Phi-entropy inequalities and Fokker-Planck equations

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Abstract. We present new Φ-entropy inequalities for diffusion semigroups under the curvature-dimension criterion. They include the isoperimetric function of the Gaussian measure. Applications to the long time behaviour of solutions to Fokker-Planck equations are given.

Mathematics Subject Classification (2000). Primary 35B40, 35K10, 60J60.

Keywords. Logarithmic Sobolev inequality, Poincaré inequality, Φ-entropies, Bakry-Emery criterion, diffusion semigroups, Fokker-Planck equation.

We consider a Markov semigroup $(\mathbf{P}_t)_{t\geq 0}$ on $\mathbb{R}^n$, acting on functions on $\mathbb{R}^n$ by $\mathbf{P}_t f(x) = \int_{\mathbb{R}^n} f(y)\, p_t(x, dy)$ for x in $\mathbb{R}^n$. The kernels $p_t(x, dy)$ are probability measures on $\mathbb{R}^n$ for all x and $t \geq 0$, called transition kernels. Moreover we assume that the Markov infinitesimal generator $L = \frac{\partial}{\partial t}\Big|_{t=0^+} \mathbf{P}_t$ is given by

$$Lf(x) = \sum_{i,j=1}^{n} D_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) - \sum_{i=1}^{n} a_i(x) \frac{\partial f}{\partial x_i}(x)$$

where $D(x) = (D_{ij}(x))_{1\leq i,j\leq n}$ is a symmetric $n \times n$ matrix, nonnegative in the sense of quadratic forms on $\mathbb{R}^n$ and with smooth coefficients; also the $a_i, 1 \leq i \leq n$, are smooth. Such a semigroup or generator is called a *diffusion*, and we refer to [5], [6] or [13] for backgrounds on these semigroups.

If μ is a Borel probability measure on $\mathbb{R}^n$ and f a μ-integrable function on $\mathbb{R}^n$ we let $\mu(f) = \int_{\mathbb{R}^n} f(x)\, \mu(dx)$. If moreover Φ is a convex function on an interval I of $\mathbb{R}$ and f an I-valued function with f and $\Phi(f)$ μ-integrable, we let

$$\mathbf{Ent}_\mu^\Phi(f) = \mu(\Phi(f)) - \Phi(\mu(f))$$

be the Φ-entropy of f under μ (see [10] for instance). Two fundamental examples are $\Phi(x) = x^2$ on $\mathbb{R}$, for which $\mathbf{Ent}_\mu^\Phi(f)$ is the variance of f, and $\Phi(x) = x \ln x$ on $]0, +\infty[$, for which $\mathbf{Ent}_\mu^\Phi(f)$ is the Boltzmann entropy of a positive function f. By

Jensen's inequality the Φ-entropy $\mathbf{Ent}_\mu^\Phi(f)$ is always a nonnegative quantity and, if moreover Φ is strictly convex, it is positive unless f is a constant, equal to $\mu(f)$.

The semigroup $(\mathbf{P_t})_{t\geq 0}$ is said *μ-ergodic* if $\mathbf{P_t} f$ converges to $\mu(f)$ as t tends to infinity, in $L^2(\mu)$ for all functions f.

In Section 1 we shall derive bounds on $\mathbf{Ent}_\mu^\Phi(f)$ or $\mathbf{Ent}_{\mathbf{P}_t}^\Phi(f)(x)$ which will measure the convergence of $\mathbf{P_t} f$ to $\mu(f)$ in the ergodic setting. This is motivated by the study of the long time behaviour of solutions to Fokker-Planck equations, which will be discussed in Section 2.

Some results of this note with their proofs are detailled in [9].

1. Phi-entropy inequalities

Bounds on $\mathbf{Ent}_{\mathbf{P}_t}^\Phi(f)$ and assumptions on L will be given in terms of the *carré du champ* and Γ_2 operators associated to L, defined by

$$\Gamma(f,g) = \frac{1}{2}\Big(L(fg) - f\,Lg - g\,Lf\Big),\ \Gamma_2(f) = \frac{1}{2}\Big(L\Gamma(f) - 2\Gamma(f, Lf)\Big).$$

If ρ is a real number, we say that the semigroup $(\mathbf{P}_t)_{t\geq 0}$ (or the infinitesimal generator L) satisfies the $CD(\rho,\infty)$ *curvature-dimension criterion* (see [7]) if

$$\Gamma_2(f) \geq \rho\,\Gamma(f)$$

for all functions f, where $\Gamma(f) = \Gamma(f,f)$.

The carré du champ is explicitely given by

$$\Gamma(f,g)(x) = < \nabla f(x), D(x)\,\nabla g(x) > .$$

Expressing Γ_2 is more complex in the general case (see [2, 4]) but, for instance:

• if D is constant, then L satisfies the $CD(\rho,\infty)$ criterion if and only if

$$\frac{1}{2}\big(Ja(x)D + (Ja(x)D)^*\big) \geq \rho\,D \qquad (1)$$

for all x, as quadratic forms on $\mathbb{R}^n$, where Ja is the Jacobian matrix of a and M^* is the transposed matrix of a matrix M;

• if $D(x) = d(x)I$ is a scalar matrix, then L, which is $d(x)\Delta - < a, \nabla >$ in this case, satisfies the $CD(\rho,\infty)$ criterion if and only if

$$\frac{1}{2}\big(M(x) + M(x)^*\big) \geq \rho\,d(x)\,I \qquad (2)$$

for all x, as quadratic forms on $\mathbb{R}^n$, where

$$M = \frac{1}{2}(d\,\Delta d - < a, \nabla d > - \|\nabla d\|^2)I + (\frac{1}{2} - \frac{n}{4})\nabla d \otimes \nabla d + d^2\,Ja.$$

Poincaré and logarithmic Sobolev inequalities for the semigroup $(\mathbf{P}_t)_{t\geq 0}$ are known to be implied by the $CD(\rho,\infty)$ criterion. More generally, and following [6, 7, 10], let $\rho > 0$ and Φ be a C^4 strictly convex function on an interval I of $\mathbb{R}$

such that $-1/\Phi''$ be convex. If the semigroup $(\mathbf{P_t})_{t\geq 0}$ is μ-ergodic and satisfies the $CD(\rho,\infty)$ criterion, then μ satisfies the Φ-entropy inequality

$$\mathbf{Ent}_\mu^\Phi(f) \leq \frac{1}{2\rho}\,\mu(\Phi''(f)\,\Gamma(f)) \tag{3}$$

for all I-valued functions f.

The main instances of such Φ's are the maps $x \mapsto x^2$ on $\mathbb{R}$ and $x \mapsto x\ln x$ on $]0,+\infty[$ or more generally, for $1 \leq p \leq 2$ the map

$$\Phi_p(x) = \begin{cases} \frac{x^p - x}{p(p-1)}, & x > 0 \quad \text{if } p \in]1,2] \\ x\ln x, & x > 0 \quad \text{if } p = 1 \end{cases} \tag{4}$$

For this entropy Φ_p with p in $]1,2]$ the Φ-entropy inequality (3) writes

$$\frac{\mu(g^2) - \mu(g^{2/p})^p}{p-1} \leq \frac{2}{p\rho}\,\mu(\Gamma(g)) \tag{5}$$

for all positive functions g. It is called a *Beckner inequality.* For given g the map $p \mapsto \frac{\mu(g^2)-\mu(g^{2/p})^p}{p-1}$ is nonincreasing with respect to $p>0, p\neq 1$. Moreover its limit for p tending to 1 is $\mathbf{Ent}_\mu\big(g^2\big)$, so that the Beckner inequalities (5) for p in $]1,2]$ give a natural monotone interpolation between the weaker Poincaré inequality (for $p=2$), and the stronger logarithmic Sobolev inequality (in the limit $p \to 1$).

Long time behaviour of the semigroup. The Φ-entropy inequalities are adapted to obtain estimates on the long time behaviour of diffusion semigroups. Let indeed $(\mathbf{P_t})_{t\geq 0}$ be such a semigroup, ergodic for the probability measure μ. If Φ is a $\mathcal{C}^2$ function on an interval I, then

$$\frac{d}{dt}\mathbf{Ent}_\mu^\Phi(\mathbf{P_t}f) = -\mu(\Phi''(\mathbf{P_t}f)\,\Gamma(\mathbf{P_t}f)) \tag{6}$$

for all $t \geq 0$ and all I-valued functions f. As a consequence, if C is a positive number, then the semigroup converges in Φ-entropy with exponential rate:

$$\mathbf{Ent}_\mu^\Phi(\mathbf{P_t}f) \leq e^{-\frac{t}{C}}\mathbf{Ent}_\mu^\Phi(f) \tag{7}$$

for all $t \geq 0$ and all I-valued functions f, if and only if the measure μ satisfies the Φ-entropy inequality for all I-valued functions f,

$$\mathbf{Ent}_\mu^\Phi(f) \leq C\mu(\Phi''(f)\,\Gamma(f)). \tag{8}$$

1.1. Refined Φ-entropy inequalities

We now give and study improvements of inequality (3) for the main family of function Φ, namely the Φ_p functions given by (4) for $p \in]1,2[$.

Theorem 1 ([9]). Let ρ be a real number and p in $]1,2[$. Then the following assertions are equivalent, with $\big(1-e^{-2\rho t}\big)/\rho$ and $\big(e^{2\rho t}-1\big)/\rho$ replaced by $2t$ if $\rho = 0$:

(i) the semigroup $(\mathbf{P}_t)_{t\geq 0}$ satisfies the $CD(\rho,\infty)$ criterion;

(ii) the semigroup $(\mathbf{P}_t)_{t\geq 0}$ satisfies the refined local Φ_p-entropy inequality

$$\frac{1}{(p-1)^2}\left[\mathbf{P_t}(f^p)-\mathbf{P_t}(f)^p\left(\frac{\mathbf{P_t}(f^p)}{\mathbf{P_t}(f)^p}\right)^{\frac{2}{p}-1}\right]\leq\frac{1-e^{-2\rho t}}{\rho}\mathbf{P_t}\big(f^{p-2}\,\Gamma(f)\big)$$

for all positive t and all positive functions f;

(iii) the semigroup $(\mathbf{P}_t)_{t\geq 0}$ satisfies the reverse refined local Φ_p-entropy inequality

$$\frac{1}{(p-1)^2}\left[\mathbf{P_t}(f^p)-\mathbf{P_t}(f)^p\left(\frac{\mathbf{P_t}(f^p)}{\mathbf{P_t}(f)^p}\right)^{\frac{2}{p}-1}\right]\geq\frac{e^{2\rho t}-1}{\rho}\left(\frac{(\mathbf{P_t}f)^p}{\mathbf{P_t}(f^p)}\right)^{\frac{2}{p}-1}(\mathbf{P_t}f)^{p-2}\,\Gamma(\mathbf{P_t}f)$$

for all positive t and all positive functions f.

If moreover $\rho > 0$ and the probability measure μ is ergodic for the semigroup $(\mathbf{P_t})_{t\geq 0}$, then μ satisfies the refined Φ_p-entropy inequality for all positive maps g,

$$\frac{p^2}{(p-1)^2}\left[\mu(g^2)-\mu(g^{2/p})^p\left(\frac{\mu(g^2)}{\mu(g^{2/p})^p}\right)^{\frac{2}{p}-1}\right]\leq\frac{4}{\rho}\mu(\Gamma(g)). \tag{9}$$

Inequality (9) has been obtained in [3] for the generator L defined by $Lf = \mathrm{div}(D\nabla f) - <D\nabla V, \nabla f>$ with $D(x)$ a scalar matrix and for the ergodic measure $\mu = e^{-V}$, and under the corresponding $CD(\rho,\infty)$ condition (2).

It improves on the Beckner inequality (5) since

$$\frac{\mu(g^2)-\mu(g^{2/p})^p}{p-1}\leq\frac{p}{2(p-1)^2}\Big[\mu(g^2)-\mu(g^{2/p})^p\left(\frac{\mu(g^2)}{\mu(g^{2/p})^p}\right)^{\frac{2}{p}-1}\Big]. \tag{10}$$

In the first section we have noticed that for all g the map $p\mapsto\frac{\mu(g^2)-\mu(g^{2/p})^p}{p-1}$ is continuous and nonincreasing on $]0,+\infty[$ and takes the values $\mathbf{Ent}_\mu\big(g^2\big)$ at $p=1$ and $\mathbf{Var}_\mu(g)$ at $p=2$. Similarly, for the larger functional introduced in (10), the map

$$p\mapsto\frac{p}{2(p-1)^2}\Big[\mu(g^2)-\mu(g^{2/p})^p\left(\frac{\mu(g^2)}{\mu(g^{2/p})^p}\right)^{\frac{2}{p}-1}\Big]$$

is nonincreasing on $]1,+\infty[$ (see [9, Proposition 11]). Moreover it takes the value $\mathbf{Var}_\mu(g)$ at $p=2$ and tends to $\mathbf{Ent}_\mu\big(g^2\big)$ as p tends to 1, hence providing a new monotone interpolation between Poincaré and logarithmic Sobolev inequalities.

Note that the pointwise $CD(\rho,\infty)$ criterion can be replaced by the integral criterion

$$\mu\Big(g^{\frac{2-p}{p-1}}\Gamma_2(g)\Big)\geq\rho\,\mu\Big(g^{\frac{2-p}{p-1}}\Gamma(g)\Big)$$

for all positive functions g to get the refined Φ_p-entropy inequality (3), even for non reversible semigroups (see [9, Proposition 14]).

Remark 2. For $\rho=0$, and following [3], the convergence of $\mathbf{P_t}f$ towards $\mu(f)$ can be measured as

$$|H'(t)|\leq\frac{|H'(0)|}{1+\alpha t},\quad t\geq 0$$

where $\alpha = \frac{2-p}{p}|H'(0)|/H(0)$ and $H(t) = \mathbf{Ent}_\mu^\Phi(\mathbf{P_t} f)$. This is an illustration of the improvement induced by (9) instead of (5), which in this case does not give any estimate on the convergence.

1.2. The particular case of the Gaussian isoperimetry function

Let F be the distribution function of the one dimensional standard Gaussian measure. The map $\mathcal{U} = f' \circ F^{-1}$, which is the isoperimetry function of the Gaussian distribution, satisfies $\mathcal{U}'' = -1/\mathcal{U}$ on the set $[0,1]$, so that the map $\Phi = -\mathcal{U}$ is convex with $-1/\Phi''$ also convex on $[0,1]$.

Theorem 3. Let ρ be a real number. Then the following three assertions are equivalent, with $\big(1 - e^{-2\rho t}\big)/\rho$ and $\big(e^{2\rho t} - 1\big)/\rho$ replaced by $2t$ if $\rho = 0$:

(i) the semigroup $(\mathbf{P}_t)_{t\geq 0}$ satisfies the $CD(\rho, \infty)$ criterion;

(ii) the semigroup $(\mathbf{P}_t)_{t\geq 0}$ satisfies the local Φ-entropy inequality

$$\mathbf{Ent}_{\mathbf{P_t}}^\Phi(f) \leq \frac{1}{\Phi''(\mathbf{P_t} f)} \log\left(1 + \frac{1 - e^{-2\rho t}}{2\,\rho} \Phi''(\mathbf{P_t} f)\, \mathbf{P_t}(\Phi''(f)\Gamma(f))\right)$$

for all positive t and all $[0,1]$-valued functions f;

(iii) the semigroup $(\mathbf{P}_t)_{t\geq 0}$ satisfies the reverse local Φ-entropy inequality

$$\mathbf{Ent}_{\mathbf{P_t}}^\Phi(f) \geq \frac{1}{\Phi''(\mathbf{P_t} f)} \log\left(1 + \frac{e^{2\rho t} - 1}{2\,\rho} \Phi''(\mathbf{P_t} f)^2 \Gamma(\mathbf{P_t} f)\right)\right)$$

for all positive t and all $[0,1]$-valued functions f.

If moreover $\rho > 0$ and the probability measure μ is ergodic for the semigroup $(\mathbf{P_t})_{t\geq 0}$, then μ satisfies the Φ-entropy inequality for all $[0,1]$-valued functions f.

$$\mathbf{Ent}_\mu^\Phi(f) \leq \frac{1}{\Phi''(\mu(f))} \log\left(1 + \frac{\Phi''(\mu(f))}{2\rho}\Gamma(f))\right).$$

For this $\Phi = -\mathcal{U}$ it improves on the general Φ-entropy inequality (3) since $\log(1+x) \leq x$ for all $x > -1$. The proof is based on [9, Lemma 4]. It would be interesting to understand the link between this optimal bound and the isoperimetric statements of [8] for instance.

2. Long time behaviour for Fokker-Planck equations

Let us consider the linear Fokker-Planck equation

$$\frac{\partial u_t}{\partial t} = \operatorname{div}[D(x)(\nabla u_t + u_t(\nabla V(x) + F(x)))], \quad t \geq 0,\, x \in \mathbb{R}^n \tag{11}$$

where $D(x)$ is a positive symmetric $n \times n$ matrix and the vector field F satisfies

$$\operatorname{div}\big(e^{-V} DF\big) = 0. \tag{12}$$

It is one of the purposes of [2] and [4] to rigorously study the asymptotic behaviour of solutions to (11)-(12). Let us formally rephrase the argument in our semigroup terminology.

We let L be the Markov diffusion generator defined by

$$Lf = \mathrm{div}(D\nabla f) - < D(\nabla V - F), \nabla f > . \tag{13}$$

Assume that L satisfies the $CD(\rho, \infty)$ criterion with $\rho > 0$, given by (1) if D is constant, by (2) if $D(x)$ is a scalar matrix, and so on. Then the semigroup $(\mathbf{P_t})_{t\geq 0}$ associated to L is μ-ergodic with $d\mu = e^{-V}/Z dx$ where Z is a normalization constant. Moreover a Φ-entropy inequality (8) holds with $C = 1/(2\rho)$ by (3), so that the semigroup converges to μ according to (7).

But, under the condition (12), the solution to (11) for the initial datum u_0 is given by $u_t = e^{-V}\,\mathbf{P_t}(e^V u_0)$. Then, if $e^V u_0$ is I-valued, we can deduce the convergence of the solution u_t towards the stationary state e^{-V} (up to a multiplicative constant) from the convergence estimate (7) for the Markov semigroup, in the form

$$\mathbf{Ent}_\mu^\Phi\Big(\frac{u_t}{e^{-V}}\Big) \leq e^{-2\rho t}\mathbf{Ent}_\mu^\Phi\Big(\frac{u_0}{e^{-V}}\Big), \quad t \geq 0.$$

In fact such a result holds in the general situation of the Fokker-Planck equation

$$\frac{\partial u_t}{\partial t} = \mathrm{div}[D(x)(\nabla u_t + u_t a(x))], \quad t \geq 0,\, x \in \mathbb{R}^n \tag{14}$$

where again $D(x)$ is a positive symmetric $n\times n$ matrix and $a(x) \in \mathbb{R}^n$. Its generator is the dual (with respect to the Lebesgue measure) of the generator

$$Lf = \mathrm{div}(D\nabla f) - < Da, \nabla f > . \tag{15}$$

Assume that the semigroup associated to L is ergodic and that its invariant probability measure μ satisfies a Φ-entropy inequality (8) with a constant $C \geq 0$: this holds for instance if L satisfies the $CD(1/(2C), \infty)$ criterion.

In this setting when $a(x)$ is not the gradient of a potential, the invariant measure is not explicit. Moreover the explicit relation between the solution of the linear Fokker-Planck and the semigroup associated to L does not hold anymore, which lead above from the asymptotic behaviour of the semigroup to that of the solutions of the Fokker-Planck equation. This can be replaced by the following argument, for which we only assume that the ergodic measure has a positive density u_∞ with respect to the Lebesgue measure:

Let u be a solution of (14) for the initial datum u_0. Then

$$\frac{d}{dt}\mathbf{Ent}_\mu^\Phi\big(\frac{u_t}{u_\infty}\big) = \int \Phi'\big(\frac{u_t}{u_\infty}\big) L^* u_t\, dx = \int L\Big[\Phi'\big(\frac{u_t}{u_\infty}\big)\Big]\,\frac{u_t}{u_\infty} d\mu = -\int \Phi''\big(\frac{u_t}{u_\infty}\big)\Gamma\big(\frac{u_t}{u_\infty}\big) d\mu$$

by [9, Lemma 7]. Then a Φ-Entropy inequality (8) for μ implies the exponential convergence. Hence we have formally obtained:

Theorem 4. In the above notation, let Φ such that a Φ-entropy inequality (8) holds for μ and with a constant C. Then all solutions $u = (u_t)_{t\geq 0}$ to the Fokker-Planck equation (14) converge to u_∞ in Φ-entropy, with

$$\mathbf{Ent}_\mu^\Phi\left(\frac{u_t}{u_\infty}\right) \leq e^{-t/C}\mathbf{Ent}_\mu^\Phi\left(\frac{u_0}{u_\infty}\right), \quad t \geq 0.$$

Acknowledgment: This work was presented during the 7th ISAAC conference held in Imperial College, London in July 2009. It is a pleasure to thank the organizers for giving us this opportunity.

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