New sharp Gagliardo-Nirenberg-Sobolev inequalities and an improved Borell-Brascamp-Lieb inequality

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Abstract

We propose a new Borell-Brascamp-Lieb inequality which leads to novel sharp Euclidean inequalities such as Gagliardo-Nirenberg-Sobolev inequalities in $\mathbb{R}^n$ and in the half-space $\mathbb{R}^n_+$. This gives a new bridge between the geometric point of view of the Brunn-Minkowski inequality and the functional point of view of the Sobolev type inequalities. In this way we unify, simplify and generalize results by S. Bobkov - M. Ledoux, M. del Pino - J. Dolbeault and B. Nazaret.

Key words: Sobolev inequality, Gagliardo-Nirenberg inequality, Brunn-Minkowski inequality, Hamilton-Jacobi equation, Hopf-Lax solution

1 Introduction

Sharp inequalities are interesting not only because they correspond to exact solutions of variational problems (often related to problems in physics) but also because they encode in general deep geometric information on the underneath space. In the present paper, we are interested in new functional inequalities of Sobolev type, and their links with the Brunn-Minkowski inequality

$$\mathrm{vol}_n(A+B)^{1/n} \geq \mathrm{vol}_n(A)^{1/n} + \mathrm{vol}_n(B)^{1/n} \tag{1}$$

for non-empty Borel sets A, B in $\mathbb{R}^n$; here $\mathrm{vol}_n(\cdot)$ denotes the n-dimensional Lebesgue measure. Whereas it is known since [BL08] that sharp Sobolev and Gagliardo-Nirenberg inequalities in $\mathbb{R}^n$ may be derived using Brunn-Minkowski type inequalities, we will see that a new functional version of (1) provides a more direct and simple answer, that allows to tackle both the cases of $\mathbb{R}^n$ and the half-space $\mathbb{R}^n_+$.

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Before presenting this new functional inequality, let us discuss new sharp Sobolev type inequalities in $\mathbb{R}^n$ which have inspired our line of thought.

To simplify the notation, let $\|f\|_p = \|f\|_{L^p(\mathbb{R}^n)}$ denote the L^p-norm with respect to Lebesgue measure. The sharp classical Sobolev inequalities state that for $n \geqslant 2, p \in [1, n)$, $p^* = \frac{np}{n-p}$, and any smooth enough function f on $\mathbb{R}^n$ (that is for f belonging to the correct Sobolev space ensuring that both integrals are finite),

$$\|f\|_{p^*} \leq \frac{\|h_p\|_{p^*}}{\left(\int_{\mathbb{R}^n} |\nabla h_p|^p\right)^{1/p}} \left(\int_{\mathbb{R}^n} |\nabla f|^p\right)^{1/p}; \tag{2}$$

here

$$h_p(x) := (1 + |x|^{\frac{p}{p-1}})^{\frac{p-n}{p}}.$$

The optimal constants in the Sobolev inequalities have been first exhibited in [Aub76, Tal76]. Quite naturally, these inequalities admit a generalization when the Euclidean norm $|\cdot|$ on $\mathbb{R}^n$ is replaced by any norm or quasi-norm $\|\cdot\|$ on $\mathbb{R}^n$. Indeed, if we use a norm $\|\cdot\|$ to compute the size of the differential in (2), then the result remains true, namely

$$\|f\|_{p^*} \leq \frac{\|h_p\|_{p^*}}{\left(\int_{\mathbb{R}^n} ||\nabla h_p||_*^p\right)^{1/p}} \left(\int_{\mathbb{R}^n} ||\nabla f||_*^p\right)^{1/p} \tag{3}$$

where $\|y\|_* := \sup_{\|x\| \leq 1} x \cdot y$. In this case, $h_p(x) := (1 + ||x||^{\frac{p}{p-1}})^{\frac{p-n}{p}}$.

In turn, a natural extension of this problem may then be the minimization, under integrability constraints on a function g, of more general quantities like

$$\int_{\mathbb{R}^n} F(\nabla g)\, g^\alpha$$

where $F : \mathbb{R}^n \to \mathbb{R}$ is a convex function ($F = W^*$ below). Note that we have to allow a g^α term, $\alpha \in \mathbb{R}$ since it can no longer be absorbed in the gradient term when F is not homogeneous.

A first answer in this direction is the following optimal Sobolev type inequality.

Theorem 1 (A convex Sobolev inequality) *Let $n \geqslant 2$ and $W : \mathbb{R}^n \to (0, +\infty)$ such that $\liminf_{+\infty} \frac{W(x)}{|x|^\gamma} > 0$ for some $\gamma > \frac{n}{n-1}$. For any $g : \mathbb{R}^n \to (0, +\infty)$ with $g^{-n}|\nabla g|^{\gamma/(\gamma-1)} \in L^1$ and*

$$\int g^{-n} = \int W^{-n} = 1,$$

one has

$$\int W^*(\nabla g) g^{-n} \geq \frac{1}{n-1} \int W^{1-n}. \tag{4}$$

Moreover, equality holds in (4) when g is equal to W and is convex.

Here W^* is the Legendre transform of the function W, see below for details. This result admits a "concave" analogue, as we shall see.

We shall see that the sharp Sobolev inequalities (3), for $p \in (1, n)$, easily follow from this theorem when applied to $W(x) = C(1 + \|x\|^q/q)$, $q = p/(p-1) > n/(n-1)$ ($\gamma = q$ in the assumptions) and to $g = f^{p/(p-n)}$. Let us mention that the coefficients n and $n-1$ in this theorem are not arbitrary at all: in some aspects, they are the "good" ones to reach the Sobolev inequality, as we shall see. This may be compared to Corollary 2 of [BL08] which was derived via a more involved formulation of the Prékopa-Leindler inequality, leading to a less direct proof of the Sobolev inequalities.

As mentioned above, our work is inspired by the Brunn-Minkowski-Borell theory. In turn, we will propose a new functional viewpoint on this theory. As already said, it has been observed by S. Bobkov and M. Ledoux in [BL00, BL08] that Sobolev inequalities can be reached through a functional version of the Brunn-Minkowski inequality, the so-called Borell-Brascamp-Lieb (BBL) inequality, due to C. Borell and H. J. Brascamp - E. H. Lieb ([Bor75, BL76]). However, one can not use its standard functional form, as there is a subtle game with the dimension.

The standard (BBL) inequality states that, for $n \geqslant 1$, given $s \in [0,1], t = 1-s$, and three nonnegative functions $u, v, w : \mathbb{R}^n \to [0, +\infty]$ such that $\int u = \int v = 1$ and

$$\forall x, y \in \mathbb{R}^n, \qquad w(sx+ty) \geq (s\, u^{-1/n}(x) + t\, v^{-1/n}(y))^{-n},$$

then

$$\int w \geq 1.$$

This is the "strongest" version of (BBL) inequality, see e.g. [Gar02, Th. 10.2]. By a simple change of functions, the result can be re-stated as follows: let three nonnegative functions $g, W, H : \mathbb{R}^n \to [0, +\infty]$ be such that

$$\forall x, y \in \mathbb{R}^n, \qquad H(sx+ty) \leq s\, g(x) + t\, W(y)$$

and $\int W^{-n} = \int g^{-n} = 1$. Then

$$\int H^{-n} \geq 1. \tag{5}$$

One observes that (5) is not well adapted to the Sobolev inequality, but that a version with $n-1$ instead of n would do the job. To solve this issue, in [BL08] S. Bobkov and M. Ledoux cleverly used a classical geometric strengthening of the Brunn-Minkowski inequality, for sets having an hyperplane section of same volume.

A natural question raised by S. Bobkov and M. Ledoux is whether the Sobolev inequality can be proved directly from a new (BBL) inequality, which moreover would be well adapted to a monotone mass transport argument. In this work we propose an answer in the following form.

Theorem 2 (An extended Borell-Brascamp-Lieb inequality) *Let $n \geqslant 2$. Let $g, W, H : \mathbb{R}^n \to [0, +\infty]$ be Borel functions and $s \in [0,1], t = 1-s$ be such that*

$$\forall x, y \in \mathbb{R}^n, \qquad H(sx+ty) \leq s\, g(x) + t\, W(y)$$

and $\int W^{-n} = \int g^{-n} = 1$. Then

$$\int H^{1-n} \geq s \int g^{1-n} + t \int W^{1-n}. \tag{6}$$

We shall see in Section 2.4 that, for small t, the optimal H satisfies $H = g - tW^*(\nabla g) + o(t)$, so that (6) gives the above (4) in Theorem 1 and therefore the Sobolev inequalities (3) at the first order for $t \to 0$; as mentioned the Sobolev inequalities correspond to the case $W(x) = C(1 + \|x\|^q/q), q = p/(p-1), g = f^{p/(p-n)}$. More generally we shall see that sharp (classical and trace) Sobolev inequalities and new (trace) Gagliardo-Nirenberg inequalities follow from it. Moreover it can be easily proved using a mass transport argument, and we believe that this is a way of closing the circle of ideas relating Brunn-Minkowski and Sobolev inequalities.

The Sobolev inequalities in $\mathbb{R}^n$ belong to the larger family of Gagliardo-Nirenberg inequalities

$$\|f\|_\alpha \leq C\,\|\nabla f\|_p^\theta\,\|f\|_\beta^{1-\theta}.$$

Here the coefficients α, β, p belong to an adequate range and $\theta \in [0,1]$ is fixed by scaling invariance. These inequalities have attracted much attention these past years. Sharp inequalities are known for a certain family of parameters since the pioneering work of M. del Pino and J. Dolbeault [dD02]: namely, for $p > 1$, $\alpha = ap/(a-p)$ and $\beta = p(a-1)/(a-p)$ where $a > p$ is a free parameter.

This family can be recovered from Theorem 1, or rather an extension of it (see Theorem 3 and its "concave" counterpart Theorem 5). In fact this extension turns out not only to be a natural way of recovering this family, but also allows to extend the family to parameters $a < p$ leading to new sharp Gagliardo-Nirenberg inequalities with negative powers

$$\|f\|_{p\frac{a-1}{a-p}} \leq C\,\|\nabla f\|_p^\theta\,\|f\|_{\frac{ap}{a-p}}^{1-\theta}.$$

Here $p > a$ if $a \geq n+1$, or $p \in (a, \frac{n}{n+1-a})$ if $a \in [n, n+1)$, and θ is fixed by a scaling condition. Let us note that partial results for a narrower range of such $a < p$ have been proved by V.-H. Nguyen [Ngu15], by another approach.

A crucial advantage of our approach is also its robustness: it can be applied to reach a new family of sharp trace Gagliardo-Nirenberg inequalities which extend the trace Sobolev inequality proved by B. Nazaret [Naz06]. Indeed, letting $\mathbb{R}^n_+ = \{(u,x),\, u \geqslant 0,\, x \in \mathbb{R}^{n-1}\}$ we obtain the sharp family of inequalities

$$\|f\|_{L^\alpha(\partial\mathbb{R}^n_+)} \leq C\|\nabla f\|^\theta_{L^p(\mathbb{R}^n_+)}\|f\|^{1-\theta}_{L^\beta(\mathbb{R}^n_+)}.$$

Here $p > 1$, $\alpha = p(a-1)/(a-p)$ and $\beta = p(a-1)/(a-p)$ where $a > p$ is a free parameter and again $\theta \in [0,1]$ is fixed by a scaling argument. This is thus the analog of the del Pino-Dolbeaut family in the trace case.

The paper is organized as follows. In the next section we state and prove the main results, namely generalizations of Theorem 1 and 2. In Section 3 we show how these results lead to Gagliardo-Nirenberg inequalities in $\mathbb{R}^n$, including and extending the del Pino-Dolbeault family, whereas in Section 4 we follow the same procedure to reach trace Gagliardo-Nirenberg inequalities. Section 5 is devoted to limit forms of the (BBL) and Gagliardo-Nirenberg inequalities, namely the classical Prékopa-Leindler inequality and classical or new trace logarithmic Sobolev inequalities. Finally Appendix A deals with a general result on the infimum convolution, which is a crucial tool for our proofs.

Notation: When the measure is not mentioned, an integral is understood with respect to Lebesgue measure. For $x, y \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm of x and $x \cdot y$ the Euclidean scalar product. As already used, $\|f\|_p$ stands for the $L^p(\mathbb{R}^n)$ norm of a function f.

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2 Main results and proofs

Our results have two formulations, as a convex (or concave) Sobolev type inequality illustrated by Theorem 1, and as a Borell-Brascamp-Lieb type inequality like Theorem 2.

2.1 Setting and additional tools

Our setting splits in two separate cases, the origin of which will be explained below. We shall measure the gradient using a function W on $\mathbb{R}^n$ in one of the following two categories:

i. Either $W : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a *convex* fonction, with Legendre transform W^* defined by

$$W^*(y) = \sup_{x \in \mathbb{R}^n} \{x \cdot y - W(x)\}.$$

The function W is differentiable at almost every x in its domain, with

$$W^*(\nabla W(x)) + W(x) = x \cdot \nabla W(x). \tag{7}$$

ii. Either W is a nonnegative function that is *concave* on its support $\Omega_W = \{W > 0\}$. More precisely, W is a nonnegative function such that the function $\tilde{W}$ defined on $\mathbb{R}^n$ by $\tilde{W}(x) = W(x)$ if $x \in \Omega_W$ and $-\infty$ otherwise, is concave. In particular Ω_W is a convex set. The corresponding Legendre transform is defined by

$$W_*(y) = \inf_{x \in \Omega_W} \{x \cdot y - W(x))\} = \inf_{x \in \mathbb{R}^n} \{x \cdot y - \tilde{W}(x)\}. \tag{8}$$

Likewise, W is differentiable at almost every $x \in \Omega_W$, with

$$W_*(\nabla W(x)) + W(x) = x \cdot \nabla W(x). \tag{9}$$

We will later assume that W is continuous on $\mathbb{R}^n$ to avoid jumps on $\partial\Omega_W$.

We refer to [Roc70] for instance for these classical definitions and properties.

One rather naturally comes to such a setting if one has in mind the Brunn-Minkowski theory of convex measures on $\mathbb{R}^n$ as put forward by C. Borell. We briefly recall it to put our results in perspective, although we will not explicitly use it. A nonnegative function G on $\mathbb{R}^n$ is said to be *κ-concave* with $\kappa \in \mathbb{R}$ if $\kappa\, G^\kappa$ is concave on its support. In other words:

i. If $\kappa < 0$, then $G = W^{1/\kappa}$ with W convex on $\mathbb{R}^n$. The Brunn-Minkowski-Borell theory shows that one should consider the range $\kappa \in [-\frac{1}{n}, 0)$. Below we shall let $\kappa = -1/a$ for $a \geqslant n$ with the typical exemples $W(x) = 1 + |x|^q, q \geqslant 1$ and then $G(x) = (1 + |x|^q)^{-a}$. The results above in Theorems 1 and 2 correspond to the extremal case $a = n$.

ii. If $\kappa > 0$, $G = W^{1/\kappa}$ with W concave on its support. Below we shall let $\kappa = 1/a$ for $a > 0$ with the typical examples $W(x) = (1 - |x|^q)_+, q \geqslant 1$ and $G(x) = (1 - |x|^q)_+^a$.

The limit case $\kappa = 0$ is defined as the log-concavity of G.

A central tool in our work will be monotone transportation, which by now has become a cornerstone of many proofs of functional inequalities. So let us briefly describe the mathematical setting and notation on this topic we shall use below, see [Vil03, Vil09] for instance.

Given μ and ν two (Borel) probability measure on $\mathbb{R}^n$ with μ absolutely continuous with respect to Lebesgue measure, a result of Brenier [Bre91], in a form improved by McCann [McC95], states that there exists a convex function φ (the so-called Brenier map) on $\mathbb{R}^n$ such that ν is the image measure $\nabla\varphi\#\mu$ of μ by $\nabla\varphi$, i.e. for any positive or bounded Borel function H on $\mathbb{R}^n$,

$$\int H d\nu = \int H(\nabla\varphi) d\mu.$$

Assuming that $d\mu = f dx$ and $d\nu = g dx$ then [McC97] ensures that the Monge-Ampère equation

$$f(x) = g(\nabla\varphi(x)) \det(\nabla^2\varphi(x)) \tag{10}$$

holds $f dx$-almost surely. Here $\nabla^2\varphi$ is the Alexandrov Hessian of φ, which is the absolutely continuous part of the distributional Hessian of the convex function φ (but below φ will belong to $W^{2,1}_{loc}$ so there will be no singular part).

A second classical and elementary tool will be the convexity of the determinant of nonnegative symmetric matrices, such as $\nabla^2\varphi(x)$. This splits in two cases, in accordance to the cases discussed above.

- For every $k \in (0, 1/n]$, the map $H \to \det^k H$ is concave over the set of positive symmetric matrices. Concavity inequality around the identity implies

$$\det{}^k H \le 1 - nk + k \operatorname{tr} H \tag{11}$$

  for all positive symmetric matrix H.

- For every $k < 0$, the map $H \to \det^k H$ is convex over the set of positive symmetric matrices. Convexity inequality around the identity implies

$$\det{}^k H \geqslant 1 - nk + k \operatorname{tr} H \tag{12}$$

  for all positive symmetric matrix H.

2.2 Convex and concave Sobolev inequalities

We start with a generalization of Theorem 1 and we will next establish its "concave" counterpart.

The result involves a "measurement" function $W : \mathbb{R}^n \to \mathbb{R}^+$ that will be convex in applications, and actually of the form

$$W(x) = 1 + \|x\|^q/q \tag{13}$$

for a norm $\|\cdot\|$ on $\mathbb{R}^n$ and $q > 1$; its Legendre transform is $W^*(y) = \|y\|_*^p/p - 1$ with $p = q/(q-1)$ and $\|\cdot\|_*$ the dual norm. We assume that negative powers of W are integrable, so when W is convex this implies already that W is greater than $|x|$ at infinity. We actually require a slightly stronger super-linearity, which is trivially fullfilled in the applications of type (13).

Theorem 3 (Convex inequalities) *Let $n \geqslant 1$. Let $a \geqslant n$ (and $a > 1$ if $n = 1$) and let $W : \mathbb{R}^n \to (0, +\infty)$ such that*

$$\int W^{-a} = 1$$

and

$$\exists \gamma > \max\left\{\frac{n}{a-1}, 1\right\}, \qquad \liminf_{x\to+\infty} \frac{W(x)}{|x|^\gamma} > 0. \tag{14}$$

For any positive function $g \in W^{1,1}_{loc}$ such that $g^{-a}|\nabla g|^{\gamma/(\gamma-1)}$ is integrable and

$$\int g^{-a} = 1,$$

one has

$$(a-1)\int W^*(\nabla g)g^{-a} + (a-n)\int g^{1-a} \geq \int W^{1-a}. \tag{15}$$

Moreover, there is equality in (15) if $g = W$ and is convex.

Theorem 1 and the classical Sobolev inequalities correspond to the extremal case $a = n$.

Much could be said regarding the assumptions on W and g in the theorem.

First, the condition (14) and $\int W^{-a} < +\infty$ ensure that $\int W^{1-a} < +\infty$. Actually, $W > 0$ continuous (for instance convex) and (14) ensure that $\int W^{1-a}$ and $\int W^{-a}$ are finite.

Next, the integrability assumption $g^{-a}|\nabla g|^{\gamma/(\gamma-1)} \in L^1(\mathbb{R}^n)$ is here for technical reasons, in order to justify an integration by parts; we believe that the correct assumption should simply be that $\int W^*(\nabla g)g^{-a} < +\infty$. Note that a convex W itself has no reason to match this integrability assumption (although it is $W^{1,1}_{loc}$). When we write that there is equality in (15) for $g = W$, it is by direct computation and integration by parts, as we shall see; then the assumption (14) appears as the natural requirement to justify the computation.

Note that the condition $\gamma > 1$ in (14), already needed for the condition on g to make sense, ensures that W^* is well defined (i.e. finite) on $\mathbb{R}^n$.

Analogously, we assume that W is finite (i.e. the convex function W has a domain equal to $\mathbb{R}^n$); this prevents us from reaching the 1-homogeneous case $W^*(x) = C + \|x\|_*$, which corresponds to the L^1 Sobolev inequality. In this case, extremal functions are given by indicators of sets (given by the domain of W), and it requires to work with functions of bounded variation and related notions of capacity. Therefore, it is to be expected that this degenerate case should be treated separately when it comes to identifying the extremal functions.

Proof

$\triangleleft$ Let φ be Brenier's map such that $\nabla\varphi \# g^{-a} = W^{-a}$. Then, from (10), almost everywhere

$$W(\nabla\varphi) = g\,\left(\det \nabla^2\varphi\right)^{1/a}.$$

Moreover, since $a \geqslant n$, from (11) with $k = 1/a$ we have almost everywhere

$$\left(\det \nabla^2\varphi\right)^{1/a} \leq 1 - \frac{n}{a} + \frac{1}{a}\,\Delta\varphi,$$

where here and below $\Delta\varphi = \operatorname{tr}\left(\nabla^2\varphi\right)$. Integrating with respect to the measure $g^{-a}dx$ we get

$$\int W(\nabla\varphi)g^{-a} \leq \left(1 - \frac{n}{a}\right)\int g^{1-a} + \frac{1}{a}\int \Delta\varphi\, g^{1-a}.$$

Let us assume we can integrate by parts the second term; this only requires to put some suitable condition on g^{1-a} (in our situation φ is at least $W^{2,1}_{loc}$, see e.g. [Fig17]). Actually, we can for instance establish, when $a > \gamma/(\gamma-1)$, the following sufficient inequality

$$\int \Delta\varphi\, g^{1-a} \leq (a-1)\int \nabla\varphi\cdot\nabla g\; g^{-a}. \tag{16}$$

Assuming (16) we have

$$a\int W(\nabla\varphi)g^{-a}\le(a-n)\int g^{1-a}+(a-1)\int\nabla g\cdot\nabla\varphi\,g^{-a}.$$

But by definition of Legendre's transform

$$\nabla g\cdot\nabla\varphi\le W(\nabla\varphi)+W^*(\nabla g)$$

so collecting terms we have

$$\int W(\nabla\varphi)g^{-a}\le(a-1)\int W^*(\nabla g)g^{-a}+(a-n)\int g^{1-a}.$$

Finally $\int W(\nabla\varphi)g^{-a}=\int W^{1-a}$ since $\nabla\varphi\#g^{-a}=W^{-a}$, so we have

$$(a-1)\int W^*(\nabla g)g^{-a}+(a-n)\int g^{1-a}\ge\int W^{1-a}\tag{17}$$

as claimed.

This ends the proof of the inequality in the Theorem when $\gamma':=\gamma/(\gamma-1)<a$, provided we justify the integration by parts (16). For this, we extend the argument in [CNV04, Lemma 7] which is given for $W(x)=1+\|x\|^\gamma$ and $a=n$. We introduce the function $g_\varepsilon^{1-\frac{a}{\gamma'}}(x):=\min\{g^{1-\frac{a}{\gamma'}}(x/(1-\varepsilon)),g^{1-\frac{a}{\gamma'}}(x)\chi(\varepsilon x)\}$ for a cut-off function χ, for instance such that $0\le\chi\le1$, $\chi(x)=1$ if $|x|\le1/2$ and $\chi(x)=0$ is $|x|\ge1$. The argument is then identical to the one in [CNV04]: we first justify (16) for the function g_ε instead of g, then we let ε tend to 0. For this a key fact is that the sequence $\nabla g_\varepsilon^{1-\frac{a}{\gamma'}}$ is bounded in $L^{\gamma'}$. To see this fact we observe that the sequence $\nabla\Big(g^{1-\frac{a}{\gamma'}}(x/(1-\varepsilon))\Big)$ is bounded in $L^{\gamma'}$ by change of variable $y=x/(1-\varepsilon)$. So is the sequence $\nabla\Big(g^{1-\frac{a}{\gamma'}}(x)\chi(\varepsilon x)\Big)$ since

$$2^{1-\gamma'}\int\Big|\nabla\Big(g^{1-\frac{a}{\gamma'}}(x)\chi(\varepsilon x)\Big)\Big|^{\gamma'}dx\le\int\big|\nabla g^{1-\frac{a}{\gamma'}}(x)\big|^{\gamma'}|\chi(\varepsilon x)|^{\gamma'}dx+\varepsilon^{\gamma'}\int g^{\gamma'-a}(x)\,|\nabla\chi|^{\gamma'}(\varepsilon x)dx.$$

There, for the first term, $|\chi|\le1$ and $\nabla g^{1-\frac{a}{\gamma'}}\in L^{\gamma'}$ since $g^{-a}|\nabla g|^{\gamma'}\in L^1$ by assumption. Moreover by Hölder's inequality for the power $a/(a-\gamma')$ with $a>\gamma'$ and then change of variable $y=\varepsilon\,x$ the second term is bounded by

$$\varepsilon^{\gamma'(1-\frac{n}{a})}\Big(\int g^{-a}\Big)^{1-\frac{\gamma'}{a}}\Big(\int|\nabla\chi|^a\Big)^{\frac{\gamma'}{a}}$$

and hence uniformly bounded in ε for $a\ge n$.

Next, we extend the result to the case $\gamma'\ge a$ by reducing to the previous case as follows. Fix any $s>a/(a-1)$, that is $1<s':=s/(s-1)<a$. Define $W_\varepsilon(x):=Z_\varepsilon\big(W(x)+\varepsilon|x|^s\big)$ with Z_ε such that $\int W_\varepsilon^{-a}=1$. Since $s'\le\gamma'$, Hölder's inequality and the integrability of g^{-a} ensure that $g^{-a}|\nabla g|^{s'}$ is integrable. Therefore g and W_ε match the hypotheses of the previous case, so (17) gives

$$(a-1)\int(W_\varepsilon)^*(\nabla g)g^{-a}+(a-n)\int g^{1-a}\ge\int W_\varepsilon^{1-a}.$$

Note that $Z_\varepsilon\to1$ and $W_\varepsilon\to W$. The right-hand side converges to $\int W^{1-a}$ by dominated or monotone convergence. For the left-hand side, since $W_\varepsilon\ge Z_\varepsilon W$, we have $\int(W_\varepsilon)^*\big(\nabla g\big)g^{-a}\le$

$Z_\varepsilon \int W^*\big(\frac{\nabla g}{Z_\varepsilon}\big)g^{-a}$ which converges to $\int W^*\big(\nabla g\big)g^{-a}$ by dominated convergence. This gives the desired inequality (17) for W and g.

Finally, it is easily proved that equality holds in (17) when $g = W$ with W convex. In this case $\nabla\varphi(x) = x$ in the argument above. The growth condition (14) allows to perform the integration by parts $(a-1)\int(x\cdot\nabla W)W^{-a} = n\int W^{1-a}$, which means equality in (16); together with the crucial relation (7), this ensures equality in the argument above and in (15). $\triangleright$

Remark 4 • *In the proof above, we have separated the cases when $\gamma' := \gamma/(\gamma-1)$ is above or below a for technical reasons. This dichotomy will actually come back when we will study the related Gagliardo-Nirenberg inequalities.*

- *In Section 2.4 we will see how to derive inequality (15) from the new Borell-Brascamp-Lieb inequality (25) of Theorem (8).*

The companion "concave" case is as follows. The notation are those given in Section 2.1. For any nonnegative W we let $W_*(y) = \inf\limits_{W(x)>0}\{x\cdot y - W(x)\}$. Note that W_* is a negative function in our case of interest when W is a nonnegative continuous function concave on its support.

Theorem 5 (Concave inequalities) *Let $n \geqslant 1, a > 0$, and $W : \mathbb{R}^n \to [0,+\infty)$. Then for any compactly supported function $g : \mathbb{R}^n \to [0,+\infty)$ with $g^{a+1} \in W^{1,1}$ such that*

$$\int g^a = \int W^a = 1$$

we have

$$(a+1)\int(-W_*)(\nabla g)g^a - (a+n)\int g^{1+a} \geq \int W^{1+a}. \tag{18}$$

Moreover, there is equality if $g = W$, with W continuous on $\mathbb{R}^n$ and concave on its support.

Proof

$\triangleleft$ The proof follows the previous one. Let φ be Brenier's map such that $\nabla\varphi\#g^a = W^a$. Then, from (12), g^a-almost everywhere

$$W(\nabla\varphi) = g\left(\det\nabla^2\varphi\right)^{-1/a} \geq \Big(1+\frac{n}{a}\Big)g - \frac{1}{a}g\,\Delta\varphi.$$

Integrating with respect to the measure $g^a dx$ and then by parts, we find

$$\int W(\nabla\varphi)g^a \geq \Big(1+\frac{n}{a}\Big)\int g^{a+1} + \frac{a+1}{a}\int g^a\nabla g\cdot\nabla\varphi.$$

We obtain inequality (18) using the g^a-a.e. inequality

$$\nabla g\cdot\nabla\varphi \geqslant W(\nabla\varphi) + W_*(\nabla g),$$

which is valid since $W(\nabla\varphi(x))^a > 0$ for g^a-almost all x, and the fact that $\nabla\varphi\#g^a = W^a$.

When $g = W$ and is continuous and concave on its support, the proof above with $\nabla\varphi(x) = x$ gives equality at all steps. Note that integration by parts is valid because W is continuous and therefore equal to zero on $\partial\{W > 0\}$ and that we can invoque (9) in the last step. $\triangleright$

2.3 A new family of functional Brunn-Minkowski inequalities

Whereas Theorems 3 and 5 are convex or concave generalizations of Theorem 1 (which is Theorem 3 for $a = n$), we now present two generalizations of Theorem 2.

The first one concerns the convex case.

Theorem 6 (Φ-Borell-Brascamp-Lieb inequality) *Let $a \geqslant n \geqslant 1$ (and $a > 1$ if $n = 1$) and let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a concave function.*

Let also $g, W, H : \mathbb{R}^n \to [0, +\infty]$ be Borel functions and $s \in [0, 1]$, $t = 1 - s$, be such that

$$\forall x, y \in \mathbb{R}^n, \qquad H(sx + ty) \leq sg(x) + tW(y) \tag{19}$$

and $\int W^{-a} = \int g^{-a} = 1$. Then

$$\int \Phi(H)H^{-a} \geq s \int \Phi(g)g^{-a} + t \int \Phi(W)W^{-a}. \tag{20}$$

Observe that Theorem 2 is Theorem 6 in the case when $\Phi(x) = x$ and $a = n$, while the classical Borell-Brascamp-Lieb inequality (5) is recovered for $\Phi(x) \equiv 1$ and $a = n$. Roughly speaking, there is a hierarchy between all the inequalities (20), and inequality (6) (when $a = n$) appears as the strongest one.

Proof

$\lhd$ The theorem can be proved in two ways, following the ideas from R. J. McCann's or F. Barthe's PhDs [Bar97, McC94].

Let φ be Brenier's map such that $\nabla\varphi \# g^{-a} = W^{-a}$. Then from the Monge-Ampère equation (10), we have that almost everywhere

$$W(\nabla\varphi) = g \ \det(\nabla^2\varphi)^{1/a}.$$

Moreover, it follows from the assumptions that Φ is non-decreasing and $x \mapsto \frac{\Phi(x)-\Phi(0)}{x}$ is non-increasing, so that $x \mapsto \Phi(x)x^{-a}$ is non-increasing.

First proof: This proof is a little bit formal since we use a change of variables formula without proof. However, it is useful to fix the ideas, and helps to follow the rigorous proof below.

So, by the change of variable $z = sx + t\nabla\varphi(x)$, and using both assumptions on Φ we have

$$\begin{aligned}
\int \Phi(H)H^{-a} &= \int \Phi(H(sx + t\nabla\varphi(x)))H^{-a}(sx + t\nabla\varphi(x)) \det(s\mathrm{Id} + t\nabla^2\varphi(x))dx \\
&\geqslant \int \Phi(sg + tW(\nabla\varphi))\big(sg + tW(\nabla\varphi)\big)^{-a} \det(s\mathrm{Id} + t\nabla^2\varphi). \\
&\geqslant \int \big[s\Phi(g) + t\Phi(W(\nabla\varphi))\big] \Big(s + t\det(\nabla^2\varphi)^{1/a}\Big)^{-a} \det(s\mathrm{Id} + t\nabla^2\varphi)\, g^{-a}.
\end{aligned}$$

Since $a \geqslant n$, the concavity of $\det^k$ with $k = 1/a$, recalled before (11), yields

$$\det(s\mathrm{Id} + t\nabla^2\varphi) \geqslant \Big(s + t\det(\nabla^2\varphi)^{1/a}\Big)^a. \tag{21}$$

Finally $\int \Phi(W(\nabla\varphi))g^{-a} = \int \Phi(W)W^{-a}$ by image measure property since $\nabla\varphi \# g^{-a} = W^{-a}$. This concludes the argument, as

$$\int \Phi(H)H^{-a} \geqslant \int \big[s\Phi(g) + t\Phi(W(\nabla\varphi))\big]g^{-a} = s\int \Phi(g)g^{-a} + t\int \Phi(W)W^{-a}.$$

Second proof: We use the idea of R. J. McCann. From [McC94, Lem. D.1], let $(\rho_t)_{t\in[0,1]}$ be the density of the path between g^{-a} and W^{-a} defined as follows: for each t, ρ_t is the density of the image measure of ρ_0 under $s\,\mathrm{Id}+tT = \nabla\varphi_t$, where $\varphi_t(x) = s\frac{|x|^2}{2} + t\varphi(x)$, $x \in \mathbb{R}^n$. Then, using twice the associated Monge-Ampère equations (10) for $\rho_1 = \nabla\varphi\#\rho_0$ and $\rho_t = \nabla\varphi_t\#\rho_0$, together with the determinant inequality (21), we find that ρ_0-almost everywhere

$$\rho_t(\nabla\varphi_t) \leq (sg + tW(\nabla\varphi))^{-a}.$$

Multiplying the inequality by $\Phi(sg + tW(\nabla\varphi))$, then ρ_0-a.e.

$$\Phi(sg + tW(\nabla\varphi))\rho_t(\nabla\varphi_t) \leq \Phi(sg + tW(\nabla\varphi))(sg + tW(\nabla\varphi))^{-a}.$$

Hence, using that Φ is concave and $x \mapsto \Phi(x)\,x^{-a}$ non-increasing, we get that ρ_0-a.e.

$$\big[s\Phi(g) + t\Phi(W(\nabla\varphi))\big]\rho_t(\nabla\varphi_t) \leq \Phi(H(sx + \nabla\varphi))\,H(sx + \nabla\varphi)^{-a} = \Phi(H(\nabla\varphi_t(x))\,H(\nabla\varphi_t(x))^{-a}.$$

Since $\rho_t(\nabla\varphi_t(x)) > 0$ for ρ_0-almost every x, we can rewrite the previous inequality as

$$s\Phi(g(x)) + t\Phi(W(\nabla\varphi(x))) \leq \frac{\Phi(H(\nabla\varphi_t(x))\,H(\nabla\varphi_t(x))^{-a}}{\rho_t(\nabla\varphi_t(x))}1_{\rho_t(\nabla\varphi_t(x))>0}, \quad \rho_0(x) - a.e.$$

Integrating with respect to $\rho_0 = g^{-a}$ we find, using $\nabla\varphi\#g^{-a} = W^{-a}$, for the left-hand side

$$\int \big[s\Phi(g(x)) + t\Phi(W(\nabla\varphi(x)))\big]\rho_0(x)\,dx = s\int\Phi(g)g^{-a} + t\int\Phi(W)W^{-a}$$

and, using $\nabla\varphi_t\#\rho_0 = \rho_t$, for the right-hand side

$$\int\Big[\frac{\Phi(H(\nabla\varphi_t(x))\,H(\nabla\varphi_t(x))^{-a}}{\rho_t(\nabla\varphi_t(x))}1_{\rho_t(\nabla\varphi_t(x))>0}\Big]\rho_0(x)\,dx = \int_{\{\rho_t>0\}}\Phi(H)H^{-a}\,dy \leq \int\Phi(H)H^{-a}.$$

This concludes the argument. $\triangleright$

The concave inequality in Theorem 5 also has a Borell-Brascamp-Lieb formulation. We only state it for power functions Φ since the general case seems less appealing.

Theorem 7 (A concave Borell-Brascamp-Lieb inequality) *Let $n \geqslant 1$ and $a > 0$. Let also $g, W, H : \mathbb{R}^n \to [0, +\infty]$ be Borel functions and $t \in [0,1]$ and $s = 1-t$ be such that*

$$\forall x, y \in \mathbb{R}^n, \qquad H(sx + ty) \geqslant sg(x) + tW(y) \tag{22}$$

and $\int W^a = \int g^a = 1$. Then

$$\int H^{1+a} \geq s^{n+a+1}\int g^{1+a} + s^{n+a}t\int W^{1+a} + (n+a)s^{n+a}t\int g^{1+a}. \tag{23}$$

Inequality (20) is optimal in the sense that if $g = W$ and is convex, then one can exhibit a map H which depends on s such that inequality (20) is an equality. This is not the case for inequality (23) which is less powerful than (20). Nevertheless the linearization of (23), for t going to 0, becomes optimal and gives optimal Gagliardo-Nirenberg inequalities in the concave case: see Section 3.2.

Proof

◁ We start as in the proof of Theorem 6, sticking to the first formal argument for size limitation. As above, the argument can be made rigorous following McCann's argument.

Let φ be Brenier's map such that $\nabla\varphi\# g^a = W^a$. Then almost surely,

$$g^a = W(\nabla\varphi)^a \ \det(\nabla^2\varphi).$$

By assumption on φ and the concavity inequality (21) we have

$$\begin{aligned}\int H^{1+a} \ &= \int H^{1+a}(sx + t\nabla\varphi(x))\det(s\mathrm{Id} + t\nabla^2\varphi(x))dx\\ &\geqslant \int (sg + tW(\nabla\varphi))^{1+a}\det(s\mathrm{Id} + t\nabla^2\varphi)\\ &\geqslant \int (sg + tW(\nabla\varphi))^{1+a}\Big(s + t(\det\nabla^2\varphi)^{1/n}\Big)^n.\end{aligned}$$

Now we keep only the order zero and one terms in the Taylor expansion in t of both terms above:

$$(sg + tW(\nabla\varphi)^{1+a} = (sg)^{1+a}\Big(1 + \frac{t}{s}\frac{W(\nabla\varphi)}{g}\Big)^{1+a} \geqslant s^{1+a}g^{1+a} + (a+1)s^a t g^a W(\nabla\varphi);$$

$$\Big(s + t(\det\nabla^2\varphi)^{1/n}\Big)^n = s^n\left(1 + \frac{t}{s}\Big(\frac{g}{W(\nabla\varphi)}\Big)^{a/n}\right)^n \geqslant s^n + ns^{n-1}t\Big(\frac{g}{W(\nabla\varphi)}\Big)^{a/n}.$$

Hence

$$\int H^{1+a} \geqslant s^{n+a+1}\int g^{1+a} + (1+a)s^{n+a}t\int g^a W(\nabla\varphi) + ns^{n+a}t\int g^a W(\nabla\varphi)\Big(\frac{g}{W(\nabla\varphi)}\Big)^{\frac{n+a}{n}}.$$

Then in the last term we apply the inequality

$$nX^{\frac{n+a}{n}} \geqslant (n+a)X - a, \qquad X \geqslant 0$$

with $X = g/W(\nabla\varphi)$. We obtain the desired inequality. ▷

2.4 Dynamical formulation of generalized Borell-Brascamp-Lieb inequalities and derivation of Sobolev inequalities

Borell-Brascamp-Lieb inequalities admit an equivalent dynamical formulation given by the largest possible function H given g and W. For that, consider the following inf-convolution, defined for functions $W, g : \mathbb{R}^n \to (0, +\infty]$, $h \geqslant 0$ and $x \in \mathbb{R}^n$ by

$$Q_h^W(g)(x) = \begin{cases} \inf_{y\in\mathbb{R}^n}\Big\{g(y) + hW\Big(\frac{x-y}{h}\Big)\Big\} & \text{if } h > 0,\\ g(x) & \text{if } h = 0\end{cases} \tag{24}$$

or equivalently

$$Q_h^W(g)(x) = \inf_{z\in\mathbb{R}^n}\{g(x-hz) + hW(z)\}.$$

Then the constraint (19) implies that the inf-convolution

$$H(x) = sQ_{t/s}^W(g)(x/s), \qquad x \in \mathbb{R}^n$$

if the largest function H satisfying (19). From this observation, the Φ-Borell-Brascamp-Lieb inequality (20) can be rewritten as follows.

Theorem 8 (Dynamical reformulation of Φ-Borell-Brascamp-Lieb inequalities) *Let* $a \geqslant n \geqslant 1$ *(and* $a > 1$ *if* $n = 1$*) and let* $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a concave function. Let also* $g, W : \mathbb{R}^n \to [0, +\infty]$ *be Borel functions such that* $\int W^{-a} = \int g^{-a} = 1$.

Then for any $h \geqslant 0$ *the Φ-Borell-Brascamp-Lieb inequality (20) is equivalent to*

$$(1+h)^{a-n} \int \Phi\Big(\frac{1}{1+h} Q_h^W(g)\Big) Q_h^W(g)^{-a} \geq \frac{1}{1+h} \int \Phi(g) g^{-a} + \frac{h}{1+h} \int \Phi(W) W^{-a}. \qquad (25)$$

In particular, when $a = n$ *and* $\Phi(x) = x$*, the extended Borell-Brascamp-Lieb inequality (6) is equivalent to*

$$\forall h \geqslant 0, \qquad \int Q_h^W(g)^{1-n} \geq \int g^{1-n} + h \int W^{1-n}. \qquad (26)$$

Moreover equality holds in inequalities (25) and (26) when $g = W$ *and is convex.*

For the equality case, note from (24) that

$$Q_h^W(g)(x) = (1+h) W\left(\frac{x}{h+1}\right), \qquad x \in \mathbb{R}^n$$

when $g = W$ and is convex. Hence, equality holds in (25) and (26) in this case, as claimed.

Inequalities (25) and (26) are equalities when $h = 0$. Moreover, for $h \to 0$ we have in general that

$$Q_h^W g = g - h W^*(\nabla g) + o(h)$$

as explained in Appendix A, so that Theorem 8 admits a linearization as a convex inequality. With the same conditions on the function Φ as in Theorem 8, from inequality (25) we obtain

$$\int W^*(\nabla g)\left(a\frac{\Phi(g)}{g} - \Phi'(g)\right) g^{-a} + \int \big((a-n+1)\Phi(g) - g\Phi'(g)\big) g^{-a} \geqslant \int \Phi(W) W^{-a} \qquad (27)$$

for a class of functions g and W (which we do not try to carefully describe for a general Φ). Of course again inequality (27) is optimal: equality holds when $g = W$ and is convex.

In the case $\Phi(x) = x$, it is shown in Appendix A how to deduce the inequality (15) of Theorem 3 (and therefore Theorem 1) from (25) for a restricted class $\mathcal{F}^a$ of functions (g, W), inspired by [BL08] and given in Appendix A.2, Definition 25. In the case of interest of the Sobolev inequality (2) for $W(x) = C(1 + \|x\|^q/q), q = p/(p-1)$, it is shown in [BL08] how to recover the Sobolev inequality from this restricted class. For, it is classical to be sufficient to prove (2) for C^1, nonnegative and compactly functions f, and this case can be recovered by using

$$g_\varepsilon(x) = \Big(f(x) + \varepsilon(1 + \|x\|^q)^{(p-n)/p}\Big)^{p/(p-n)} + \varepsilon\left(1 + \|x\|^q\right)$$

which is in the restricted class.

Remark 9 *Likewise, the classical Borell-Brascamp-lieb inequality (5) admits the following dynamical formulation: if* $W, g : \mathbb{R}^n \to (0, +\infty)$ *are such that* $\int W^{-n} = \int g^{-n} = 1$*, then*

$$\int Q_h^W(g)^{-n} \geq 1, \qquad h \geqslant 0.$$

For h tending to 0 we recover the convexity inequality (15) with $a = n + 1$, namely

$$\int_{\mathbb{R}^n} \frac{W^*(\nabla g)}{g^{n+1}} \geqslant 0, \tag{28}$$

which had been derived in [BGG15]. As can be seen from Section 3 below, this inequality implies the Gagliardo-Nirenberg inequalities only for the parameters $a \geqslant n+1$. In particular, it does not imply the Sobolev inequality, as pointed out in [BL08], and this was a motivation to our work.

It has recently been proved in [Zug17] that the two formulations (5) and (28) are in fact equivalent.

The *concave* Borell-Brascamp-Lieb inequality (23) also admits a dynamical formulation with the sup-convolution instead of the inf-convolution. For $W, g : \mathbb{R}^n \to [0, +\infty)$ and $h \geqslant 0$ we let

$$R_h^W(g)(x) = \begin{cases} \sup_{y \in \mathbb{R}^n} \left\{ g(y) + hW\left(\frac{x-y}{h}\right) \right\} & \text{if } h > 0, \\ g(x) & \text{if } h = 0, \end{cases} \qquad x \in \mathbb{R}^n.$$

Then the constraint (22) implies that the best function H if given by the sup-convolution,

$$H(x) = sR_{t/s}^W(g)(x/s), \qquad x \in \mathbb{R}^n.$$

From this observation, the "concave" Borell-Brascamp-Lieb inequality (23) admits the equivalent following dynamical formulation: if $\int W^a dx = \int g^a dx = 1$ then for all $h \geqslant 0$,

$$\int R_h^W(g)^{1+a} \geq \int g^{1+a} + h \int W^{1+a} + (n+a)h \int g^{1+a}. \tag{29}$$

Similarly to the convex case, inequality (18) can be recovered from (29) by taking the derivative in h, at $h = 0$. We do not give more details on this computation.

3 Sharp Gagliardo-Nirenberg-Sobolev inequalities in $\mathbb{R}^n$

A family of sharp Gagliardo-Nirenberg inequalities in $\mathbb{R}^n$ was first obtained by M. del Pino and J. Dolbeault in [dD02]. The family was generalized to an arbitrary norm in [CNV04] by using the mass transport method proposed in [CE02] by the second author.

The del Pino-Dolbeault Gagliardo-Nirenberg family of inequalities, which includes the Sobolev inequality, is a consequence of Theorems 3 and 5. In this section we prove in a rather direct and easy way that our extended Borell-Brascamp-Lieb inequality (6) implies the del Pino - Dolbeault Gagliardo-Nirenberg family of inequalities, but also a new family. As recalled in the introduction, S. Bobkov and M. Ledoux [BL08] have also derived the Sobolev inequality from the Brunn-Minkowski inequality, but we believe that our method is more intuitive than theirs.

Below, $||\cdot||$ denotes an arbitrary norm in $\mathbb{R}^n$ and for $y \in \mathbb{R}^n$ we let $||y||_* = \sup_{||x|| \leq 1} x \cdot y$ its dual norm. Recall that the Legendre transform of $x \mapsto ||x||^q/q$ (with $q > 1$) is the function $y \mapsto ||y||_*^p/p$ for $1/p + 1/q = 1$.

3.1 From Theorem 3 to convex Gagliardo-Nirenberg inequalities

Let $n \geqslant 1$, $a \geqslant n$ ($a > 1$ if $n = 1$) and $q > 1$. Let W be defined by

$$W(x) = \frac{||x||^q}{q} + C, \qquad x \in \mathbb{R}^n$$

where the constant $C > 0$ is such that $\int W^{-a} = 1$. Then

$$W^*(y) = \frac{||y||_*^p}{p} - C, \qquad y \in \mathbb{R}^n$$

where $1/p + 1/q = 1$.

We apply Theorem 3 with this fixed function W. First, let us notice that C is well defined and $\int W^{1-a}$ is finite whenever

$$\begin{cases} \text{If } \ a \geqslant n+1 \ \text{ then } \ p > 1 \\ \text{If } \ a \in [n, n+1) \ \text{ then } \ 1 < p < \frac{n}{n+1-a} = \bar{p} \ \ (\bar{p} = n \ \text{ when } \ a = n). \end{cases} \tag{30}$$

These constraints are illustrated in Figure 1 in the case $n = 4$: Equation (30) is satisfied whenever the couple (a, p) is in the grey or black area.

Let us note that, under (30), the condition (14) on W in Theorem 3 is satisfied with $\gamma = q$.

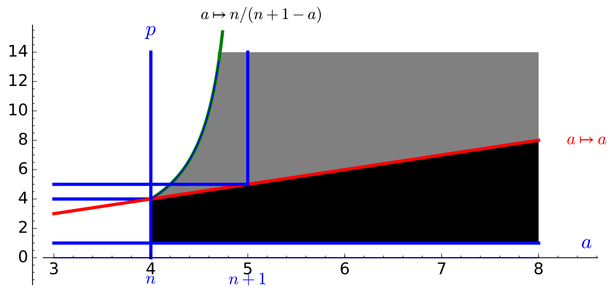

Figure 1: Ranges of admissible parameters (a, p) with $n = 4$

Assuming that the parameters a and p are in this admissible set, then for any function $g : \mathbb{R}^n \to]0, +\infty[$ such that $\int g^{-a} = 1$ and $\nabla g^{1-\frac{a}{p}} \in L^p$, inequality (15) in Theorem 3 becomes

$$D \leq \frac{a-1}{p} \int \frac{||\nabla g||_*^p}{g^a} + (a-n) \int g^{1-a}. \tag{31}$$

Here $D = (a-1)C + \int W^{1-a}$ is well defined, W and $a > 1$ being fixed. This inequality is the cornerstone of this section.

Sobolev inequalities: As a warm up, let us consider the case $a = n$, $n \geqslant 2$ and $p \in (1, n)$. Then inequality (31) becomes

$$\frac{Dp}{n-1} \leq \int \frac{||\nabla g||_*^p}{g^n}$$

for any positive function g such that $\int g^{-n} = 1$ and $\nabla g^{1-\frac{n}{p}} \in L^p$. Letting $f = g^{\frac{p-n}{p}}$, this gives

$$\frac{Dp}{n-1}\left|\frac{n-p}{p}\right|^p \leq \int ||\nabla f||_*^p$$

for any positive function f such that $\int f^{\frac{np}{n-p}} = 1$ and $\nabla f \in L^p$. Removing the normalization we get

$$\frac{Dp}{n-1}\left|\frac{n-p}{p}\right|^p \left(\int f^{\frac{np}{n-p}}\right)^{\frac{n-p}{n}} \leq \int ||\nabla f||_*^p.$$

The inequality is of course optimal since equality holds when $g = W$ or equivalently when $f(x) = \left(C + \frac{||x||^q}{q}\right)^{\frac{p-n}{p}}$. Classically removing the sign condition we recover

Theorem 10 (Sobolev inequalities) *Let $n \geqslant 2$, $p \in (1, n)$ and $p^* = np/(n-p)$. The inequality*

$$\left(\int |f|^{p^*}\right)^{\frac{1}{p^*}} \leq C_{n,p}\left(\int ||\nabla f||_*^p\right)^{\frac{1}{p}}.$$

holds for any function $f \in L^{p^*}$ with $\nabla f \in L^p$; here $C_{n,p}$ is the optimal constant reached by the function $x \mapsto (1 + ||x||^q)^{\frac{p-n}{p}}$.

Gagliardo-Nirenberg inequalities: Consider now the case $a > n$ and $p \neq a$ satisfying conditions (30). Letting $h = g^{1-\frac{a}{p}} = g^{\frac{p-a}{p}}$, inequality (31) becomes

$$1 \leq D_2 \int ||\nabla h||_*^p + (a-n)\int h^{p\frac{a-1}{a-p}}$$

for any positive function h such that $\int h^{\frac{ap}{a-p}} = 1$ and $\nabla h \in L^p$, where D_2 is an explicit positive constant. Removing the normalization, the inequality becomes

$$\left(\int h^{\frac{ap}{a-p}}\right)^{\frac{a-p}{a}} \leq D_2 \int ||\nabla h||_*^p + (a-n)\int h^{p\frac{a-1}{a-p}} \left(\int h^{\frac{ap}{a-p}}\right)^{\frac{1-p}{a}}$$

for aby positive h for which the integrals are finite.

To obtain a compact form of this inequality, we replace $h(x) = f(\lambda x)$ and optimize over $\lambda > 0$. For another explicit constant D_3 we get

$$\left(\int f^{\frac{ap}{a-p}}\right)^{\frac{a-p}{ap}\left(1-\frac{1-p}{a-p}\omega\right)} \leq D_3\left(\int ||\nabla f||_*^p\right)^{\frac{1-\omega}{p}} \left(\int f^{p\frac{a-1}{a-p}}\right)^{\frac{a-p}{p(a-1)}\frac{a-1}{a-p}\omega}$$

where $\omega = \frac{p(a-n)}{p(a-n)+n} \in (0, 1)$. There are now two cases, depending on the sign of $\left(1 - \frac{1-p}{a-p}\omega\right) = \frac{a}{a-p}\frac{(a-n-1)p+n}{p(a-n)+n}$ and $\frac{a-1}{a-p}\omega$. If $p < a$ then both coefficients are positive, as one can check by considering the cases $a < n+1$ and $a \geqslant n+1$: this leads to the first case in Theorem 11 below. If $p > a$, then under the constraints (30) both coefficients are negative: this leads to the second case below.

Removing the sign condition we have obtained:

Theorem 11 (Gagliardo-Nirenberg inequalities) *Let $n \geqslant 1$ and $a > n$.*

- *For any* $1 < p < a$*, the inequality*

$$\left(\int |f|^{\frac{ap}{a-p}}\right)^{\frac{a-p}{ap}} \leq D^{+}_{n,p,a}\left(\int ||\nabla f||_*^p\right)^{\frac{\theta}{p}}\left(\int |f|^{p\frac{a-1}{a-p}}\right)^{\frac{a-p}{p(a-1)}(1-\theta)} \tag{32}$$

holds for any function f *for which the integrals are finite. Here* $\theta \in]0,1[$ *is given by*

$$\frac{a-p}{a} = \theta\frac{n-p}{n} + (1-\theta)\frac{a-p}{a-1} \tag{33}$$

and $D^{+}_{n,p,a}$ *is the optimal constant given by the extremal function* $x \mapsto (1+||x||^q)^{\frac{p-a}{p}}$*.*

- *If* $p > a$ *when* $a \geqslant n+1$*, or if* $p \in \left(a, \frac{n}{n+1-a}\right)$ *when* $a \in [n, n+1)$*, then the inequality*

$$\left(\int |f|^{p\frac{a-1}{a-p}}\right)^{\frac{a-p}{p(a-1)}} \leq D^{-}_{n,p,a}\left(\int ||\nabla f||_*^p\right)^{\frac{\theta'}{p}}\left(\int |f|^{\frac{ap}{a-p}}\right)^{\frac{a-p}{ap}(1-\theta')} \tag{34}$$

holds for any function f *for which the integrals are finite. Here* $\theta' \in]0,1[$ *is given by*

$$\frac{p-a}{a-1} = \theta'\frac{p-n}{n} + (1-\theta')\frac{p-a}{a}$$

and $D^{-}_{n,p,a}$ *is the optimal constant given by the extremal function* $x \mapsto (1+||x||^q)^{\frac{p-a}{p}}$*.*

Remark 12
- *Inequalities* (32) *form the del Pino-Dolbeault family of Gagliardo-Nirenberg inequalities in* $\mathbb{R}^n$*. They correspond to parameters* (a,p) *in the black area in Figure 1.*
- *Inequalities* (34) *are Gagliardo-Nirenberg inequalities with a negative exponent, that is* $p\frac{a-1}{a-p} < 0$ *and* $\frac{ap}{a-p} < 0$*. To obtain such inequalities with the same optimal functions, the range of parameters* (30) *seems to be optimal. They correspond to parameters* (a,p) *in the grey area in Figure 1. Let us note that this family, with a smaller range of parameters* (a,p)*, has been obtained by V.-H. Nguyen [Ngu15, Th. 3.1 (ii)]. To our knowledge, the family* (34) *is new except for the part of the family derived by Nguyen.*
- *In [BGL14, Th. 6.10.4] it has been shown how to deduce sharp Gagliardo-Nirenberg inequalities from the Sobolev inequality, but only for the parameters* $a = n + m/2$*,* $m \in \mathbb{N}$*. The idea is to work in higher dimensions, for instance* $\mathbb{R}^{n+m}$ *with a function* $g(x,y) = (h(x)+||y||^p)^{-(n+m-2)/2}$ *and to use the scaling property of the Lebesgue measure. From inequality* (15) *of Theorem 3 we can also use higher dimensions to reach the whole family* (32) *of Gagliardo-Nirenberg inequalities. As in [BGL14], we consider* $g(x,y) = h(x)+||y||^r$ *and* $W(x,y) = ||x||^p + ||y||^r + C$ *in* $\mathbb{R}^{n+m}$ *for a parameter* $r > 1$*. The additional parameter* $r > 1$ *allows to reach the full sharp family* (32)*.*

3.2 From Theorem 5 to concave Gagliardo-Nirenberg inequalities

Let $n \geqslant 1$. Let $a > 0$ and $q > 1$, and define

$$W(x) = \frac{C}{q}(1-||x||^q)_+, \qquad x \in \mathbb{R}^n$$

where C is such that $\int W^a = 1$. From the definition (8), we have

$$W_*(y) = \begin{cases} -\frac{C^{1-p}}{p}||y||_*^p - \frac{C}{q} & \text{if } ||y||_* \le C \\ -||y||_* & \text{if } ||y||_* \geqslant C, \end{cases} \qquad y \in \mathbb{R}^n$$

where $1/p + 1/q = 1$. In particular from the Young inequality

$$W_*(y) \geqslant -\frac{C^{1-p}}{p}||y||_*^p - \frac{C}{q}, \qquad y \in \mathbb{R}^n. \tag{35}$$

Then the inequality (18) with this function W gives

$$(a+n)\int g^{1+a} \le (a+1)\frac{C^{1-p}}{p}\int ||\nabla g||_*^p g^a + \frac{C}{q}(a+1) - \int W^{a+1}$$

for any nonnegative and compactly supported function g such that $\int g^a = 1$ and $g^{1+a} \in W^{1,1}$.

Let us notice that $\frac{C}{q}(a+1) - \int W^{a+1}dx > 0$. Letting now $f = g^{\frac{a+p}{p}}$ we obtain

$$\int f^{p\frac{1+a}{a+p}} \le D_1 \int ||\nabla f||_*^p + D_2,$$

for any nonnegative and compactly supported function f such that $\int f^{\frac{ap}{a+p}} = 1$ and $f^{p\frac{1+a}{a+p}} \in W^{1,1}$; here D_1 and D_2 are explicit constants. Removing the normalization, this gives

$$\int f^{p\frac{1+a}{a+p}} \le D_1 \int ||\nabla f||_*^p \left(\int f^{\frac{ap}{a+p}}\right)^{\frac{1-p}{a}} + D_2\Big(\int h^{\frac{ap}{a+p}}\Big)^{\frac{1+a}{a}}.$$

We can now remove the sign condition and optimize by scaling to recover the following result of [dD02] (and [CNV04] for an arbitrary norm).

Theorem 13 (Concave Gagliardo-Nirenberg inequalities) *Let $n \geqslant 1$. For any $p > 1$ and $a > 0$ the inequality*

$$\left(\int |f|^{p\frac{a+1}{a+p}}\right)^{\frac{a+p}{p(a+1)}} \le D_{n,p,a}\Big(\int ||\nabla f||_*^p\Big)^{\frac{\theta}{p}}\left(\int |f|^{\frac{ap}{a+p}}\right)^{\frac{a+p}{ap}(1-\theta)}$$

holds for any compactly supported function f with $\nabla f \in L^p$. Here $\theta \in]0,1[$ is given by

$$\frac{a+p}{a+1} = \theta\frac{n-p}{n} + (1-\theta)\frac{a+p}{a}$$

and $D_{n,p,a}$ is the optimal constant given by the extremal function $x \mapsto (1-||x||^q)_+^{\frac{a+p}{p}}$.

The obtained inequality is optimal since (18) is an equality when $g = W$. Moreover, when $g = W$, then almost surely $||\nabla g||_* \le C$, so that (35) is an equality.

4 Sharp trace Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{R}^n_+$

Our aim here is to explain how our framework allows to recover known and obtain new trace Sobolev and Gagliardo-Nirenberg inequalities on $\mathbb{R}^n_+$, in sharp form. In the above denomination we shall restrict to the convex case. As before, we have two possible, equivalent, routes. One is to establish an abstract convex Sobolev type inequality using mass transport, and the other one is to establish a new functional Brunn-Minkowski type inequality on $\mathbb{R}^n_+$, and derive Sobolev inequalities from it, by linearisation. Since the second one is formally more general (although it requires technical, non-essential, assumptions on the functions), we will favour it.

Let us fix some notation. For any $n \geqslant 2$, we let

$$\mathbb{R}^n_+ = \{z = (u, x),\ u \geqslant 0, x \in \mathbb{R}^{n-1}\}.$$

Then $\partial\mathbb{R}^n_+ = \{(0, x),\ x \in \mathbb{R}^{n-1}\} = \mathbb{R}^{n-1}$. For $e = (1, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $h \in \mathbb{R}$ we let

$$\mathbb{R}^n_{+he} = \mathbb{R}^n_+ + he = \{(u, x),\ u \geqslant h, x \in \mathbb{R}^{n-1}\}.$$

The Borell-Brascamp-Lieb inequality (20) with $\Phi(x) = x$ takes the following form in $\mathbb{R}^n_+$.

Theorem 14 (Trace Borell-Brascamp-Lieb inequality) *Let $a \geqslant n$, $g : \mathbb{R}^n_+ \to (0, +\infty)$ and $W : \mathbb{R}^n_{+e} \to (0, +\infty)$ such that $\int_{\mathbb{R}^n_+} g^{-a} = \int_{\mathbb{R}^n_{+e}} W^{-a} = 1$. Then, for all $h > 0$,*

$$(1 + h)^{a-n} \int_{\mathbb{R}^n_{+he}} Q_h^W(g)^{1-a} \geqslant \int_{\mathbb{R}^n_+} g^{1-a} + h \int_{\mathbb{R}^n_{+e}} W^{1-a} \tag{36}$$

where, for $(u, x) \in \mathbb{R}^n_{+he}$,

$$Q_h^W(g)(u, x) = \inf_{(v,y)\in\mathbb{R}^n_+,\, 0\leq v\leq u-h} \left\{ g(v, y) + hW\left(\frac{u - v}{h}, \frac{x - y}{h}\right) \right\}.$$

Moreover (36) is an equality when $g(z) = W(z + e)$ for any $z \in \mathbb{R}^n_+$ and is convex.

Proof
◁ Let $\tilde{g} : \mathbb{R}^n \to (0, +\infty]$ and $\tilde{W} : \mathbb{R}^n \to (0, +\infty]$ be defined by

$$\tilde{g}(x) = \begin{cases} g(x) \text{ if } x \in \mathbb{R}^n_+ \\ +\infty \text{ if } x \notin \mathbb{R}^n_+ \end{cases} \quad \text{and} \quad \tilde{W}(x) = \begin{cases} W(x) \text{ if } x \in \mathbb{R}^n_{+e} \\ +\infty \text{ if } x \notin \mathbb{R}^n_{+e}. \end{cases} \tag{37}$$

Then $\int_{\mathbb{R}^n} \tilde{g}^{-a} = \int_{\mathbb{R}^n} \tilde{W}^{-a} = 1$. Hence, we can apply the dynamical formulation (25) of Theorem 6 with $\Phi(x) = x$ and the functions $\tilde{g}$, $\tilde{W}$. For any $h \geqslant 0$ we obtain

$$(1 + h)^{a-n} \int_{\mathbb{R}^n} Q_h^{\tilde{W}}(\tilde{g})^{1-a} \geq \int_{\mathbb{R}^n_+} g^{1-a} + h \int_{\mathbb{R}^n_{+e}} W^{1-a},$$

where

$$Q_h^{\tilde{W}}(\tilde{g})(u, x) = \inf_{(v,y)\in\mathbb{R}^n} \left\{ \tilde{g}(v, y) + h\tilde{W}\left(\frac{u - v}{h}, \frac{x - y}{h}\right) \right\}, \qquad (u, x) \in \mathbb{R}^n.$$

From the definition of $\tilde{g}$ and $\tilde{W}$, the infimum can be restricted to $0 \leq v \leq u - h$, so that $Q_h^{\tilde{W}}(\tilde{g})(u,x)$ is equal to $+\infty$ when $u < h$, and to $Q_h^W(g)(x)$ otherwise. It implies

$$\int_{\mathbb{R}^n} Q_h^{\tilde{W}}(\tilde{g})^{1-a} = \int_{\mathbb{R}^n_{+he}} Q_h^{\tilde{W}}(\tilde{g})^{1-a} = \int_{\mathbb{R}^n_{+he}} Q_h^W(g)^{1-a},$$

which gives inequality (36).

When $g(z) = W(z+e)$ and W is convex, then by convexity

$$Q_h^W(g)(u,x) = (h+1)W\left(\frac{u+1}{h+1}, \frac{x}{h+1}\right)$$

for any $(u,x) \in \mathbb{R}^n_{+he}$. Then inequality (36) is an equality. $\triangleright$

As observed in Section 2.4, a Borell-Brascamp-Lieb type inequality on $\mathbb{R}^n$ implies a convex inequality. It is also the case on $\mathbb{R}^n_+$, by computing the derivative of (36) at $h = 0$ and using the identity

$$\int_{\mathbb{R}^n_{+he}} Q_h^W(g)^{1-a} = \int_h^\infty \int_{\mathbb{R}^{n-1}} Q_h^W(g)^{1-a}(u,x)dudx.$$

Assume now that (g, W) is in $\mathcal{F}_+^a$ as in Definition 21. Then by Theorem 24 in the appendix,

$$\frac{d}{dh}\Big|_{h=0} \int_h^\infty \int_{\mathbb{R}^{n-1}} Q_h^W(g)^{1-a}(u,x)dudx = -\int_{\partial\mathbb{R}^n_+} g^{1-a}dx + (a-1)\int_{\mathbb{R}^n_+} \frac{W^*(\nabla g)}{g^a}dz,$$

where we recall the definition of the Legendre transform:

$$W^*(y) = \inf_{x \in \mathbb{R}^n_{+e}} \{x \cdot y - W(x)\}, \qquad y \in \mathbb{R}^n. \tag{38}$$

So we have obtained:

Proposition 15 (Trace convex inequality) *Let $a \geqslant n$. Let $g : \mathbb{R}^n_+ \to (0, +\infty)$ and $W : \mathbb{R}^n_{+e} \to (0, +\infty)$ belong to $\mathcal{F}_+^a$ (see Definition 21 of Section A.1) with W convex and $\int_{\mathbb{R}^n_+} g^{-a} = \int_{\mathbb{R}^n_{+e}} W^{-a} = 1$. Then*

$$(a-1)\int_{\mathbb{R}^n_+} \frac{W^*(\nabla g)}{g^a} + (a-n)\int_{\mathbb{R}^n_+} g^{1-a} \geqslant \int_{\mathbb{R}^n_{+e}} W^{1-a}dz + \int_{\partial\mathbb{R}^n_+} g^{1-a}. \tag{39}$$

Moreover (39) is an equality when $g(z) = W(z+e)$ for $z \in \mathbb{R}^n_+$ and is convex.

Remark 16 *Inequality (39) can also be directly proved by mass transport and integration by parts.*

We follow Section 3 to get trace Gagliardo-Nirenberg inequalities from Proposition 15.

Let $a \geqslant n, p \in (1,n)$ and $q = p/(p-1)$. Let also $W(z) = C\frac{||z||^q}{q}$ for $z \in \mathbb{R}^n_{+e}$, where the constant $C > 0$ is such that $\int_{\mathbb{R}^n_{+e}} W^{-a} = 1$. We first observe that Conditions *(C1)-(C2)* of Definition 21 hold with $\gamma = q$ since $q > n/(a-1)$ for $a \geqslant n$. Moreover, for $y \in \mathbb{R}^n$,

$$W^*(y) = \sup_{x \in \mathbb{R}^n_{+e}} \left\{x \cdot y - C\frac{||x||^q}{q}\right\} \leq \sup_{x \in \mathbb{R}^n} \left\{x \cdot y - C\frac{||x||^q}{q}\right\} = C^{1-p}\frac{||y||_*^p}{p}. \tag{40}$$

Hence Proposition 15 implies

$$C^{1-p}\frac{a-1}{p}\int_{\mathbb{R}^n_+}\frac{||\nabla g||_*^p}{g^a}+(a-n)\int_{\mathbb{R}^n_+}g^{1-a}\geqslant\int_{\mathbb{R}^n_{+e}}W^{1-a}+\int_{\partial\mathbb{R}^n_+}g^{1-a} \tag{41}$$

for any function g satisfying $\int_{\mathbb{R}^n_+}g^{-a}=1$ and *(C3)-(C4)* with $\gamma=q$, so that (g,W) belongs to $\mathcal{F}^a_+$.

It has to be mentioned that inequality (41) is still optimal, despite inequality (40). For, when $g(x)=W(x+e)$ for $x\in\mathbb{R}^n_+$, then the minimum in (38) at the point $\nabla g(x)$ is reached in $\mathbb{R}^n_{+e}$ and then (40) is an equality.

Inequality (41) is again the cornerstone of this section.

Trace Sobolev inequalities: Again, as a warm up, let us assume that $a=n$. Then (41) gives

$$\int_{\partial\mathbb{R}^n_+}g^{1-n}\leq C^{1-p}\,\frac{n-1}{p}\int_{\mathbb{R}^n_+}\frac{||\nabla g||_*^p}{g^n}-\int_{\mathbb{R}^n_{+e}}W^{1-n}$$

for any function g satisfying $\int_{\mathbb{R}^n_+}g^{-n}=1$ and *(C3)-(C4)* for $\gamma=q$. For $f=g^{\frac{p-n}{p}}$, so that $\int_{\mathbb{R}^n_+}f^{\frac{pn}{n-p}}=1$, this inequality becomes

$$\int_{\partial\mathbb{R}^n_+}f^{\frac{p(n-1)}{n-p}}\leq C^{1-p}\,\frac{n-1}{p}\left(\frac{p}{n-p}\right)^p\int_{\mathbb{R}^n_+}||\nabla f||_*^p-\int_{\mathbb{R}^n_{+e}}W^{1-n}. \tag{42}$$

We now need to extend this inequality to all $\mathcal{C}^1$ and compactly supported functions f in $\mathbb{R}^n_+$ (it does not mean that f vanishes in $\partial\mathbb{R}^n_+$). For this, consider a $\mathcal{C}^1$ and compactly supported function f in $\mathbb{R}^n_+$ and let

$$f_\varepsilon(x)=\varepsilon|x+e|^{-\frac{n-p}{p-1}}+c_\varepsilon f(x),$$

where c_ε is such that $\int_{\mathbb{R}^n_+}f_\varepsilon^{\frac{pn}{n-p}}=1$. Then $g_\varepsilon=f_\varepsilon^{-\frac{p}{n-p}}$ satisfies *(C3)* and *(C4)*. Moreover $c_\varepsilon\to 1$ when ε goes to 0 and then inequality (42) holds for the function f.

Removing the normalization in (42) we have, for any f,

$$\int_{\partial\mathbb{R}^n_+}f^{\tilde p}dx\leq A\int_{\mathbb{R}^n_+}||\nabla f||_*^p dz\,\beta^{\tilde p-p}-B\beta^{\tilde p},$$

where

$$\tilde p=\frac{p(n-1)}{n-p},\quad \beta=\left(\int_{\mathbb{R}^n_+}f^{\frac{pn}{n-p}}dz\right)^{\frac{n-p}{np}},\quad A=C^{1-p}\frac{n-1}{p}\left(\frac{p}{n-p}\right)^p\ \text{ and }\ B=\int_{\mathbb{R}^n_{+e}}W^{1-n}dz.$$

Equivalently, with $u=\frac{\tilde p}{p}=\frac{n-1}{n-p}$ and $v=\frac{\tilde p}{\tilde p-p}$ (which satisfy $u,v>1$ and $1/u+1/v=1$) ,

$$\int_{\partial\mathbb{R}^n_+}f^{\tilde p}\leq Bv\left[\frac{A}{Bv}\int_{\mathbb{R}^n_+}||\nabla f||_*^p\,\beta^{\tilde p-p}-\frac{1}{v}\beta^{\tilde p}\right].$$

Now the Young inequality $xy\leq x^u/u+y^v/v$ with

$$x=\frac{A}{Bv}\int_{\mathbb{R}^n_+}||\nabla f||_*^p\quad\text{and}\quad y=\beta^{\tilde p-p}$$

yields

$$\left(\int_{\partial\mathbb{R}^n_+} f^{\tilde{p}}\right)^{1/\tilde{p}} \leq \frac{A^{1/p}}{(Bv)^{\frac{p-1}{p(n-1)}}}\left(\frac{n-p}{n-1}\right)^{\frac{n-p}{p(n-1)}}\left(\int_{\mathbb{R}^n_+} ||\nabla f||_*^p\right)^{1/p}.$$

The proof of optimality it is a little bit technical and will be given below in the more general case of Theorem 18. It is also given in [Naz06]. Equality holds when $g(z) = W(z+e)$ or equivalently when $f(z) = \left(C\frac{||z+e||^q}{q}\right)^{-\frac{n-p}{p}} = C'||z+e||^{-\frac{n-p}{p-1}}$ for $z \in \mathbb{R}^n_+$. Removing the sign condition we have thus obtained the following result by B. Nazaret [Naz06], who promoted the idea of adding a vector e to the map W. We have derived it for $\mathcal{C}^1$ and compactly supported functions, but by approximation it is possible to extends it to the appreciate space.

Theorem 17 (Trace Sobolev inequalities from [Naz06]) *For any $1 < p < n$ and for $\tilde{p} = p(n-1)/(n-p)$ the Sobolev inequality*

$$\left(\int_{\partial\mathbb{R}^n_+} |f|^{\tilde{p}} dx\right)^{1/\tilde{p}} \leq D_{n,p}\left(\int_{\mathbb{R}^n_+} ||\nabla f||_*^p dz\right)^{1/p}$$

holds for any $\mathcal{C}^1$ and compactly supported function f on $\mathbb{R}^n_+$. Here $D_{n,p}$ is the optimal constant given by the extremal function

$$h_p(z) = ||z+e||^{-\frac{n-p}{p-1}}, \quad z \in \mathbb{R}^n_+.$$

Trace Gagliardo-Nirenberg inequalities: Assume now that $a \geqslant n > p > 1$ and let $h = g^{\frac{p-a}{p}}$. Then the inequality (41) can be written as

$$\int_{\partial\mathbb{R}^n_+} h^{p\frac{a-1}{a-p}} dx \leq C^{1-p}\frac{a-1}{p}\left(\frac{p}{a-p}\right)^p \int_{\mathbb{R}^n_+} ||\nabla h||_*^p dz + (a-n)\int_{\mathbb{R}^n_+} h^{p\frac{a-1}{a-p}} dz - \int_{\mathbb{R}^n_{+e}} W^{1-a} dz$$

for any $\mathcal{C}^1$ and compactly supported function h in $\mathbb{R}^n_+$ such that $\int_{\mathbb{R}^n_+} h^{\frac{ap}{a-p}} = 1$. In this case we use the same argument as for the Sobolev inequality above to replace the conditions *(C3)* and *(C4)* of Definition 21.

Removing the normalization, we get

$$\int_{\partial\mathbb{R}^n_+} h^{p\frac{a-1}{a-p}} \leq C^{1-p}\frac{a-1}{p}\left(\frac{p}{a-p}\right)^p \int_{\mathbb{R}^n_+} ||\nabla h||_*^p \,\beta^{p\frac{p-1}{a-p}} - \int_{\mathbb{R}^n_{+e}} W^{1-a}\,\beta^{p\frac{a-1}{a-p}} + (a-n)\int_{\mathbb{R}^n_+} h^{p\frac{a-1}{a-p}} \tag{43}$$

with now

$$\beta = \left(\int_{\mathbb{R}^n_+} h^{\frac{pa}{a-p}} dz\right)^{\frac{a-p}{ap}}.$$

Let $u = \frac{a-1}{a-p}$ and $v = \frac{a-1}{p-1}$, which satisfy $u, v > 1$ and $1/u + 1/v = 1$. As for the Sobolev inequality we rewrite the right-hand side of (43) as

$$C^{1-p}\frac{a-1}{p}\left(\frac{p}{a-p}\right)^p \int_{\mathbb{R}^n_+} ||\nabla h||_*^p \,\beta^{p\frac{p-1}{a-p}} - \int_{\mathbb{R}^n_{+e}} W^{1-a}\,\beta^{p\frac{a-1}{a-p}}$$

$$= Bv\left[\frac{A}{Bv}\int_{\mathbb{R}^n_+} ||\nabla h||_*^p \,\beta^{p\frac{p-1}{a-p}} - \frac{1}{v}\beta^{p\frac{a-1}{a-p}}\right],$$

with

$$A = C^{1-p}\frac{a-1}{p}\left(\frac{p}{a-p}\right)^p \text{ and } B = \int_{\mathbb{R}^n_{+e}} W^{1-a}.$$

From the Young inequality applied to the parameters u, v and

$$x = \frac{A}{Bv}\int_{\mathbb{R}^n_+} ||\nabla h||_*^p \text{ and } y = \beta^{p\frac{p-1}{a-p}} \tag{44}$$

we get

$$C^{1-p}\frac{a-1}{p}\left(\frac{p}{a-p}\right)^p \int_{\mathbb{R}^n_+} ||\nabla h||_*^p \,\beta^{p\frac{p-1}{a-p}} - \int_{\mathbb{R}^n_{+e}} W^{1-a}\,\beta^{p\frac{a-1}{a-p}} \leq \frac{A^{\frac{a-1}{a-p}}}{(Bv)^{\frac{p-1}{a-p}}}\frac{a-p}{a-1}\left(\int_{\mathbb{R}^n_+} ||\nabla h||_*^p\right)^{\frac{a-1}{a-p}} \tag{45}$$

and then

$$\int_{\partial\mathbb{R}^n_+} h^{p\frac{a-1}{a-p}}dx \leq \frac{A^{\frac{a-1}{a-p}}}{(Bv)^{\frac{p-1}{a-p}}}\frac{a-p}{a-1}\left(\int_{\mathbb{R}^n_+} ||\nabla h||_*^p dz\right)^{\frac{a-1}{a-p}} + (a-n)\int_{\mathbb{R}^n_+} h^{p\frac{a-1}{a-p}}dz \tag{46}$$

from (43). For any $\lambda > 0$, we replace $h(z) = f(\lambda z)$ for $z \in \mathbb{R}^n_+$. We obtain

$$\int_{\partial\mathbb{R}^n_+} f^{p\frac{a-1}{a-p}}dx \leq \lambda^{\frac{(a-n)(p-1)}{a-p}}\frac{A^{\frac{a-1}{a-p}}}{(Bv)^{\frac{p-1}{a-p}}}\frac{a-p}{a-1}\left(\int_{\mathbb{R}^n_+} ||\nabla f||_*^p dz\right)^{\frac{a-1}{a-p}} + \lambda^{-1}(a-n)\int_{\mathbb{R}^n_+} f^{p\frac{a-1}{a-p}}dz.$$

Taking the infimum over $\lambda > 0$ gives

$$\left(\int_{\partial\mathbb{R}^n_+} f^{p\frac{a-1}{a-p}}dx\right)^{\frac{a-p}{p(a-1)}} \leq D\left(\int_{\mathbb{R}^n_+} ||\nabla f||_*^p dz\right)^{\frac{\theta}{p}}\left(\int_{\mathbb{R}^n_+} f^{p\frac{a-1}{a-p}}dz\right)^{(1-\theta)\frac{a-p}{p(a-1)}}.$$

for an explicit constant D and $\theta \in]0,1]$ being the unique parameter satisfying

$$\frac{n-1}{n}\frac{a-p}{a-1} = \theta\frac{n-p}{n} + (1-\theta)\frac{a-p}{a-1}. \tag{47}$$

Removing the sign condition, we have obtained:

Theorem 18 (Trace Gagliardo-Nirenberg inequalities) *For any $a \geqslant n > p > 1$, the Gagliardo-Nirenberg inequality*

$$\left(\int_{\partial\mathbb{R}^n_+} |f|^{p\frac{a-1}{a-p}}dx\right)^{\frac{a-p}{p(a-1)}} \leq D_{n,p,a}\left(\int_{\mathbb{R}^n_+} ||\nabla f||_*^p dz\right)^{\frac{\theta}{p}}\left(\int_{\mathbb{R}^n_+} f^{p\frac{a-1}{a-p}}\right)^{(1-\theta)\frac{a-p}{p(a-1)}} \tag{48}$$

holds for any $\mathcal{C}^1$ and compactly supported function f on $\mathbb{R}^n_+$. Here θ is defined in (47) and $D_{n,p,a}$ is the optimal constant, reached when

$$f(z) = h_p(z) = ||z+e||^{-\frac{a-p}{p-1}}, \quad z \in \mathbb{R}^n_+.$$

When $a = n$, then $\theta = 1$ and we recover the trace Sobolev inequality of Theorem 17.

Proof

$\lhd$ From the above computation we only have to prove that the inequality (48) is optimal. First, it follows from Proposition 15 that inequality (43) is an equality when

$$\forall z \in \mathbb{R}^n_+, \;\; h(z) = h_p(z) = ||z + e||^{-\frac{a-p}{p-1}},$$

the function h_p not needing to be normalized. Moreover, if inequality (45) is an equality, then it is also the case for (46) and then (48). So, we only have to prove that (45) is an equality when $h = h_p$, which sums up to the fact that the Young inequality is an equality. This is the case when $x = y^{v-1}$ in (44), that is,

$$\frac{A}{Bv} \int_{\mathbb{R}^n_+} ||\nabla h||_*^p dz = \left(\beta^{p\frac{p-1}{a-p}}\right)^{v-1},$$

or equivalently

$$\frac{A}{Bv} \int_{\mathbb{R}^n_+} ||\nabla h||_*^p dz = \left(\int_{\mathbb{R}^n_+} ||z+e||^{-\frac{ap}{p-1}}\right)^{\frac{a-p}{a}}.$$

Let now $\mathcal{I}_\alpha = \int_{\mathbb{R}^n_+} ||z+e||^{-\alpha} dz$ for $\alpha > 0$. Then

$$C = \frac{p}{p-1} \mathcal{I}^{1/a}_{\frac{ap}{p-1}}, \quad B = \mathcal{I}^{\frac{1-a}{a}}_{\frac{ap}{p-1}} \mathcal{I}_{p\frac{a-1}{p-1}} \quad \text{and} \quad \int_{\mathbb{R}^n_+} ||\nabla h||_*^p dz = \left(\frac{a-p}{p-1}\right)^p \mathcal{I}_{p\frac{a-1}{p-1}}$$

from their respective definition. Then, from the definition of A, equality in the Young inequality indeed holds. This finally gives equality for the map h. It has to be mentioned that the case $a = n$ gives the optimality of the trace Sobolev inequality of Theorem 17. $\rhd$

Remark 19

- *In a first version of this work we conjectured that the function h_p is the only optimal function up to dilatation and translation. Since then, in [Ngu17] V.-H. Nguyen has answered affirmatively to this conjecture.*
- *As for the Gagliardo-Nirenberg in $\mathbb{R}^n$, the inequality (48) can be proved by using inequality (39) with $a = n$ in higher dimension, as proposed in Remark 12.*
- *Trace Gagliardo-Nirenberg inequalities on convex cones have recently been derived in [Zug17].*

5 Remarks on the logarithmic Sobolev inequality

In their work [dD03] on Gagliardo-Nirenberg inequalities (where only the Euclidean norm is considered), M. del Pino and J. Dolbeault observed that when the parameter a goes to infinity the sharp Gagliardo-Nirenberg inequality (32) in $\mathbb{R}^n$ yields the L^p-Euclidean logarithmic Sobolev inequality

$$\mathrm{Ent}_{dx}(f^p) \le \frac{n}{p} \int_{\mathbb{R}^n} f^p dx \; \log\left(\mathcal{L}_p \frac{\int ||\nabla f||_*^p dx}{\int f^p dx}\right) \tag{49}$$

for any positive function f. Here $1/p + 1/q = 1$, $\mathcal{L}_p$ is the optimal constant attained for $f(x) = e^{-||x||^q}$ and

$$\mathrm{Ent}_{dx}(f^p) := \int_{\mathbb{R}^n} f^p \log \frac{f^p}{\int f^p} dx.$$

This bound is an instance, when $V(x) = ||x||^q + C$, of the following general inequality of [Gen03, Gen08]: for any $V, f : \mathbb{R}^n \to (0, +\infty)$ such that $\int e^{-f} = \int e^{-V} = 1$, there holds

$$\int_{\mathbb{R}^n} (f + V^*(\nabla f))e^{-f} \geqslant n, \tag{50}$$

with equality when $f = V$ and is convex. Inequality (50) has been derived in [Gen03, Gen08] from the Prékopa-Leindler inequality, which is a consequence of the classical Borell-Brascamp-Lieb inequality (5). It says that for $F, V, f : \mathbb{R}^n \to \mathbb{R}$ and $u \in [0, 1]$ such that $\int e^{-f} = \int e^{-V} = 1$ and

$$\forall x, y \in \mathbb{R}^n, \qquad F((1-u)x + uy) \leq (1-u)f(x) + uV(y), \tag{51}$$

then

$$\int_{\mathbb{R}^n} e^{-F} \geq 1. \tag{52}$$

As above for the Borell-Brascamp-Lieb inequalities, (52) can be rewritten in a dynamical form:

$$\int_{\mathbb{R}^n} e^{-\frac{1}{1+h}Q_h^V(f)} \geq (1+h)^n, \qquad h \geqslant 0. \tag{53}$$

Then, as for above inequalities, this formulation can be linearized as $h \to 0$, recovering (50).

Our new Borell-Brascamp-Lieb inequality (20) also yields the Prékopa-Leindler inequality (51)-(52) for $\Phi = 1$ and a going to infinity. For that, it suffices to apply (20) with $g = Z_g^{-1/a}(1 + f/a), W = Z_V^{-1/a}(1 + V/a)$ for $Z_g = \int(1 + g/a)^{-a}$ and $Z_V = \int(1 + V/a)^{-a}$, $s = uZ_g^{1/a}/(uZ_g^{1/a} + (1-u)Z_V^{1/a})$ and $H = (1 + F/a)/(uZ_g^{1/a} + (1-u)Z_V^{1/a})$, and then to let a go to infinity.

In the derivation of (49) from the sharp Gagliardo-Nirenberg inequality, the argument in [dD03] is based on the key fact that the exponent θ in equation (33) goes to 0 when $a \to +\infty$. In the case of the half-space $\mathbb{R}^n_+$, the exponent θ in equation (47) goes to $1/p$ when $a \to +\infty$: hence this method does not seem to adapt easily to the $\mathbb{R}^n_+$ case. Hence, to get a trace logarithmic Sobolev inequality in $\mathbb{R}^n_+$ we rather resort to the argument of Section 4, as follows.

Let then $W : \mathbb{R}^n_{+e} \to \mathbb{R}$ and $g : \mathbb{R}^n_+ \to \mathbb{R}$ such that $\int_{\mathbb{R}^n_{+e}} e^{-W} = \int_{\mathbb{R}^n_+} e^{-g} = 1$, and define $\tilde{W}$ and $\tilde{g}$ as in (37). Then by (53)

$$\int_{\mathbb{R}^n} e^{-\frac{1}{1+h}Q_h^{\tilde{W}}(\tilde{g})}(z)dz = \int_h^\infty \int_{\mathbb{R}^{n-1}} e^{-\frac{1}{1+h}Q_h^W(g)(u,x)}dudx \geqslant (1+h)^n.$$

In the limit $h \to 0$ we get the bound corresponding to (50) in the trace case, namely

$$\int_{\mathbb{R}^n_+} (g + W^*(\nabla g))e^{-g} \geqslant n + \int_{\partial\mathbb{R}^n_+} e^{-g} \tag{54}$$

whenever the function g is in a appropriate set of functions. We will not give here more details.

As in Section 4 again, let now $q > 1$, $|| \cdot ||$ be a norm in $\mathbb{R}^n$, and let $W(z) = C||z||^q/q$ for $z \in \mathbb{R}^n_{+e}$, where $C = \big(\int_{\mathbb{R}^n_+} e^{-\frac{||x||^q}{q}}dx\big)^{q/n}$ is such that $\int_{\mathbb{R}^n_{+e}} e^{-W} = 1$. Then $W^*(y) \leq C^{1-p}||y||_*^p/p$ for $y \in \mathbb{R}^n_+$, with $1/p + 1/q = 1$. Let then f be a positive function on $\mathbb{R}^n_+$ such that $\int_{\mathbb{R}^n_+} f^p = 1$, and apply inequality (54) to $g = -p\log f$. After removing the normalization we obtain

$$\mathrm{Ent}_{dx}(f^p) \leq \left(\frac{C}{p}\right)^{1-p} \int_{\mathbb{R}^n_+} ||\nabla f||_*^p dx - n\int_{\mathbb{R}^n_+} f^p dx - \int_{\partial\mathbb{R}^n_+} f^p dx. \tag{55}$$

Inequality (55) is a trace logarithmic Sobolev inequality. It does not have a compact expression as does inequality (49) in the case of $\mathbb{R}^n$, where the scaling optimization can be performed. Nevertheless, in $\mathbb{R}^n_+$, it improves upon the usual (49) if we consider functions on $\mathbb{R}^n_+$.

A Time derivative of the infimum-convolution

The time derivative of the Hopf-Lax formula (24) has been treated in different contexts, namely for Lipschitz (as in [Eva98]) or bounded (as in [Vil09]) initial data. In our case the function g grows as $|x|^p$ with $p > 1$ at infinity and thus these classical results can not be applied. We will thus follow the method proposed by S. Bobkov and M. Ledoux [BL08], extending it to more general functions W and also to the half-space $\mathbb{R}^n_+$.

We give all the details for the half-space $\mathbb{R}^n_+$ which are more intricate.

A.1 The $\mathbb{R}^n_+$ case

Let $a \geqslant n$ and let $g : \mathbb{R}^n_+ \to (0, +\infty)$, $W : \mathbb{R}^n_{+e} \to (0, +\infty)$ such that $\int_{\mathbb{R}^n_+} g^{-a}$ and $\int_{\mathbb{R}^n_{+e}} W^{-a}$ are finite. The functions g and W are assumed to be $\mathcal{C}^1$ in the interior of their respective domain of definition. Moreover we assume that W goes to infinity faster that linearly:

$$\lim_{x\in\mathbb{R}^n_{+e}, |x|\to\infty} \frac{W(x)}{|x|} = +\infty. \tag{56}$$

Our objective is to give sufficient conditions such that the derivative at $h = 0$ of the function

$$\mathbb{R}^+ \ni h \mapsto \int_h^\infty \int_{\mathbb{R}^{n-1}} Q_h^W(g)^{1-a}(u, x)dudx$$

is equal to

$$-\int_{\partial\mathbb{R}^n_+} g^{1-a}dz + (a-1)\int_{\mathbb{R}^n_+} \frac{W^*(\nabla g)}{g^a}dz$$

where

$$W^*(y) = \sup_{x\in\mathbb{R}^n_{+e}} \{x \cdot y - W(x)\}, \qquad y \in \mathbb{R}^n. \tag{57}$$

For this, let us first recall the definition of $Q_h^W g$: for $x \in \mathbb{R}^n_{+he}$,

$$Q_h^W g(x) = \begin{cases} \inf_{y\in\mathbb{R}^n_+,\, x-y\in\mathbb{R}^n_{+he}} \left[g(y) + hW\left(\frac{x-y}{h}\right)\right] & \text{if } h > 0, \\ g(x) & \text{if } h = 0. \end{cases}$$

or equivalently, for $h > 0$ and $x \in \mathbb{R}^n_{+he}$,

$$Q_h^W g(x) = \inf_{z\in\mathbb{R}^n_{+e},\, x-hz\in\mathbb{R}^n_+} \{g(x - hz) + hW(z)\} = \inf_{z\in\mathbb{R}^n_{+he},\, x-z\in\mathbb{R}^n_+} \left\{g(x - z) + hW\left(\frac{z}{h}\right)\right\}$$

First, we have

Lemma 20 *In the above notation and assumptions, for all* $x \in \overset{\circ}{\mathbb{R}^n_+}$

$$\frac{\partial}{\partial h}\Big|_{h=0} Q_h^W g(x) = -W^*(\nabla g(x)). \tag{58}$$

Proof

◁ We follow and adapt the proof proposed in [BL08]. Let $x \in \overset{\circ}{\mathbb{R}^n_+}$ be fixed.
By definition of $Q_h^W g$, for any $z \in \mathbb{R}^n_{+e}$ and $h > 0$ small enough so that $x - hz \in \mathbb{R}^n_+$, one has

$$\frac{Q_h^W g(x) - g(x)}{h} \le \frac{g(x-hz) - g(x)}{h} + W(z).$$

Since g is $\mathcal{C}^1$, then for all $z \in \mathbb{R}^n_{+e}$

$$\limsup_{h \to 0} \frac{Q_h^W g(x) - g(x)}{h} \le -\nabla g(x) \cdot z + W(z).$$

Then, from the definition (57) of W^*,

$$\limsup_{h \to 0} \frac{Q_h^W g(x) - g(x)}{h} \le -W^*(\nabla g(x)).$$

We now prove the converse inequality. Let

$$A_{x,h} = \{z \in \mathbb{R}^n_{+e},\ hW(z) \le g(x - he) + hW(e)\}.$$

For a small enough $h > 0$ such that $x - he \in \mathbb{R}^n_+$ we have $Q_h^W g(x) \le g(x-he) + hW(e)$, so

$$Q_h^W g(x) = \inf_{z \in A_{x,h},\, x - hz \in \mathbb{R}^n_+} \{g(x-hz) + hW(z)\}.$$

Hence

$$\begin{aligned} \frac{Q_h^W g(x) - g(x)}{h} &= \inf_{z \in A_{x,h},\, x-hz \in \mathbb{R}^n_+} \left\{ \frac{g(x-hz) - g(x)}{h} + W(z) \right\} \\ &= \inf_{z \in A_{x,h},\, x-hz \in \mathbb{R}^n_+} \left\{ -\nabla g(x) \cdot z + z\varepsilon_x(hz) + W(z) \right\}, \end{aligned}$$

where $\varepsilon_x(hz) \to 0$ when $hz \to 0$. It implies

$$\frac{Q_h^W g(x) - g(x)}{h} \geqslant \inf_{z \in A_{x,h}} \left\{ -\nabla g(x) \cdot z + z\varepsilon_x(hz) + W(z) \right\}.$$

By the coercivity condition (56) on W and since g is locally bounded, the set $A_{x,h}$ is bounded by a constant C, uniformly in $h \in (0,1)$. In particular for every $\eta > 0$, there exists $h_\eta > 0$ such that for all $h \le h_\eta$ and $z \in A_{x,h}$, $|\varepsilon_x(hz)| \le \eta$. Moreover, for all $h \le h_\eta$,

$$\begin{aligned} \frac{Q_h^W g(x) - g(x)}{h} \geqslant \inf_{z \in A_{x,h}} \{-\nabla g(x) \cdot z + W(z)\} - C\eta &\geqslant \inf_{z \in \mathbb{R}^n_{+e}} \{-\nabla g(x) \cdot z + W(z)\} - C\eta \\ &= -W^*(\nabla g(x)) - C\eta. \end{aligned}$$

Let us take the limit when h goes to 0,

$$\liminf_{h \to 0} \frac{Q_h^W g(x) - g(x)}{h} \geqslant -W^*(\nabla g(x)) - C\eta.$$

As η is arbitrary, we finally get equality (58). ▷

Our assumptions on the couple (g, W) are summarized in the following definition.

Definition 21 (the set $\mathcal{F}_+^a$ of admissible couples in $\mathbb{R}_+^n$) *Let $n \geqslant 2$, $g : \mathbb{R}_+^n \to (0, +\infty)$ and $W : \mathbb{R}_{+e}^n \to (0, +\infty)$. We say that the couple (g, W) belongs to $\mathcal{F}_+^a$ with $a \geqslant n$ if the following four conditions are satisfied for some γ:*

(C1) $\gamma > \max\{\frac{n}{a-1}, 1\}$.

(C2) There exists a constant $A > 0$ such that $W(x) \geqslant A|x|^\gamma$ for all $x \in \mathbb{R}_{+e}^n$.

(C3) There exists a constant $B > 0$ such that $|\nabla g(x)| \leq B(|x|^{\gamma-1} + 1)$ for all $x \in \mathbb{R}_+^n$.

(C4) There exists a constant C such that $C(|x|^\gamma + 1) \leq g(x)$ for all $x \in \mathbb{R}_+^n$.

In the following, we let C_j denote several constants which are independent of $h > 0$ and $x \in \mathbb{R}_{+he}^n$, but may depend on γ, A, B.

Lemma 22 *Assume (C1)∼(C4). Then, we find a constant $h_1 > 0$ such that, for all $h \in (0, h_1)$ and $x \in \mathbb{R}_{+he}^n$*

$$-C_1 h(1 + |x|^\gamma) \leq Q_h^W g(x) - g(x) \leq C_2 h(|x|^{\gamma-1} + 1).$$

Proof

◁ **1.** Let us first consider the easier upper bound. For any $h > 0$ and $x \in \mathbb{R}_{+he}^n$ then $x - he \in \mathbb{R}_+^n$, so that

$$Q_h^W g(x) - g(x) \leq g(x - he) - g(x) + hW(e).$$

On the other hand, for any $x \in \mathbb{R}_+^n$ and $y \in \mathbb{R}^n$ such that $x + y \in \mathbb{R}_+^n$ we have from *(C3)*,

$$\begin{aligned}&|g(x+y) - g(x)| \\ &\quad = \left|\int_0^1 \nabla g(x + \theta y) \cdot y d\theta\right| \leq |y| \int_0^1 |\nabla g(x + \theta y)| d\theta \leq C_3 |y|(|x|^{\gamma-1} + |y|^{\gamma-1} + 1). \quad (59)\end{aligned}$$

From this remark applied to $y = -he$ with $h \in (0, 1)$, one gets for any $x \in \mathbb{R}_{+he}^n$

$$Q_h^W g(x) - g(x) \leq C_4 h(|x|^{\gamma-1} + 1) + hW(e) \leq C_5 h(|x|^{\gamma-1} + 1). \quad (60)$$

2. For the lower bound, we first need some preparation. Thus, fix $h \in (0, 1)$ and $x \in \mathbb{R}_{+he}^n$ arbitrarily. Let $\hat{y} \in \mathbb{R}_{+he}^n$ be a minimizer of the infimum convolution

$$Q_h^W g(x) = \inf_{y \in \mathbb{R}_{+he}^n} \left[g(x - y) + hW\left(\frac{y}{h}\right)\right] = g(x - \hat{y}) + hW\left(\frac{\hat{y}}{h}\right).$$

Such a $\hat{y}$ surely exists by *(C2)* and *(C4)*. From (60) and *(C2)* we have (recall that $h < 1$),

$$\frac{A}{h^{\gamma-1}} |\hat{y}|^\gamma \leq hW\left(\frac{\hat{y}}{h}\right) \leq g(x) - g(x - \hat{y}) + C_5(|x|^{\gamma-1} + 1). \quad (61)$$

From inequality (59),

$$|g(x) - g(x - \hat{y})| \leq C_6 |\hat{y}| \left[|x|^{\gamma-1} + |\hat{y}|^{\gamma-1} + 1\right]. \quad (62)$$

From (62) and (61),

$$\frac{A}{h^{\gamma-1}} |\hat{y}|^\gamma \leq C_6 |\hat{y}|(|x|^{\gamma-1} + |\hat{y}|^{\gamma-1} + 1) + C_5(|x|^{\gamma-1} + 1)$$

Choose a small constant $0 < h_1 \leq 1$ so that

$$1 < \frac{A}{h_1^{\gamma-1}} - C_6.$$

When $0 < h < h_1$, we have

$$\frac{|\hat{y}|^\gamma}{|\hat{y}|+1} \leq C_7 \left[1 + |x|^{\gamma-1}\right]$$

so that

$$|\hat{y}| \leq C_8 \left(1 + |x|\right).$$

3. Then, fix $h \in (0, h_1)$ and $x \in \mathbb{R}^n_{+he}$ arbitrarily, where h_1 is the constant defined in step 2. By the arguments in step 2, we see that

$$Q_h^W g(x) - g(x) = \inf_{y \in \mathbb{R}^n_{+he},\, x-y \in \mathbb{R}^n_+,\, |y| \leq C_8(1+|x|)} \left[g(x-y) - g(x) + hW\left(\frac{y}{h}\right)\right]. \tag{63}$$

As in (59), we have

$$g(x) - g(x-y) \leq |y| \int_0^1 |\nabla g(x - \theta y)| d\theta. \tag{64}$$

When $|y| \leq C_8(1+|x|)$ and $0 < \theta < 1$, we have $|x - \theta y| \leq (1+C_8)(1+|x|)$, so that $|\nabla g(x-\theta y)| \leq C_9(1+|x|^{\gamma-1})$ by *(C3)*, uniformly in $0 < \theta < 1$. Thus, when $|y| \leq C_8(1+|x|)$, we have, by (64),

$$g(x) - g(x-y) \leq C_9(1 + |x|^{\gamma-1})\,|y|.$$

Hence, by (63) and *(C1)*, we obtain

$$\begin{aligned}
Q_h^W g(x) - g(x) &\geqslant \inf_{y \in \mathbb{R}^n_{+he},\, |y| \leq C_8(1+|x|)} \left[-C_9(1+|x|^{\gamma-1})\,|y| + hW\left(\frac{y}{h}\right)\right] \\
&\geqslant \inf_{y \in \mathbb{R}^n_{+he},\, |y| \leq C_8(1+|x|)} \left[-C_9(1+|x|^{\gamma-1})\,|y| + \frac{A}{h^{\gamma-1}}|y|^\gamma\right] \\
&\geqslant \inf_{y \in \mathbb{R}^n} \left[-C_9(1+|x|^{\gamma-1})\,|y| + \frac{A}{h^{\gamma-1}}|y|^\gamma\right] \\
&= -C_{10} h (1+|x|^{\gamma-1})^{\frac{\gamma}{\gamma-1}}.
\end{aligned}$$

The last equality is a direct computation. Therefore, we conclude that

$$Q_h^W g(x) - g(x) \geqslant -C_{11} h (1 + |x|^\gamma).$$

The proof is complete. $\triangleright$

Lemma 23 *Assume (C1)∼(C4). Then, we find constants $C_0, h_2 > 0$ such that for all $h \in (0, h_2)$ and $x \in \mathbb{R}^n_{+he}$*

$$\left|\frac{Q_h^W g(x)^{1-a} - g(x)^{1-a}}{h}\right| \leq \frac{C_0}{1 + |x|^{\gamma(a-1)}}.$$

Proof

$\triangleleft$ First, for any $\alpha, \beta > 0$ and $a > 1$, then

$$|\alpha^{1-a} - \beta^{1-a}| \leq (a-1)|\alpha - \beta|(\alpha^{-a} + \beta^{-a}). \tag{65}$$

Indeed, if for instance $\beta > \alpha > 0$, then for some $\theta \in (\alpha, \beta)$ we have

$$\alpha^{1-a} - \beta^{1-a} = (a-1)(\beta - \alpha)\theta^{-a} \le (a-1)(\beta-\alpha)\alpha^{-a}.$$

By inequality (65) and Lemma 22, we have

$$\begin{aligned} \left|\frac{Q_h^W g(x)^{1-a} - g(x)^{1-a}}{h}\right| &\le (a-1)\left|\frac{Q_h^W g(x) - g(x)}{h}\right| \left[Q_h^W g(x)^{-a} + g(x)^{-a}\right] \\ &\le K_1(1+|x|^\gamma)\left[Q_h^W g(x)^{-a} + g(x)^{-a}\right] \end{aligned}$$

for all $h \in (0, h_1)$ and $x \in \mathbb{R}^n_{+he}$.

On the other hand, by *(C4)* and Lemma 22, we have for all $h \in (0, h_1)$ and $x \in \mathbb{R}^n_{+he}$

$$Q_h^W g(x) \geqslant g(x) - C_1 h(1+|x|^\gamma) \geqslant (C - C_1 h)(|x|^\gamma + 1).$$

Choose a small constant h_3 so that

$$\frac{C}{2} \le C - C_1 h_3.$$

and let $h_2 \min\{h_1, h_3\}$. Then, for all

$$Q_h^W g(x) \geqslant \frac{C}{2}(|x|^\gamma + 1)$$

whence, again using *(C4)*,

$$\left|\frac{Q_h^W g(x)^{1-a} - (g(x))^{1-a}}{h}\right| \le C_2(1+|x|^\gamma)^{1-a}$$

for all $h \in (0, h_2)$ and $x \in \mathbb{R}^n_{+he}$. $\triangleright$

We can now state and prove the main result of this section:

Theorem 24 *In the above notation, assume that the couple (g, W) is in $\mathcal{F}^a_+$. Then*

$$\frac{d}{dh}\Big|_{h=0} \int_h^\infty \int_{\mathbb{R}^{n-1}} Q_h^W(g)^{1-a}(u,x)\,du\,dx = -\int_{\partial\mathbb{R}^n_+} g^{1-a}dx + (a-1)\int_{\mathbb{R}^n_+} \frac{W^*(\nabla g)}{g^a}dz.$$

Proof

$\triangleleft$ One can write the h-derivative as follows:

$$\begin{aligned} &\frac{1}{h}\left(\int_h^\infty \int_{\mathbb{R}^{n-1}} Q_h^W(g)^{1-a}(u,x)\,du\,dx - \int_{\mathbb{R}^n_+} g^{1-a}(u,x)\,du\,dx\right) \\ &\qquad = \frac{1}{h}\left(\int_h^\infty \int_{\mathbb{R}^{n-1}} g^{1-a}(u,x)\,du\,dx - \int_{\mathbb{R}^n_+} g^{1-a}(u,x)\,du\,dx\right) \\ &\qquad\quad + \frac{1}{h}\left(\int_h^\infty \int_{\mathbb{R}^{n-1}} Q_h^W(g)^{1-a}(u,x)\,du\,dx - \int_h^\infty \int_{\mathbb{R}^{n-1}} g^{1-a}(u,x)\,du\,dx\right). \end{aligned}$$

First

$$\frac{1}{h}\left(\int_h^\infty \int_{\mathbb{R}^{n-1}} g^{1-a}(u,x)dudx - \int_{\mathbb{R}^n_+} g^{1-a}(u,x)dudx\right) = -\frac{1}{h}\int_0^h \int_{\mathbb{R}^{n-1}} g^{1-a}(u,x)dudx,$$

which goes to $-\int_{\mathbb{R}^{n-1}} g^{1-a}(0,x)dx = -\int_{\partial\mathbb{R}^n_+} g^{1-a}$ when h goes to 0. Secondly,

$$\frac{1}{h}\left(\int_h^\infty \int_{\mathbb{R}^{n-1}} Q_h^W(g)^{1-a}(u,x)dudx - \int_h^\infty \int_{\mathbb{R}^{n-1}} g^{1-a}(u,x)dudx\right)$$
$$= \int_{\mathbb{R}^n_+} \left[\frac{Q_h^W(g)^{1-a}(u,x) - g^{1-a}(u,x)}{h}\right] 1_{u\geqslant h} dudx. \quad (66)$$

By Lemma 20 the function in the right-hand side of (66) converges pointwisely to $W^*(\nabla g)g^{-a}$ as $h \to 0$. Moreover, since $\gamma(a-1) > n$, by Lemma 23 it is bounded uniformly in h by an integrable function. Hence by the Lebesgue dominated convergence Theorem the left-hand-side of (66) converges (when $h \to 0$) to

$$(a-1)\int_{\mathbb{R}^n_+} W^*(\nabla g)g^{-a}.$$

The proof is complete. $\triangleright$

A.2 The $\mathbb{R}^n$ case

We only give the result and conditions for the $\mathbb{R}^n$ case.

We let $g, W : \mathbb{R}^n \to (0,+\infty)$ such that g is $\mathcal{C}^1$ and $\int g^{-n} = \int W^{-n} = 1$.

Definition 25 (the set $\mathcal{F}^a$ of admissible couples in $\mathbb{R}^n$) *Let $g : \mathbb{R}^n \to (0,+\infty)$ and $W : \mathbb{R}^n \to (0,+\infty)$. We say that the couple (g,W) belongs to $\mathcal{F}^n$ with $a \geqslant n$ ($a > 1$ if $n = 1$) if the following four conditions are satisfied for some γ:*

(C1) $\gamma > \max\{\frac{n}{a-1}, 1\}$.

(C2) There exists a constant $A > 0$ such that $W(x) \geqslant A|x|^\gamma$ for all $x \in \mathbb{R}^n$.

(C3) There exists a constant $B > 0$ such that $|\nabla g(x)| \leq B(|x|^{\gamma-1} + 1)$ for all $x \in \mathbb{R}^n$.

(C4) There exist a constant C such that $C(|x|^\gamma + 1) \leq g(x)$ for all $x \in \mathbb{R}^n$.

Theorem 26 *Assume that the couple (g,W) is in $\mathcal{F}^a$. Then, the derivative at $h = 0$ of the map*

$$(0,+\infty) \ni h \mapsto \int Q_h^W(g)^{1-a}$$

is equal to

$$(1-a)\int \frac{W^*(\nabla g)}{g^a}.$$

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