

ON PAIRWISE SENSITIVITY

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Abstract We define the concept of pairwise sensitivity with respect to initial conditions for the endomorphisms on Lebesgue metric spaces. The idea is that the orbits of almost every pair of nearby initial points (for the product of the invariant measure) of a pairwise sensitive map may diverge from a positive quantity independent of the initial points. We study the relationships between pairwise sensitivity and weak mixing, pairwise sensitivity and positiveness of metric entropy and we compute the largest sensitivity constant.

Index Terms Sensitive dependence on initial conditions, Measure-preserving transformation, Ergodicity, Mixing, Metric entropy.

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1. Introduction

The concept of *sensitive dependence on initial conditions* has attracted much attention in recent years and several authors have tried to formalize it in various ways. The phrase -sensitive dependence on initial conditions- was first used by Ruelle (1978), to indicate some exponential rate of divergence of orbits of nearby points. More generally, it captures the idea that a very small change in the initial condition can cause a big change in the trajectory. Following the pioneer work by Guckenheimer (1979), Devaney (1989) called sensitive a self-map $T : X \rightarrow X$ on the metric space (X, d) satisfying the property : there exists $\delta > 0$ such that for all $x \in X$ and all $\varepsilon > 0$ there is some $y \in X$ which is within a distance ε of x and for some $n \geq 0$, $d(T^n x, T^n y) \geq \delta$. In the last years, several authors proposed sufficient conditions both on T and (X, d) to ensure the sensitivity property (*cf.* Abraham

et al, 2002, 2004, Banks *et al*, 1992, Glasner and Weiss, 1993, Guckenheimer, 1979).

However, sensitive dependence on initial conditions was first defined in Chaos Theory to measure the divergence of orbits of nearby points, by analogy with the *butterfly effect* described by the meteorologist Ed Lorenz (for an overview in Chaos Theory, we refer the reader to the book by Devaney, 1989). From this point of view, we think that the dissymmetric feature of the above definition of sensitivity should be avoided. In order to follow the sensitivity idea drawn by the butterfly effect, one could say that T is sensitive if there exists $\delta > 0$ such that for all x, y in X , one can find $n \geq 0$ with $d(T^n x, T^n y) \geq \delta$. However, this property appears to be too strong because it is never satisfied by the non injective maps, such as the archetype of a chaotic map, namely the quadratic one $Tx = 4x(1 - x)$ on $X = [0, 1]$.

As an attempt to weaken the previous definition of sensitivity, we make use of tools from Ergodic Theory. From now on, we consider an endomorphism T on a probability Lebesgue space (X, \mathcal{B}, μ) (*cf.* Petersen, 1983, page 16) and we fix a metric d on X . For simplicity, we assume throughout that the support of μ , denoted $\text{supp } \mu$, is not reduced to a single point.

Our sensitivity property described below is easily shown to be stronger than Guckenheimer's one.

Definition *The endomorphism T is said to be pairwise sensitive (with respect to initial conditions) if there exists $\delta > 0$ -a sensitivity constant- such that for $\mu^{\otimes 2}$ -a.e. $(x, y) \in X^2$, one can find $n \geq 0$ with $d(T^n x, T^n y) \geq \delta$.*

Equivalently, T is pairwise sensitive if there exists $\delta > 0$ with

$$\mu^{\otimes 2} \left(\bigcap_{n \geq 0} \overline{T}^{-n} \mathcal{A}_\delta \right) = 0,$$

where, here and in the following, $\overline{T} = T \times T$ is the map on X^2 defined by $\overline{T} : (x, y) \mapsto (Tx, Ty)$ and, for any $r > 0$, \mathcal{A}_r stands for the set :

$$\mathcal{A}_r := \{(x, y) \in X^2 : d(x, y) < r\}.$$

We note that this sensitivity property is the exact measure theoretic equivalent of the concept studied in Akin and Kolyada (2003) in a topological

dynamic context. (See also the article by Blanchard *et al* (2002a) on the abundance of Li-Yorke pairs.)

Section 2 is devoted to the computation of the sensitivity constant and to the case where T is weakly mixing. The case where T is of positive metric entropy is studied in Section 3.

2. Pairwise sensitivity, weak mixing and the sensitivity constant

Observe that if δ is a sensitivity constant for T , then so is any positive $\delta' \leq \delta$. This leads to consider the following quantity, denoted $\Delta(T)$:

$$\Delta(T) = \sup \{ \delta : \delta \text{ is a sensitivity constant for } T \}.$$

From now on, $\text{diam}(A)$ stands for the diameter of $A \subset X$ and for all $z \in X$, $r > 0$, $B(z, r)$ is the open ball :

$$B(z, r) = \{x \in X : d(z, x) < r\}.$$

Theorem 2.1 *Assume that T is pairwise sensitive. Then,*

- (i) *There exists $\delta > 0$ such that for $\mu^{\otimes 2}$ -a.e. $(x, y) \in X^2$, one can find a sequence $(n_k)_{k \geq 0}$ with $d(T^{n_k}x, T^{n_k}y) \geq \delta$ for all $k \geq 0$;*
- (ii) *For $\mu^{\otimes 2}$ -a.e. $(x, y) \in X^2$, one has $\sup_{n \geq 0} d(T^n x, T^n y) \geq \Delta(T)$;*
- (iii) $\Delta(T) \leq \text{diam}(\text{supp } \mu)$.

Notice that, since the support of μ is not reduced to a single point, there exists $\delta > 0$ with $\mu^{\otimes 2}(\mathcal{A}_\delta) < 1$. Hence, the quantity

$$a(\mu) := \sup \{ \delta : \mu^{\otimes 2}(\mathcal{A}_\delta) < 1 \}$$

is positive.

Lemma 2.1 *One has*

$$a(\mu) = \text{diam}(\text{supp } \mu).$$

Proof First notice that $a(\mu) \leq D := \text{diam}(\text{supp } \mu)$ because for all $\varepsilon > 0$, $\mu^{\otimes 2}(\mathcal{A}_{D+\varepsilon}) = 1$. Moreover, let for all $\varepsilon > 0$,

$$F_\varepsilon = \left\{ (x, y) \in \text{supp } \mu^{\otimes 2} : d(x, y) \leq D - \frac{\varepsilon}{2} \right\},$$

where $\text{supp } \mu^{\otimes 2}$ denotes the support of $\mu^{\otimes 2}$. Then, $\mu^{\otimes 2}(F_\varepsilon) < 1$ because F_ε is a closed set and

$$F_\varepsilon \subsetneq \text{supp } \mu^{\otimes 2} = \text{supp } \mu \times \text{supp } \mu.$$

Since $\mathcal{A}_{D-\varepsilon} \subset F_\varepsilon$, one deduces that $\mu^{\otimes 2}(\mathcal{A}_{D-\varepsilon}) < 1$ and hence, that $a(\mu) \geq D$. \square

Proof of Theorem 2.1 (i) It is a straightforward consequence of Halmos Recurrence Theorem (cf. Petersen, 1983, page 39).

(ii) If T is pairwise sensitive, then for all $\varepsilon > 0$ small enough :

$$\mu^{\otimes 2}\left(\bigcap_{n \geq 0} \bar{T}^{-n} \mathcal{A}_{\Delta(T)-\varepsilon}\right) = 0.$$

We get from a monotonicity argument that :

$$\mu^{\otimes 2}\left(\bigcup_{\varepsilon > 0} \bigcap_{n \geq 0} \bar{T}^{-n} \mathcal{A}_{\Delta(T)-\varepsilon}\right) = \lim_{\varepsilon \searrow 0} \mu^{\otimes 2}\left(\bigcap_{n \geq 0} \bar{T}^{-n} \mathcal{A}_{\Delta(T)-\varepsilon}\right) = 0,$$

hence Assertion (ii), because

$$\bigcup_{\varepsilon > 0} \bigcap_{n \geq 0} \bar{T}^{-n} \mathcal{A}_{\Delta(T)-\varepsilon} = \left\{ (x, y) \in X^2 : \sup_{n \geq 0} d(T^n x, T^n y) < \Delta(T) \right\}.$$

(iii) Assume that $a(\mu) < \Delta(T)$. For any $r \in]a(\mu), \Delta(T)[$, one has simultaneously :

$$\mu^{\otimes 2}(\mathcal{A}_r) = 1 \text{ and } \mu^{\otimes 2}\left(\bigcap_{n \geq 0} \bar{T}^{-n} \mathcal{A}_r\right) = 0,$$

which is a contradiction. Therefore, $a(\mu) \geq \Delta(T)$ and (iii) is now straightforward from Lemma 2.1. \square

Theorem 2.2 Assume that T is weakly mixing. Then, T is pairwise sensitive and moreover :

$$\Delta(T) = \text{diam}(\text{supp } \mu).$$

Remarks

- Ergodicity is not strong enough in order to ensure the pairwise sensitivity property. Indeed, consider the case $X = \mathbb{R}/\mathbb{Z}$, μ the Haar-Lebesgue measure and d the natural metric on X . The self-map T defined by $Tx = x + \theta \pmod{1}$, where θ is an irrational number, being an isometry for d , can not be pairwise sensitive. However, it is known to be ergodic.
- In the case of a Guckenheimer's type definition of sensitivity, Abraham *et al* (2002, 2004) also provide some bounds for the largest sensitivity constant.
- Thus, for the classical dynamical systems such as r -adic maps, tent map or quadratic map, one has $\Delta(T) = 1$. This means that the orbits of almost all pairs of nearby initial points ultimately move away one from another as far as possible.

Proof of Theorem 2.2 Let $\delta < \text{diam}(\text{supp } \mu)$. Since \bar{T} is an ergodic endomorphism on $(X^2, \mathcal{B} \otimes \mathcal{B}, \mu^{\otimes 2})$ (*cf.* Petersen, 1983, page 65) and

$$\bigcap_{n \geq 0} \bar{T}^{-n} \mathcal{A}_\delta$$

is a \bar{T} -invariant set, one has

$$\mu^{\otimes 2} \left(\bigcap_{n \geq 0} \bar{T}^{-n} \mathcal{A}_\delta \right) = 0,$$

because $\mu^{\otimes 2}(\mathcal{A}_\delta^d) < 1$ by Lemma 2.1. Hence, T is pairwise sensitive and $\Delta(T) \geq \text{diam}(\text{supp } \mu)$. Apply now Theorem 2.1 (*iii*), and the theorem is proved. \square

Observe now that for $\mu^{\otimes 2}$ -a.e. $(x, y) \in X^2$, one has

$$\sup_{n \geq 0} d(T^n x, T^n y) \leq \text{diam}(\text{supp } \mu).$$

Corollary 2.1 below is then a straightforward consequence of Theorem 2.1 (*ii*) and Theorem 2.2.

Corollary 2.1 *If T is weakly mixing, then for $\mu^{\otimes 2}$ -a.e. $(x, y) \in X^2$:*

$$\sup_{n \geq 0} d(T^n x, T^n y) = \text{diam}(\text{supp } \mu).$$

3. Pairwise sensitivity and metric entropy

For any measurable countable partition α of X , we denote by $h(T, \alpha)$ the metric entropy of the transformation T with respect to the partition α (cf. Petersen, 1983, Chapter 5). Whatever the definition of sensitivity, it is expected that positiveness of metric entropy implies the sensitivity property (cf. Glasner and Weiss, 1993, Blanchard *et al*, 2002b, and Abraham *et al*, 2004, in which positiveness of the Lyapunov exponent is considered). Theorem 4.1 below gives an answer to this problem in the case of pairwise sensitivity.

Recall that a measurable set A in X is a μ -continuity set if its boundary ∂A satisfies $\mu(\partial A) = 0$ (see Billingsley, 1968, page 11).

Theorem 3.1 *Assume that T is ergodic and consider a finite measurable partition $\alpha = \{P_1, \dots, P_l\}$ of X . If P_1, \dots, P_l are μ -continuity sets for d and if $h(T, \alpha) > 0$, then T is pairwise sensitive.*

A very similar conclusion is obtained in Blanchard *et al* (2002b), but these authors consider the case where T is a homeomorphism on a compact space.

Proof Without loss of generality, we can assume that $h(T, \alpha) < \infty$. For all $D \in \mathcal{B}$ and $\varepsilon > 0$, denote by $D^{-\varepsilon}$ the internal ε -boundary of D :

$$D^{-\varepsilon} = \{x \in D : d(x, D^c) < \varepsilon\},$$

and moreover :

$$K_\varepsilon = \exp\left(2l \sum_{i=1}^l \mu(P_i^{-\varepsilon})\right).$$

Since the P_i 's are μ -continuity sets, $K_\varepsilon \rightarrow 1$ as $\varepsilon \rightarrow 0$. Hence, one can choose $\delta > 0$ such that

$$K_\delta 2^{-h(T, \alpha)/2} < 1. \quad (4.1)$$

The map

$$x \mapsto \mu\left(\bigcap_{n \geq 0} T^{-n} B(T^n x, \delta)\right)$$

defined on X is T -invariant and moreover, according to the Fubini Theorem,

$$\mu^{\otimes 2}\left(\bigcap_{n \geq 0} \overline{T}^{-n} \mathcal{A}_\delta\right) = \int_X \mu\left(\bigcap_{n \geq 0} T^{-n} B(T^n x, \delta)\right) \mu(dx).$$

Consequently, by ergodicity of T , we have for μ -a.e. $x \in X$:

$$\mu^{\otimes 2} \left(\bigcap_{n \geq 0} \bar{T}^{-n} \mathcal{A}_\delta \right) = \mu \left(\bigcap_{n \geq 0} T^{-n} B(T^n x, \delta) \right). \quad (4.2)$$

We deduce from the von Neumann Ergodic Theorem that for μ -a.e. $x \in X$:

$$\frac{1}{n} \text{card} \left(k \in \{0, \dots, n\} : T^k x \in \bigcup_{i=1}^l P_i^{-\delta} \right) \rightarrow \sum_{i=1}^l \mu(P_i^{-\delta}). \quad (4.3)$$

We now fix a point $x \in X$ which satisfies both (4.2) and (4.3). It is associated with it a sequence $(i_n)_{n \geq 0} \in \{1, \dots, l\}^{\mathbb{N}}$ such that $T^n x \in P_{i_n}$ for all $n \geq 0$. For all $n \geq 0$, we let :

$$\mathcal{Q}_n = \{k \in \{0, \dots, n\} : B(T^k x, \delta) \not\subseteq P_{i_k}\}.$$

For all $n \geq 0$,

$$\begin{aligned} \text{card } \mathcal{Q}_n &\leq \text{card} \left(k \in \{0, \dots, n\} : T^k x \in P_{i_k}^{-\delta} \right) \\ &\leq \text{card} \left(k \in \{0, \dots, n\} : T^k x \in \bigcup_{i=1}^l P_i^{-\delta} \right) \end{aligned}$$

By (4.3), there exists $N_1 \geq 0$ such that for all $n \geq N_1$:

$$\text{card } \mathcal{Q}_n \leq 2n \sum_{i=1}^l \mu(P_i^{-\delta}).$$

For all $n \geq 0$, we let :

$$\mathcal{S}_n = \left\{ (s_k)_{k=0, \dots, n} : s_k \in \{1, \dots, l\} \text{ if } k \in \mathcal{Q}_n \text{ and } s_k = i_k \text{ if } k \notin \mathcal{Q}_n \right\}.$$

This set satisfies, for all $n \geq N_1$:

$$\text{card } \mathcal{S}_n = \exp \left(\text{card } \mathcal{Q}_n \log l \right) \leq \exp \left(2nl \sum_{i=1}^l \mu(P_i^{-\delta}) \right) = K_\delta^n. \quad (4.4)$$

Now denote, for $n \geq 0$ and $s \in \mathcal{S}_n$:

$$L_{n,s} = \bigcap_{k=0}^n T^{-k} P_{s_k}.$$

For all $n \geq 0$, we have the following inclusions :

$$\begin{aligned}
\bigcap_{k=0}^n T^{-k} B(T^k x, \delta) &= \bigcap_{k \in \mathcal{Q}_n^c \cup \mathcal{Q}_n} T^{-k} B(T^k x, \delta) \\
&\subset \bigcap_{k \in \mathcal{Q}_n^c} T^{-k} P_{i_k} \\
&\subset \bigcup_{s \in \mathcal{S}_n} L_{n,s}. \quad (4.5)
\end{aligned}$$

Now fix $\varepsilon \in]0, h(T, \alpha)/2[$. By the Entropy Equipartition Property (*cf.* Petersen, 1983, page 263), there exists $N_2 \geq 0$ such that for all $n \geq N_2$, the elements of $\{1, \dots, l\}^{n+1}$ can be divided into two disjoint classes, \mathcal{G}_n and \mathcal{B}_n , such that :

$$\mu\left(\bigcup_{s \in \mathcal{B}_n} L_{n,s}\right) \leq \varepsilon,$$

and, for all $s \in \mathcal{G}_n$:

$$\mu(L_{n,s}) \leq 2^{-n(h(T, \alpha) - \varepsilon)}.$$

We then deduce from (4.4) and (4.5) that for all $n \geq \max(N_1, N_2)$:

$$\begin{aligned}
\mu\left(\bigcap_{k=0}^n T^{-k} B(T^k x, \delta)\right) &\leq \mu\left(\bigcup_{s \in \mathcal{S}_n \cap \mathcal{B}_n} L_{n,s}\right) + \sum_{s \in \mathcal{S}_n \cap \mathcal{G}_n} \mu(L_{n,s}) \\
&\leq \varepsilon + \text{card } \mathcal{S}_n \max_{s \in \mathcal{G}_n} \mu(L_{n,s}) \\
&\leq \varepsilon + K_\delta^n 2^{-n(h(T, \alpha) - \varepsilon)} \\
&\leq \varepsilon + K_\delta^n 2^{-nh(T, \alpha)/2},
\end{aligned}$$

where the latter inequality comes from the fact that $\varepsilon < h(T, \alpha)/2$. Letting $n \rightarrow \infty$, we deduce from (4.1) that for all $\varepsilon > 0$ small enough :

$$\lim_n \mu\left(\bigcap_{k=0}^n T^{-k} B(T^k x, \delta)\right) \leq \varepsilon,$$

hence, letting $\varepsilon \rightarrow 0$:

$$\mu\left(\bigcap_{n \geq 0} T^{-n} B(T^n x, \delta)\right) = 0.$$

Finally, we deduce from (4.2) and the choice of x that T is pairwise sensitive.

□

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