\begin{aligned}
&\left\|\sum_{N+1\le i\ne j\le M}\int_0^1(1-t)\mathbb{E}_\omega\left[\frac{\partial^2\varphi\circ\tilde{u}_M}{\partial y_i\partial y_j}(Y_1,...Y_N,tY_{N+1},...,tY_M,x)Y_iY_j\right]dt\right\|_{C^{1,\beta}(\bar{D})}\\
&\le C_2(\beta,\varphi)(1+\|f\|_{L^p(D)})^{6+2d_\varphi}c'(\beta,\varphi)\sum_{N+1\le i\ne j\le M}\lambda_i\lambda_j\|b_i\|^2_{C^{0,\beta}(\bar{D})}\|b_j\|^2_{C^{0,\beta}(\bar{D})}\\
&\le k_2(\beta,f,\varphi)\sum_{N+1\le i\ne j\le M}\lambda_i\lambda_j\|b_i\|^2_{C^{0,\beta}(\bar{D})}\|b_j\|^2_{C^{0,\beta}(\bar{D})},
\end{aligned}$$

where $k_2(\beta,f,p,\varphi)=C_2(\beta)(1+\|f\|_{L^p(D)})^{6+2d_\varphi}c'(\beta,\varphi)$. We have finally the following estimate for the total error:

$$\begin{aligned}
&\mathbb{E}_\omega\left[(\varphi(u_M)-\varphi(u_N))(\omega,x)\right]\\
=&\sum_{N+1\le i\le M}\int_0^1(1-t)\mathbb{E}_\omega\left[\frac{\partial^2\varphi\circ\tilde u_M}{\partial y_i^2}(Y_1,...Y_N,tY_{N+1},...,tY_M,x)Y_i^2\right]dt\\
+&\sum_{N+1\le i\neq j\le M}\int_0^1(1-t)\mathbb{E}_\omega\left[\frac{\partial^2\varphi\circ\tilde u_M}{\partial y_i\partial y_j}(Y_1,...Y_N,tY_{N+1},...,tY_M,x)Y_iY_j\right]dt.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|\mathbb{E}_\omega[(\varphi(u_M)-\varphi(u_N))(\omega,x)]\|_{C^{1,\beta}(\bar D)}\\
\le&k_1(\beta,f,a)R_N^\beta+k_2(\beta,f,a)\sum_{N+1\le i\neq j\le M}\lambda_i\lambda_j\|b_i\|^2_{C^{0,\beta}(\bar D)}\|b_j\|^2_{C^{0,\beta}(\bar D)}\\
\le&k_1(\beta,f,p,\varphi)R_N^\beta+k_2(\beta,f,p,\varphi)\left(\sum_{N+1\le i\le M}\lambda_i\|b_i\|^2_{C^{0,\beta}(\bar D)}\right)^2\\
\le&C_\varphi(k_1(\beta,f,p,\varphi)+k_2(\beta,f,p,\varphi)R_0^\beta)R_N^\beta.
\end{aligned}$$

We define then $C_{w1}(\beta,f,p,\varphi)=k_1(\beta,f,p,\varphi)+k_2(\beta,f,p,\varphi)R_0^\beta$.
We are now ready to conclude, by giving a bound for $\mathbb{E}_\omega[\varphi(u)-\varphi(u_N)]$ in $C^{1,\beta}(\bar D)$-norm.
Let $N\ge 1$, then for all $M>N$, we have:

$$\begin{aligned}
||\mathbb{E}_\omega[\varphi(u)-\varphi(u_N)]||_{C^{1,\beta}(\bar D)}\le&||\mathbb{E}_\omega[(\varphi(u)-\varphi(u_M))]||_{C^{1,\beta}(\bar D)}\\
&+||\mathbb{E}_\omega[(\varphi(u_M)-\varphi(u_N)))]||_{C^{1,\beta}(\bar D)}\\
\le&||\mathbb{E}_\omega[(\varphi(u)-\varphi(u_M))]||_{C^{1,\beta}(\bar D)}+C_{w1}(\beta,f,p,\varphi)R_N^\beta.
\end{aligned}$$

Note that $u-u_M$ satisfies the equation:

$$\operatorname{div}(a\nabla(u-u_M)(x))=\operatorname{div}((a_M-a)\nabla u_M(x))\quad\text{on } D$$

with Dirichlet boundary condition, for almost every ω. We define for almost all ω, $\kappa_\beta(\omega)=P\left(\frac{1}{a^{\min}(\omega)},\|a(\omega)\|_{\mathcal{C}^{0,\beta}}\right)$. By Proposition 2.6, we get that κ_β belongs to $L^p(\Omega)$ for any $p\ge 1$. By Theorem 3.1, we deduce:

$$\begin{aligned}
\|u-u_M\|_{C^{1,\beta}(\bar D)}\le&C_r(p,\beta,D)\,\kappa_\beta\|(a_M-a)\nabla u_M\|_{C^{0,\beta}(\bar D)}\\
\le&C_r(p,\beta,D)\,\kappa_\beta\|a_M-a\|_{C^{0,\beta}(\bar D)}\|u_M\|_{C^{1,\beta}(\bar D)}.
\end{aligned}\tag{4.3}$$

We deduce from Proposition 2.6 and Theorem 3.1 that for any β such that $0<\beta<\frac12\wedge\left(1-\frac{d}{p}\right)$, $M\in\mathbb{N}$ and $q\ge 1$, u and u_M belong to $L^q(\Omega,\mathcal{C}^{1,\beta}(\bar D))$ and $(u_M)_{M\in\mathbb{N}}$ is bounded in $L^q(\Omega,\mathcal{C}^{1,\beta}(\bar D))$. From the previous inequality, Proposition 2.7 and again Proposition 2.6, we deduce the (strong) convergence of u_M to u in $L^q(\Omega,\mathcal{C}^{1,\beta}(\bar D))$ as M goes to $+\infty$ for any $q\ge 1$. More precisely, there exists for any α with $0<\beta<\alpha<1/2$ a constant $C_s(p,q,\alpha,\beta)$ such that for any $M\in\mathbb{N}$ we have

$$\|u-u_M\|_{L^q(\Omega,\mathcal{C}^{1,\beta}(\bar D))}\le C_s(p,q,\alpha,\beta)\sqrt{R_M^\alpha}.\tag{4.4}$$

Coming back to the weak error estimate, we get

$$\begin{aligned}\|\mathbb{E}_\omega[(\varphi(u)-\varphi(u_M))]\|_{C^{1,\beta}(\bar D)} &\le \mathbb{E}_\omega[\|\varphi(u)-\varphi(u_M)\|_{\mathcal{C}^{1,\beta}}]\\ &\le \mathbb{E}_\omega[\|\varphi(u)-\varphi(u_M)\|_{\mathcal{C}^{0}(\bar D)}\\ &+2\|\varphi'\circ u\|_{\mathcal{C}^{0,\beta}(\bar D)}\|\nabla u-\nabla u_M\|_{\mathcal{C}^{0,\beta}(\bar D)}\\ &+2\|\varphi'\circ u-\varphi'\circ u_M\|_{\mathcal{C}^{0,\beta}(\bar D)}\|\nabla u_M\|_{\mathcal{C}^{0,\beta}(\bar D)}].\end{aligned}$$

In order to conclude, we notice that since the six first derivatives of φ are bounded by $C_\varphi(1+x^{2d_\varphi})$, we have

$$\|\varphi(u)-\varphi(u_M)\|_{\mathcal{C}^0(\bar D)} \le C_\varphi(1+\|u\|^{2d_\varphi}_{\mathcal{C}^0(\bar D)}+\|u_M\|^{2d_\varphi}_{\mathcal{C}^0(\bar D)})\|u-u_M\|_{\mathcal{C}^0(\bar D)},$$

and

$$\begin{aligned}\|\varphi'\circ u-\varphi'\circ u_M\|_{\mathcal{C}^{0,\beta}(\bar D)} \le\ & C_\varphi(1+\|u\|^{2d_\varphi}_{\mathcal{C}^0(\bar D)}+\|u_M\|^{2d_\varphi}_{\mathcal{C}^{0,}(\bar D)})\\ &\times(\|u-u_M\|_{\mathcal{C}^{0,\beta}(\bar D)}+\|u-u_M\|_{\mathcal{C}^0(\bar D)}\|u\|_{\mathcal{C}^{0,\beta}(\bar D)}),\end{aligned}$$

which implies that

$$\begin{aligned}\|\mathbb{E}_\omega[(\varphi(u)-\varphi(u_M))]\|_{C^{1,\beta}(\bar D)} \le\ & 2C_\varphi\mathbb{E}_\omega[(1+\|u\|^{2d_\varphi}_{\mathcal{C}^{0,\beta}(\bar D)}+\|u_M\|^{2d_\varphi}_{\mathcal{C}^{0,\beta}(\bar D)})\\ &\times\|u-u_M\|_{\mathcal{C}^{1,\beta}(\bar D)}(1+\|u\|_{\mathcal{C}^{0,\beta}(\bar D)})(1+\|u_M\|_{\mathcal{C}^{1,\beta}(\bar D)})].\end{aligned} \tag{4.5}$$

Since for any $q\ge 1$, u belongs to $L^q(\Omega,\mathcal{C}^{1,\beta}(\bar D))$, $(u_M)_{M\ge 0}$ is bounded in $L^q(\Omega,\mathcal{C}^{1,\beta}(\bar D))$ and $\|u-u_M\|_{L^p(\Omega,\mathcal{C}^{1,\beta}(\bar D))}$ goes to 0 as M goes to $+\infty$ (see (4.4)), by Hölder inequality, it can be seen that $\|\mathbb{E}_\omega[(\varphi(u)-\varphi(u_M))]\|_{C^{1,\beta}(\bar D)}$ goes to zero when $M\to+\infty$ and we obtain:

$$||\mathbb{E}_\omega[\varphi(u)-\varphi(u_N)]||_{C^{1,\beta}(\bar D)} \le C_{w1}(\beta,f,p,\varphi)R_N^\beta.$$

5. Proof of Theorem 2.11. In this paragraph, we prove Theorem 2.11. We establish a bound for a slightly different weak error, considering the case of test function ψ of a and u defined on $\mathcal{C}^{0,\beta}(\bar D)\times\mathcal{C}^{1,\beta}(\bar D)$. In order to get this new weak error bound, we need a preliminary result, similar to Proposition 4.2.

PROPOSITION 5.1. *Let β such that $\frac12\wedge\left(1-\frac{d}{p}\right)>\beta>0$ with $\psi\in\mathcal{C}^4(\mathcal{C}^{0,\beta}(\bar D)\times\mathcal{C}^{1,\beta}(\bar D),\mathbb{R})$ whose derivatives have at most polynomial growth, i.e. such that for any $u\in\mathcal{C}^{1,\beta}(\bar D)$, $a\in\mathcal{C}^{0,\beta}(D)$ and $k=0,...4$ we have*

$$\|D^k\psi(a,u)\|_{\mathcal{L}((\mathcal{C}^{0,\beta}(\bar D)\times\mathcal{C}^{1,\beta}(\bar D))^k,\mathbb{R})} \le C_\psi(1+\|u\|^{2d_\psi}_{\mathcal{C}^{1,\beta}(\bar D)}+\|a\|^{2d_\psi}_{\mathcal{C}^{0,\beta}(\bar D)}),$$

then we have the following estimates on the derivatives of $\psi\circ(\tilde a_N,\tilde u_N)$ with respect to the y_i, of order less than 4. There exists a constant $C_3(\beta,\psi)$ such that for any multi-index δ with $|\delta|\le 4$, any $N\in\mathbb{N}$ and any $y\in\mathbb{R}^N$ we have

$$\begin{aligned}\left|\frac{\partial^\delta\psi\circ(\tilde a_N,\tilde u_N)}{\partial y^\delta}\right| \le\ & C_3(\beta,\psi)(1+\tilde\kappa_{N,\beta}(y))^{2|\delta|+2d_\psi}\left(1+\|\tilde a_N(y)\|_{C^{0,\beta}(\bar D)}\right)^{|\delta|+2d_\psi}\\ &\times(1+\|f\|_{L^p(D)})^{|\delta|+2d_\psi}\prod_{i\in\mathbb{N}}\sqrt{\lambda_i^{\delta_i}}\|b_i\|^{\delta_i}_{C^{0,\beta}(\bar D)}.\end{aligned}$$

Proof. This proof is quite similar to the proof of 4.2, hence we will skip most of the details. For $|\delta| = 0$ we have

$$|\psi \circ (\tilde{a}_N, \tilde{u}_N)| \le C_\psi (1 + \|\tilde{u}_N(y)\|^{2d_\psi}_{\mathcal{C}^{1,\beta}(\bar{D})} + \|\tilde{a}_N(y)\|^{2d_\psi}_{\mathcal{C}^{0,\beta}(\bar{D})}).$$

The conclusion follows from Proposition 4.1. For $|\delta| = 1$, using Proposition 4.1 we have

$$\begin{aligned}
&\left|\frac{\partial \psi \circ (\tilde{a}_N, \tilde{u}_N)}{\partial y_i}\right| = \left| D\psi(\tilde{a}_N, \tilde{u}_N).\frac{\partial(\tilde{a}_N, \tilde{u}_N)}{\partial y_i}\right| \\
&\le C_\psi (1 + \|\tilde{u}_N(y)\|^{2d_\psi}_{\mathcal{C}^{1,\beta}(\bar{D})} + \|\tilde{a}_N(y)\|^{2d_\psi}_{\mathcal{C}^{0,\beta}(\bar{D})}) \left(\left\|\frac{\partial \tilde{u}_N(y)}{\partial y_i}\right\|_{\mathcal{C}^{1,\beta}(\bar{D})} + \left\|\frac{\partial \tilde{a}_N(y)}{\partial y_i}\right\|_{\mathcal{C}^{0,\beta}(\bar{D})} \right) \\
&\le C_\psi (1 + \|\tilde{u}_N(y)\|^{2d_\psi}_{\mathcal{C}^{1,\beta}(\bar{D})} + \|\tilde{a}_N(y)\|^{2d_\psi}_{\mathcal{C}^{0,\beta}(\bar{D})}) \sqrt{\lambda_i} \|b_i\|_{\mathcal{C}^{0,\beta}(\bar{D})} \\
&\left(C_1(\beta, 1, p, D)(1 + \tilde{\kappa}_{n,\beta}(y))^2 (1 + \|\tilde{a}_N(y)\|_{\mathcal{C}^{0,\beta}(\bar{D})}) \|f\|_{L^p(D)} + 2\|\tilde{a}_N(y)\|_{\mathcal{C}^{0,\beta}(\bar{D})} \right).
\end{aligned}$$

The conclusion follows from Proposition 4.1. The cases $|\delta| = 2, 3, 4$ are similar.
□

Using these preliminary results, we are now able to prove Theorem 2.11. The proof uses the same ideas and techniques as Theorem 2.8, hence we only give the main steps. Take $M > N$, the following inequalities will lead to the result by letting $M \to +\infty$ at the final step.

$$\begin{aligned}
&\mathbb{E}_\omega[\psi \circ (a_M, u_M)(\omega)] - \mathbb{E}_\omega[\psi \circ (a_N, u_N)(\omega)] \\
&= \mathbb{E}_\omega[\psi \circ (\tilde{a}_M, \tilde{u}_M)(Y_1(\omega), ..., Y_M(\omega))] - \mathbb{E}_\omega[\psi \circ (\tilde{a}_M, \tilde{u}_M)(Y_1(\omega), ..., Y_N(\omega), 0, ...0)] \\
&= \mathbb{E}_\omega[D_y(\psi \circ (\tilde{a}_M, \tilde{u}_M))(Y_1, ..., Y_N, 0, ...0).(0, ..., 0, Y_{N+1}, ...Y_M)] \\
&+ \mathbb{E}_\omega \left[\int_0^1 (1-t) D_y^2 (\psi \circ (\tilde{a}_M, \tilde{u}_M))(Y_1, ..., Y_N, tY_{N+1}, ...tY_M).(0, ..., 0, Y_{N+1}, ...Y_M)^2 dt \right].
\end{aligned}$$

For the same reasons as in the proof of Theorem 2.8, the first term of the sum is equal to 0. We now bound the second term, by splitting it into a term with cross derivatives $\frac{\partial^2}{\partial x_i \partial x_j}$ such that $i = j$ and a term with cross derivatives such that $i \neq j$.

$$\begin{aligned}
&\mathbb{E}_\omega[\psi \circ (a_M, u_M)(\omega)] - \mathbb{E}_\omega[\psi \circ (a_N, u_N)(\omega)] \\
&= \sum_{N+1 \le i \le M} \int_0^1 (1-t) \mathbb{E}_\omega \left[\frac{\partial^2 (\psi \circ (\tilde{a}_M, \tilde{u}_M))}{\partial y_i^2} (Y_1, ..., Y_N, tY_{N+1}, ..., tY_M) Y_i^2 \right] dt \\
&+ \sum_{N+1 \le i \neq j \le M} \int_0^1 (1-t) \mathbb{E}_\omega \left[\frac{\partial^2 (\psi \circ (\tilde{a}_M, \tilde{u}_M))}{\partial y_i \partial y_j} (Y_1, ..., Y_N, tY_{N+1}, ..., tY_M) Y_i Y_j \right] dt.
\end{aligned}$$

We first bound the term corresponding to the second order derivatives with $i = j$, using Proposition 5.1, for $N + 1 \le i \le M$ we have :

$$\begin{aligned}
&\left| \int_0^1 (1-t) \mathbb{E}_\omega \left[\frac{\partial^2 (\psi \circ (\tilde{a}_M, \tilde{u}_M))}{\partial y_i^2} (Y_1, ..., Y_N, tY_{N+1}, ..., tY_M) Y_i^2 \right] \right| \\
&\le C_3(\beta, \psi)(1 + \|f\|_{L^p(D)})^{2+2d_\psi} \lambda_i \|b_i\|^2_{\mathcal{C}^{0,\beta}(\bar{D})} \\
&\int_0^1 \mathbb{E}_\omega[(1 + \kappa_{n,\beta,\gamma_t})^{4+2d_\psi} (1 + \|a_{\gamma_t,N}\|_{\mathcal{C}^{0,\beta}(\bar{D})})^{2+2d_\psi} Y_i^2] dt.
\end{aligned}$$

The integral can be bounded independently from t and N as in the proof of Theorem 2.8 and we conclude that there exists a constant $k_3(\beta, f, \psi)$ such that

$$\left| \sum_{N+1 \leq i \leq M} \int_0^1 (1-t)\mathbb{E}_\omega \left[\frac{\partial^2(\psi \circ (\tilde{a}_M, \tilde{u}_M))}{\partial y_i^2}(Y_1, ..., Y_N, tY_{N+1}, ..., tY_M) Y_i^2 \right] dt \right|$$
$$\leq k_3(\beta, f, \psi) R_N^\beta.$$

We now bound the term corresponding to the second order derivatives with $i \neq j$. Using the same techniques and notations as in the proof of Theorem 2.8 we get for $N + 1 \leq i < j \leq M$:

$$\int_0^1 (1-t)\mathbb{E}_\omega \left[\frac{\partial^2(\psi \circ (\tilde{a}_M, \tilde{u}_M))}{\partial y_i \partial y_j}(X_{i,j}^{t,1,1}) Y_i Y_j \right] dt$$
$$= \mathbb{E}_\omega \left[\int\int\int_{[0,1]^3} (1-t)t^2 \frac{\partial^4(\psi \circ (\tilde{a}_M, \tilde{u}_M))}{\partial y_i^2 \partial y_j^2}(X_{i,j}^{t,u,s}) Y_i^2 Y_j^2 dtduds \right]$$
$$\leq C_3(\beta, \psi)(1 + \|f\|_{L^p(D)})^{4+2d_\psi} \lambda_i \|b_i\|^2_{\mathcal{C}^{0,\beta}(\bar{D})} \lambda_j \|b_j\|^2_{\mathcal{C}^{0,\beta}(\bar{D})}$$
$$\times \int\int\int_{[0,1]^3} \mathbb{E}_\omega[(1 + \kappa_{N,\beta,\gamma_{s,t,u}})^{8+2d_\psi}(1 + \|a_{\gamma_{s,t,u},N}\|_{\mathcal{C}^{0,\beta}(\bar{D})})^{4+2d_\psi} Y_i^2 Y_j^2] dtduds.$$

We deduce a bound for the sum :

$$\left| \sum_{N+1 \leq i \neq j \leq M} \int_0^1 (1-t)\mathbb{E}_\omega \left[\frac{\partial^2(\psi \circ (\tilde{a}_M, \tilde{u}_M))}{\partial y_i \partial y_j}(Y_1, ..., Y_N, tY_{N+1}, ..., tY_M) Y_i Y_j \right] dt \right|$$
$$\leq k_4(\beta, \psi) \sum_{N+1 \leq i \neq j \leq M} \lambda_i \|b_i\|^2_{\mathcal{C}^{0,\beta}(\bar{D})} \lambda_j \|b_j\|^2_{\mathcal{C}^{0,\beta}(\bar{D})}.$$

In order to conclude, it remains to notice that

$$|\mathbb{E}_\omega[\psi(a_M, u_M)] - \mathbb{E}_\omega[\psi(a_N, u_N)]|$$
$$\leq \mathbb{E}_\omega[(1 + \|u_N\|^{2d_\psi}_{\mathcal{C}^{1,\beta}(\bar{D})} + \|u_M\|^{2d_\psi}_{\mathcal{C}^{1,\beta}(\bar{D})} + \|a_N\|^{2d_\psi}_{\mathcal{C}^{0,\beta}(\bar{D})} + \|a_M\|^{2d_\psi}_{\mathcal{C}^{0,\beta}(\bar{D})})$$
$$\times \|u_N - u_M\|_{\mathcal{C}^{1,\beta}(\bar{D})} + \|a_N - a_M\|_{\mathcal{C}^{0,\beta}(\bar{D})}],$$

and to use the same arguments as in the proof of Theorem 2.8.

6. Numerics. In this section, we propose numerical results to confirm the optimality of the rate of convergence obtained for the weak errors in Theorem 2.8 and Theorem 2.11 in the cases of different test functions presented in the first section. We consider the case of an exponential covariance function, which is quite popular for applications in hydrogeology (see for instance [8, 14]) because, among others, it leads to realisations of the permeability field which are not continuously differentiable. More precisely the function cov is only a Lipschitz continuous function of $x - y$ in this case. The covariance function is then defined by

$$\text{cov}[g](x, y) = \sigma^2 e^{-\frac{\|x-y\|}{\ell}}, \tag{6.1}$$

where ℓ is the correlation length. Moreover, with such a covariance function, the eigenpairs have analytic expression when the norm $\|\cdot\|_1$ is considered and the domain is a box. We propose here to study the case of a spatial dimension $d = 1$, with a domain $D = (0,1)$ and $f = 1$, i.e. we want to compute the solution u of

$$\begin{cases} -\text{div}(a(\omega,x)\nabla u(\omega,x)) = 1 \text{ on } (0,1), \\ u(0) = 0, \ u(1) = 0. \end{cases} \tag{6.2}$$

We recall that in this particular case, there exist analytical expressions for the eigenpairs $(\lambda_n, b_n)_{n\geq 0}$ (see e.g. [27] for a proof) of the Karhunen-Loève development. More precisely, we consider the characteristic equation

$$(\ell^2 w^2 - 1)\sin(w) = 2\ell w \cos(w)$$

and denote by $(w_n)_{n\geq 0}$ the sequence of its positive roots sorted in an increasing order, then the eigenvalues of the Karhunen-Loève development can be expressed as $\lambda_n = \frac{2\ell\sigma^2}{\ell^2 w_n^2+1}$ and the eigenfunctions as

$$b_n(x) = \alpha_n(\sin(w_n x) + \ell w_n \cos(w_n x)), \text{where } \alpha_n = \frac{1}{\sqrt{(\ell^2 w_n^2 + 1)/2 + \ell}}.$$

The sequence of the roots (w_n) satisfies $w_n \underset{n\to+\infty}{\sim} n\pi$, which implies that the eigenvalues and eigenfunctions:

$$\lambda_n \underset{n\to+\infty}{\sim} \frac{2\sigma^2}{\ell\pi^2 n^2}$$

$$b_n(x) \underset{n\to+\infty}{\sim} \sqrt{2}\cos(w_n x).$$

As proved in [3], it is easy to show that assumption 2.3 is fulfilled. Moreover, we have for any $0 \leq \alpha < 1/2$

$$R_N^\alpha \leq \frac{2\sigma^2}{\ell\pi^2(1-2\alpha)} N^{2\alpha-1}.$$

The first weak error bound, given in Theorem 2.8 yields, in the case of our example, that for any $\varphi \in \mathcal{C}^6(\mathbb{R},\mathbb{R})$ whose derivatives have at most polynomial growth, and any $0 < \beta < 1/2$ there exists of a constant $C_{w1}(\alpha,\varphi,\ell,\sigma)$ such that

$$\|\mathbb{E}[\varphi(u_N) - \varphi(u)]\|_{\mathcal{C}^{1,\beta}(0,1)} \leq C_{w1}(\alpha,\varphi,\ell,\sigma)\frac{1}{N^{1-2\beta}}. \tag{6.3}$$

The second weak error bound, given in Theorem 2.11 yields, in the case of our example, that for any $0 < \beta < 1/2$, for any $\psi \in \mathcal{C}^4(\mathcal{C}^{0,\beta}(\bar{D}) \times \mathcal{C}^{1,\beta}(\bar{D}),\mathbb{R})$ whose derivatives have at most polynomial growth, there exists a constant $C_{w2}(\alpha,\varphi,\ell,\sigma)$ such that for all $N \in \mathbb{N}$ we have the following weak error bound:

$$|\mathbb{E}[\psi(a,u) - \psi(a_N,u_N)]| \leq C_{w2}(\beta,f,p,\varphi)\frac{1}{N^{1-2\beta}}. \tag{6.4}$$

We are now going to study numerically this weak convergence for our example. We will also quickly study the influence of the parameters ℓ and σ on the speed of convergence of u_N to u, this influence being strong and the values of ℓ and σ being linked to the physical properties of the porous media. In the model, σ is related to the level of uncertainty, whereas the correlation length ℓ is the characteristic length of the covariance function, i.e. represents the distance between two points necessary so that the values of the permeability are less correlated that a certain value. The constants $C_{w1}(\alpha,\varphi,\ell,\sigma)$ in (6.3) and $C_{w2}(\alpha,\varphi,\ell,\sigma)$ in (6.4) depend clearly on ℓ and σ through R_N^β, but there is also an hidden dependance though the constants appearing in Theorem 2.8 and Theorem 2.11. We quickly make some remarks on how the λ_n depend on ℓ and σ, since it turns out to give a first good idea of how the speed of convergence of the law of u_N to the law of u is influenced by these parameters, as we will see later.

FIG. 6.1. *λ_n versus n in logarithmic scale, for $\sigma = 1$ and different values of ℓ .*

First we notice that $\sigma^2 \mapsto (\lambda_n)_{n\in\mathbb{N}}$ is linear and the eigenvectors b_n do not depend on σ. The speed of convergence of the eigenvalues λ_n slows down when σ increases, and this deterioration is explicit since the eigenvalues are proportional to σ^2. The dependance on ℓ is less obvious. The term $1/\ell$ clearly appears as a multiplicative factor in the equivalent of λ_n, which can be seen on the figure 6.1, but it also impacts the behaviour of the first terms of the sequence of the eigenvalues λ_n. In particular, we observe in figure 6.1 (see also [4] for instance) a plateau in the decrease of the eigenvalues λ_n. Indeed, we can see on figure 6.1 that the asymptotic decrease $\lambda_n \underset{n\to+\infty}{\sim} \frac{2\sigma^2}{\ell\pi^2 n^2}$ is only reached after a plateau of size of order $1/\ell$, and the eigenvalues are almost constant in the beginning of this plateau.

After these preliminary remarks, we now study the weak convergence of u_N to u. In the following numerical results, a stochastic collocation method is used to compute the quantities $x \mapsto \mathbb{E}[\varphi(u_N)]$ and $\mathbb{E}[\psi(u_N)]$ and a Monte-Carlo method is used to compute the quantities $x \mapsto \mathbb{E}[\varphi(u)]$ and $\mathbb{E}[\psi(u)]$. In both cases, the spatial discretization is realised through a rectangle rule, the solution of the PDE being seen as an integral. The spatial mesh, the number of collocation points and the number of realizations in the Monte-Carlo method are chosen such that the corresponding errors are negligible with respect to the truncation error. We first consider the case $\varphi(x) = x$ in (6.3) and compare $\mathbb{E}[u_N]$ to $\mathbb{E}[u]$ by computing the L^∞-norm and then the L^2-norm, for different values of ℓ and σ, see figures 6.2 and 6.3.

FIG. 6.2. *$\|\mathbb{E}[u-u_N]\|_{L^\infty}$ versus N, in logarithmic scale.*

FIG. 6.3. *$\|\mathbb{E}[u-u_N]\|_{L^2}$ versus N, in logarithmic scale.*

We consider then the case where $\varphi : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$, and compute the L^1 norm of $\mathbb{E}[u_N^2] - \mathbb{E}[u^2]$, see figure 6.4.

FIG. 6.4. *$\|\mathbb{E}[u^2-u_N^2]\|_{L^1}$ versus N, in logarithmic scale, for $\sigma = 1$ and $\ell = 1$.*

We now study the convergence of the derivatives by computing the L^∞-norm of the derivatives of $E[u-u_N]$, that is to say the L^∞-norm of $E[u'-u'_N]$, see figure 6.5.

FIG. 6.5. $\|\mathbb{E}[u' - u'_N]\|_{L^\infty}$ *versus* N*, in logarithmic scale, for* $\sigma = 1$ *and* $\ell = 1$*.*

We finally consider the case of a test function $\psi : \mathcal{C}^1(\bar{D}) \to \mathbb{R}$ defined by $\psi(u) = \int_D u^2(x)dx$ and compare $\mathbb{E}[\|u_N\|^2_{L^2}]$ to $\mathbb{E}[\|u\|^2_{L^2}]$, see figure 6.6. We also study once again the influence of the parameters ℓ and σ on the speed of convergence.

FIG. 6.6. $|\mathbb{E}[\|u\|^2_{L^2}] - \mathbb{E}[\|u_N\|^2_{L^2}]|$ *versus* N*, in logarithmic scale.*

As we can see in Figures 6.2, 6.3, 6.4 and 6.5, the rate of convergence of the weak truncation error obtained in (6.3) from Theorem 2.8 is optimal for our example. Similarly, the numerical results of Figure 6.6 show that the rate of convergence of the weak truncation error obtained in (6.4) from Theorem 2.11 is optimal for our example. In all these cases the rate of convergence for the weak truncation error is $1/N$ (more precisely $1/(N^{1-\varepsilon})$ for any $\varepsilon > 0$).

We now comment quickly the influence of the parameters ℓ (i.e. the correlation length) and σ (which describes the intensity of the uncertainty). As expected in vue of the behaviour of the decrease of the eigenvalues λ_n studied above in Figure 6.1, the weak convergence speed of u_N to u deteriorates when σ increases or ℓ decreases. The speed of convergence is influenced by the value of ℓ both through the asymptotic rate and through the existence of a plateau for small values of ℓ, in this case the size of the plateau (see Figures 6.2,6.3 and 6.6) is similar to the one in the decrease of the eigenvalues λ_n in figure 6.1.

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