High order transmission conditions for conductive thin sheets – Asymptotic Expansions versus Thin Sheet Bases

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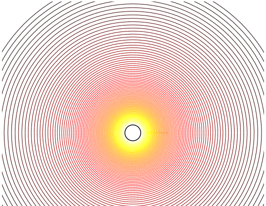
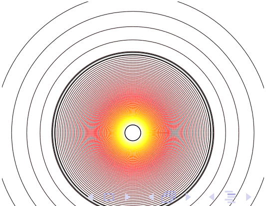
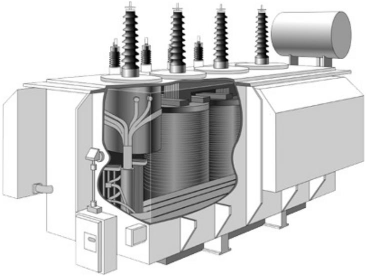


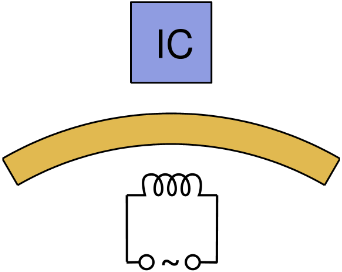


Outline

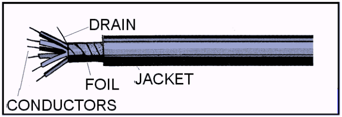
Introduction
Applications



Transformator inside a conducting casing.



Protection of integrated circuits.



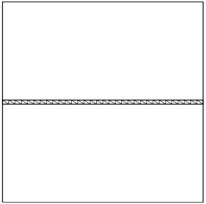
Cable with a foil shield.

Thin conducting sheets

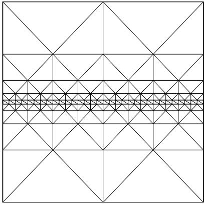
- for electromagnetic shielding,
- for mechanical stability,
- as casing for liquids and gases,
- with little material wastage.

Introduction

The meshing problem

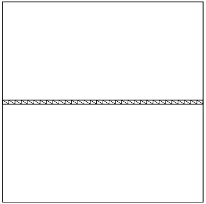


Triangle mesh inside the sheet.



Conforming triangle mesh resolving the sheet.

- Difficult to create by most mesh generators.
- High number of triangles.
- Even more triangles for resolving the sheet.



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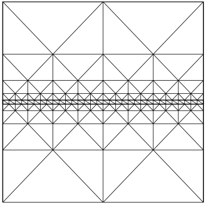

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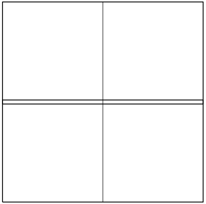

Conforming quadrilateral mesh resolving a thin sheet.

- Quadrilateral cells with large aspect ratio.
- Thin sheet basis functions inside the sheet (and no lateral refinement).

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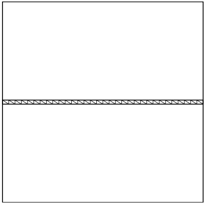
The meshing problem



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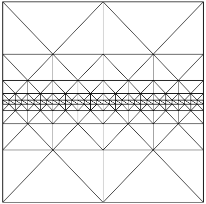

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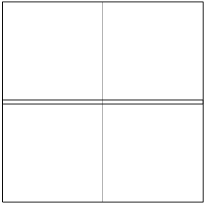

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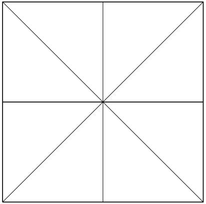


Conforming triangle mesh resolving an interface.

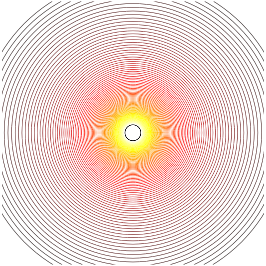
- Replace the thin sheet by an interface.
- Replace its behaviour by transmission conditions resulting from asymptotic expansions.

Time-harmonic **Eddy-current model** for low frequency applications

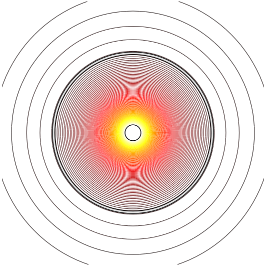
$$\mathbf{curl}\,\mathbf{e} = -i\omega\mu\mathbf{h},$$
$$\mathbf{curl}\,\mathbf{h} = \sigma\mathbf{e} + \mathbf{j}_0$$

Two important effects

- Shielding effect – in conductors induced currents diminish electromagnetic fields (behind the conductors)
- Skin effect – major current flow in a boundary layer (skins of the conductor)



Without conducting sheet.



With conducting sheet.

Field generated by an alternating current inside a cylindrical conductor.

Time-harmonic **Eddy-current model** for low frequency applications

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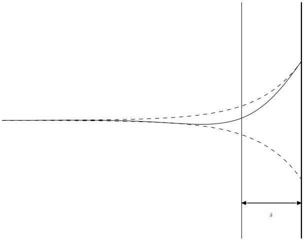
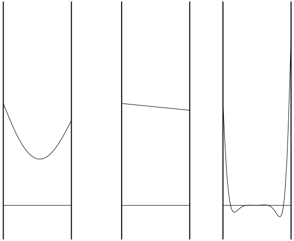


$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$$

Skin depth in solid body.



$\delta \approx d$ $\delta \gg d$ $\delta \ll d$

Skin effect in thin conducting sheets.

Time-harmonic **Eddy-current model** for low frequency applications

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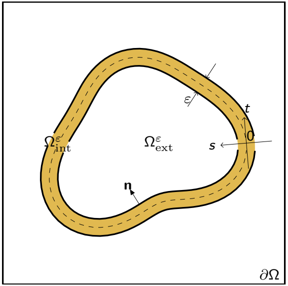
Time-harmonic **Eddy-current model** in 2D (TM mode)

$$-\Delta e(\underline{x}) + i\omega\mu_0\sigma e(\underline{x}) = i\omega\mu_0 j_0(\underline{x})$$

Consider the **Model problem** in 2D including these effects

$$-\Delta u + cu = f,$$

with "conductivity" $c(\underline{x})$ vanishing outside the thin conductive sheet.



Ω

Geometry

$\Omega_{\rm int}^{\varepsilon}$ ... thin conducting sheet of thickness ε
$\Omega_{\rm ext}^{\varepsilon}$ ... non-conducting domain
$\Omega = \overline{\Omega_{\rm int}^{\varepsilon}} \cup \overline{\Omega_{\rm ext}^{\varepsilon}}$

$\Gamma^{\rm m}$ ... midline of $\Omega_{\rm int}^{\varepsilon}$

$\exists$ smooth mapping $\widehat{\Gamma} \subset \mathbb{R} \mapsto \Gamma^{\rm m}$

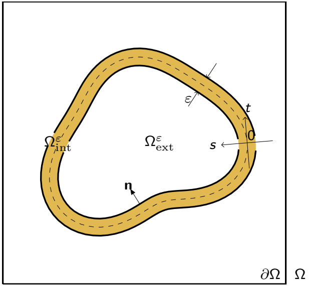
$\Rightarrow$ local orthogonal coordinate system in $\Omega_{\rm int}^{\varepsilon}$
$\Rightarrow$ local coordinates $(t, s) \in \widehat{\Gamma} \times [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$

$\underline{n}$ ... left unit normal vector

Goal : Description of thin conducting sheets of higher order.

Two strategies

A. Approximation of the solution inside the sheet by an optimal basis.
B. Asymptotic analysis of the solution inside and outside the sheet.



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Frameworks with ε varying conductivity

$$-\Delta u^\varepsilon + c(\varepsilon)\, u^\varepsilon == f$$

$c(\varepsilon) = c_0$ ... compare sheets of constant material

$c(\varepsilon) = \frac{c_0}{\varepsilon}$ ... compare sheets of constant shielding

$\Rightarrow$ Modelling error in ε related to a particular framework

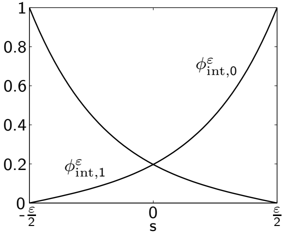
Outline

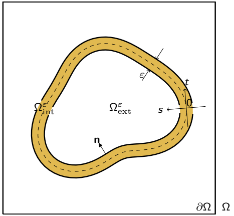
First order impedance boundary conditions

(*e. g.* Igarashi *et. al.*, 1998)

▷ use fundamental solutions of the ODE in thickness direction after neglecting longitudinal variations and the curvature.

$$-\Delta u_{\mathrm{int}}^{\varepsilon} + c u_{\mathrm{int}}^{\varepsilon} = 0 \Rightarrow -\partial_s^2 u_{\mathrm{int}}^{\varepsilon} + c u_{\mathrm{int}}^{\varepsilon} \approx 0 \Rightarrow \mathrm{e}^{\pm\sqrt{c}s}$$





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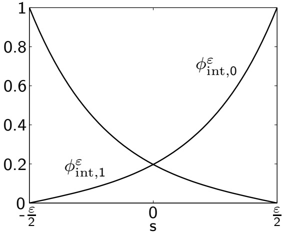
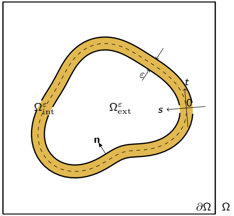
Dirichlet-to-Neumann map is derived on Γ^{ε}

$$\begin{pmatrix} \partial_n u^{\varepsilon}_{\text{int}}(t, +\frac{\varepsilon}{2}) \\ \partial_n u^{\varepsilon}_{\text{int}}(t, -\frac{\varepsilon}{2}) \end{pmatrix} \approx \mathbf{Z} \begin{pmatrix} u^{\varepsilon}_{\text{int}}(t, +\frac{\varepsilon}{2}) \\ u^{\varepsilon}_{\text{int}}(t, -\frac{\varepsilon}{2}) \end{pmatrix}$$





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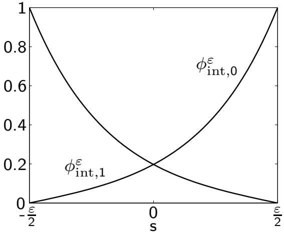
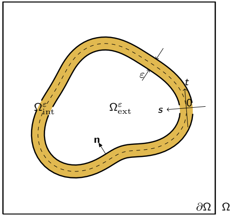
and used for transmission conditions on the mid-line Γ^{m}

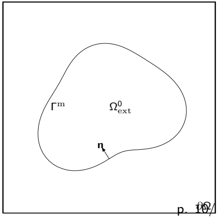
$$\begin{pmatrix} \partial_n u^{\varepsilon}_{\mathrm{ext}}(t, +0) \\ \partial_n u^{\varepsilon}_{\mathrm{ext}}(t, -0) \end{pmatrix} \approx \mathbf{Z} \begin{pmatrix} u^{\varepsilon}_{\mathrm{ext}}(t, +0) \\ u^{\varepsilon}_{\mathrm{ext}}(t, -0) \end{pmatrix}$$

Condition at Γ^{ε} used at Γ^{m} $\Rightarrow$ Modelling error of $O(\varepsilon)$

independent of any improvement in the interior





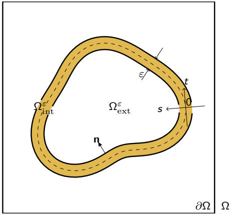


Approximation of higher order without reduction to an interface

Ansatz for the solution inside the sheet

$$u^\varepsilon_{\mathrm{int}}(t,s) \approx u^\varepsilon_{\mathrm{int},N}(t,s) = \sum_{i=0}^{N-1} \phi^\varepsilon_{\mathrm{int},i}(s,t)\, u^\varepsilon_{\mathrm{int},i}(t).$$

with $N \geq 2$ linear independent basis functions $\phi^\varepsilon_{\mathrm{int},i}$ spanning V^ε_N, and $u^\varepsilon_{\mathrm{int},i} \in H^1(\widehat{\Gamma})$.



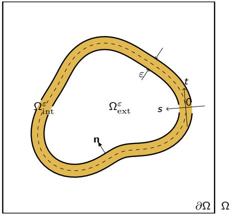
Estimate of the overall modelling error only that in the sheet

$$W^\varepsilon_N := \left\{ u : u_{\mathrm{ext}} \in H^1(\Omega^\varepsilon_{\mathrm{ext}}), u_{\mathrm{int}} \in V^\varepsilon_N \otimes H^1(\widehat{\Gamma}), u_{\mathrm{ext}} = u_{\mathrm{int}} \in H^1(\Gamma^\varepsilon) \right\} \subset H^1(\Omega)$$

$H^1(\Omega)$-ellipticity $\Rightarrow$ estimate by best-approximation by Cea's Lemma

$$\|u^\varepsilon - u^\varepsilon_N\|_{H^1(\Omega)} \leq C \inf_{u_N \in W^\varepsilon_N} \|u^\varepsilon - u_N\|_{H^1(\Omega)}$$

Approximation of higher order without reduction to an interface



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Estimate of the overall modelling error only that in the sheet

$$\widetilde{W}^\varepsilon_N := \left\{ u : u_{\text{ext}} \in H^1(\Omega^\varepsilon_{\text{ext}}), u_{\text{int}} \in V^\varepsilon_N \otimes H^1(\widehat{\Gamma}), u_{\text{ext}} = u_{\text{int}} = u^\varepsilon \in H^1(\Gamma^\varepsilon) \right\} \subset W^\varepsilon_N$$

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$$\|u^\varepsilon - u^\varepsilon_N\|_{H^1(\Omega)} \leq C \inf_{u_N \in W^\varepsilon_N} \|u^\varepsilon - u_N\|_{H^1(\Omega)} \leq C \inf_{u_N \in \widetilde{W}^\varepsilon_N} \|u^\varepsilon - u_N\|_{H^1(\Omega)}$$

$$= \underbrace{C \inf_{u_N \in \widetilde{W}^\varepsilon_N} \|u^\varepsilon - u_N\|_{H^1(\Omega^\varepsilon_{\text{ext}})}}_{0} + C \inf_{u_N \in \widetilde{W}^\varepsilon_N} \|u^\varepsilon - u_N\|_{H^1(\Omega^\varepsilon_{\text{int}})}$$

Approximation of higher order without reduction to an interface

Ansatz for the solution inside the sheet

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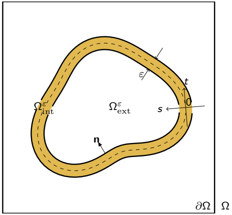

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Thin sheet basis

Extension to higher orders – for the circular sheet

Dirichlet boundary value problem in $\Omega^\varepsilon_{\rm int}$

$$-\Delta u^\varepsilon_{\rm int} + c(\varepsilon)\, u^\varepsilon_{\rm int} = 0 \qquad\qquad \text{in } \Omega^\varepsilon_{\rm int}.$$

Circular sheet $\quad -\Delta + c(\varepsilon) = \left(-\partial_s^2 - \frac{\kappa}{1+s\kappa}\partial_s + c(\varepsilon)\right) - \frac{1}{(1+s\kappa)^2}\partial_t^2$

Corresponding bilinear form ($v \in H^1_0(\Omega^\varepsilon_{\rm int})$)

$$\mathsf{a}_{\rm int}(u,v) := \int_{\Omega^\varepsilon_{\rm int}} \left(\partial_s u \partial_s u + c(\varepsilon) uv\right)(1+s\kappa) + \frac{1}{1+s\kappa}\partial_t u \partial_t v \,\mathrm{d}s\,\mathrm{d}t$$

Error of candidate function in $H^1(\Omega^\varepsilon_{\rm int})$-seminorm

$$|u^\varepsilon - w^\varepsilon_N|^2_{H^1(\Omega^\varepsilon_{\rm int})} \le |\mathsf{a}_{\rm int}(u^\varepsilon - w^\varepsilon_N, u^\varepsilon - w^\varepsilon_N)| = |\mathsf{a}_{\rm int}(w^\varepsilon_N, u^\varepsilon - w^\varepsilon_N)| \to \min$$

Thin sheet basis

Extension to higher orders – for the circular sheet

Error of **candidate function** in $H^1(\Omega^\varepsilon_{\text{int}})$-seminorm

$$|u^\varepsilon - w^\varepsilon_N|^2_{H^1(\Omega^\varepsilon_{\text{int}})} \le |\mathsf{a}_{\text{int}}(u^\varepsilon - w^\varepsilon_N, u^\varepsilon - w^\varepsilon_N)| = |\mathsf{a}_{\text{int}}(w^\varepsilon_N, u^\varepsilon - w^\varepsilon_N)| \to \min$$

Bilinear form ($u \in H^2(\Omega^\varepsilon_{\text{int}}), v \in H^1_0(\Omega^\varepsilon_{\text{int}})$)

$$\mathsf{a}_{\text{int}}(u, v) := \int_{\Omega^\varepsilon_{\text{int}}} (1 + s\kappa) \underbrace{\left(-\partial_s^2 - \frac{\kappa}{1 + s\kappa}\partial_s + c(\varepsilon)\right)}_{\text{scales with } \varepsilon} uv - \frac{1}{1 + s\kappa}\partial_t^2 uv \,\mathrm{d}s\,\mathrm{d}t$$

Candidate function for $N = 2$ satisfying boundary data with $\phi^\varepsilon_{\text{int},0}(\pm\frac{\varepsilon}{2}) = 1$, $\phi^\varepsilon_{\text{int},1}(\pm\frac{\varepsilon}{2}) = \pm\frac{1}{2}$

$$w^\varepsilon_2(s, t) = \phi^\varepsilon_{\text{int},0}(s)\{u^\varepsilon\}(t) + \phi^\varepsilon_{\text{int},1}(s) \underbrace{[u^\varepsilon](t)}_{0 \text{ for illustration}}.$$

$$\mathsf{a}_{\text{int}}(w^\varepsilon_2, v) = \int_{\Omega^\varepsilon_{\text{int}}} (1+s\kappa)\{u^\varepsilon\}(t)\underbrace{\left(-\partial_s^2 - \frac{\kappa}{1+s\kappa}\partial_s + c(\varepsilon)\right)\phi^\varepsilon_{\text{int},0}(s)}_{0} - \frac{\partial_t^2\{u^\varepsilon\}(t)}{1+s\kappa}\phi^\varepsilon_{\text{int},0}(s) v \,\mathrm{d}s\,\mathrm{d}t$$

Choose **basis functions** $\phi^\varepsilon_{\text{int},0}$, $\phi^\varepsilon_{\text{int},1}$ in the kernel of $\left(-\partial_s^2 - \frac{\kappa}{1+s\kappa} + c(\varepsilon)\right)$

Error of candidate function in $H^1(\Omega^\varepsilon_{\mathrm{int}})$-seminorm

$$|u^\varepsilon - w^\varepsilon_N|^2_{H^1(\Omega^\varepsilon_{\mathrm{int}})} \leq |\mathsf{a}_{\mathrm{int}}(u^\varepsilon - w^\varepsilon_N, u^\varepsilon - w^\varepsilon_N)| = |\mathsf{a}_{\mathrm{int}}(w^\varepsilon_N, u^\varepsilon - w^\varepsilon_N)| \to \min$$

Bilinear form $(u \in H^2(\Omega^\varepsilon_{\mathrm{int}}), v \in H^1_0(\Omega^\varepsilon_{\mathrm{int}}))$

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Candidate function for $N=4$ satisfying boundary data with $\phi^\varepsilon_{\mathrm{int},2}(\pm\frac{\varepsilon}{2}) = \phi^\varepsilon_{\mathrm{int},3}(\pm\frac{\varepsilon}{2}) = 0$

$$w^\varepsilon_4(s,t) = w^\varepsilon_2(s,t) + \phi^\varepsilon_{\mathrm{int},2}(s) u^\varepsilon_{\mathrm{int},2}(t).$$

$$\mathsf{a}_{\mathrm{int}}(w^\varepsilon_4, v) = \int_{\Omega^\varepsilon_{\mathrm{int}}} -\frac{\partial_t^2\{u^\varepsilon\}(t)}{1+s\kappa}\phi^\varepsilon_{\mathrm{int},0}(s)v + (1+s\kappa)u^\varepsilon_{\mathrm{int},2}(t)(-\partial_s^2 - \frac{\kappa}{1+s\kappa}\partial_s + c(\varepsilon))\phi^\varepsilon_{\mathrm{int},2}(s)v\,\mathrm{d}s\,\mathrm{d}t$$
$$+ \int_{\Omega^\varepsilon_{\mathrm{int}}} -\frac{\partial_t^2 u^\varepsilon_{\mathrm{int},2}(t)}{1+s\kappa}\phi^\varepsilon_{\mathrm{int},2(s)} v\,\mathrm{d}s\,\mathrm{d}t$$

Choose basis functions $\phi^\varepsilon_{\mathrm{int},2}$, $\phi^\varepsilon_{\mathrm{int},3}$ such that $(-\partial_s^2 - \frac{\kappa}{1+s\kappa} + c(\varepsilon))\phi^\varepsilon_{\mathrm{int},2}(s) = \varepsilon^2\phi^\varepsilon_{\mathrm{int},0}(s)$

Error of candidate function in $H^1(\Omega^\varepsilon_{\rm int})$-seminorm

$$|u^\varepsilon - w^\varepsilon_N|^2_{H^1(\Omega^\varepsilon_{\rm int})} \leq |\mathsf{a}_{\rm int}(u^\varepsilon - w^\varepsilon_N, u^\varepsilon - w^\varepsilon_N)| = |\mathsf{a}_{\rm int}(w^\varepsilon_N, u^\varepsilon - w^\varepsilon_N)| \to \min$$

Bilinear form $(u \in H^2(\Omega^\varepsilon_{\rm int}), v \in H^1_0(\Omega^\varepsilon_{\rm int}))$

$$\mathsf{a}_{\rm int}(u,v) := \int_{\Omega^\varepsilon_{\rm int}} (1+s\kappa)\underbrace{\left(-\partial_s^2 - \frac{\kappa}{1+s\kappa}\partial_s + c(\varepsilon)\right)}_{\text{scales with } \varepsilon} uv - \frac{1}{1+s\kappa}\partial_t^2 uv \,\mathrm{d}s\,\mathrm{d}t$$

Choose basis functions

$$(-\partial_s^2 - \frac{\kappa}{1+s\kappa} + c(\varepsilon))\phi^\varepsilon_{\rm int,0}(s) = 0$$

$$(-\partial_s^2 - \frac{\kappa}{1+s\kappa} + c(\varepsilon))\phi^\varepsilon_{\rm int,2}(s) = \varepsilon^2\phi^\varepsilon_{\rm int,0}(s)$$

Candidate function for $N=2$, $N=4$ satisfying boundary data

$$w^\varepsilon_2(s,t) = \{u^\varepsilon\}(t)\phi^\varepsilon_{\rm int,0}(s) \qquad w^\varepsilon_4(s,t) = w^\varepsilon_2(s,t) - \varepsilon^2\frac{\partial_t^2\{u^\varepsilon\}(t)}{(1+s\kappa)^2}\phi^\varepsilon_{\rm int,2}(s)$$

it remains

$$\mathsf{a}_{\rm int}(w^\varepsilon_2, v) = \int_{\Omega^\varepsilon_{\rm int}} -\frac{\partial_t^2\{u^\varepsilon\}(t)}{1+s\kappa}\phi^\varepsilon_{\rm int,0}(s) v\,\mathrm{d}s\,\mathrm{d}t$$

$$\mathsf{a}_{\rm int}(w^\varepsilon_4, v) = \varepsilon^2\int_{\Omega^\varepsilon_{\rm int}} \frac{\partial_t^4\{u^\varepsilon\}(t)}{(1+s\kappa)^3}\phi^\varepsilon_{\rm int,2}(s) v\,\mathrm{d}s\,\mathrm{d}t$$

Error of candidate function in $H^1(\Omega^\varepsilon_{\rm int})$-seminorm

$$|u^\varepsilon - w^\varepsilon_N|^2_{H^1(\Omega^\varepsilon_{\rm int})} \le |a_{\rm int}(u^\varepsilon - w^\varepsilon_N, u^\varepsilon - w^\varepsilon_N)| = |a_{\rm int}(w^\varepsilon_N, u^\varepsilon - w^\varepsilon_N)| \to \min$$

Bilinear form ($u \in H^2(\Omega^\varepsilon_{\rm int}), v \in H^1_0(\Omega^\varepsilon_{\rm int})$)

$$a_{\rm int}(u,v) := \int_{\Omega^\varepsilon_{\rm int}} (1+s\kappa) \underbrace{\left(-\partial_s^2 - \frac{\kappa}{1+s\kappa}\partial_s + c(\varepsilon)\right)}_{\text{scales with } \varepsilon} uv - \frac{1}{1+s\kappa}\partial_t^2 uv \,\mathrm{d}s\,\mathrm{d}t$$

Choose basis functions

$$\left(-\partial_s^2 - \frac{\kappa}{1+s\kappa} + c(\varepsilon)\right)\phi^\varepsilon_{{\rm int},j}(s) = 0 \qquad j = 0,1$$

$$\left(-\partial_s^2 - \frac{\kappa}{1+s\kappa} + c(\varepsilon)\right)\phi^\varepsilon_{{\rm int},j}(s) = \varepsilon^2\phi^\varepsilon_{{\rm int},j-2}(s), \qquad j = 2,\ldots,N-1$$

Candidate function satisfying boundary data

$$w^\varepsilon_N(s,t) = \sum_{i=0}^{\frac{N}{2}-1} \varepsilon^{2i} \frac{(-\partial_t)^{2i}}{(1+s\kappa)^{2i}} \left(\{u^\varepsilon\}(t)\phi^\varepsilon_{{\rm int},2i}(s) + [u^\varepsilon](t)\phi^\varepsilon_{{\rm int},2i+1}(s)\right)$$

it remains

$$a_{\rm int}(w^\varepsilon_N, v) = \varepsilon^{N-2}\int_{\Omega^\varepsilon_{\rm int}} \frac{(-\partial_t^2)^{\frac{N}{2}}}{(1+s\kappa)^{N+1}} \left(\{u^\varepsilon\}(t)\phi^\varepsilon_{{\rm int},N-2}(s) + [u^\varepsilon](t)\phi^\varepsilon_{{\rm int},N-1}(s)\right) v \,\mathrm{d}s\,\mathrm{d}t$$

Thin sheet basis

Extension to higher orders – for the circular sheet

Error of candidate function in $H^1(\Omega^\varepsilon_{\rm int})$-seminorm

$$|u^\varepsilon - w^\varepsilon_N|^2_{H^1(\Omega^\varepsilon_{\rm int})} \leq |a_{\rm int}(u^\varepsilon - w^\varepsilon_N, u^\varepsilon - w^\varepsilon_N)| = |a_{\rm int}(w^\varepsilon_N, u^\varepsilon - w^\varepsilon_N)| \to \min$$

Bilinear form $(u \in H^2(\Omega^\varepsilon_{\rm int}), v \in H^1_0(\Omega^\varepsilon_{\rm int}))$

$$a_{\rm int}(u,v) := \int_{\Omega^\varepsilon_{\rm int}} (1+s\kappa)\underbrace{\left(-\partial_s^2 - \frac{\kappa}{1+s\kappa}\partial_s + c(\varepsilon)\right)}_{\text{scales with } \varepsilon} uv - \frac{1}{1+s\kappa}\partial_t^2 uv \,\mathrm{d}s\,\mathrm{d}t$$

Choose basis functions

$$\left(-\partial_s^2 - \frac{\kappa}{1+s\kappa} + c(\varepsilon)\right)\phi^\varepsilon_{{\rm int},j}(s) = 0 \qquad j = 0, 1$$

$$\left(-\partial_s^2 - \frac{\kappa}{1+s\kappa} + c(\varepsilon)\right)\phi^\varepsilon_{{\rm int},j}(s) = \varepsilon^2 \phi^\varepsilon_{{\rm int},j-2}(s), \qquad j = 2, \ldots, N-1$$

it remains

$$a_{\rm int}(w^\varepsilon_N, v) = \varepsilon^{N-2} \int_{\Omega^\varepsilon_{\rm int}} \frac{(-\partial_t^2)^{\frac{N}{2}}}{(1+s\kappa)^{N+1}} \left(\{u^\varepsilon\}(t)\phi^\varepsilon_{{\rm int},N-2}(s) + [u^\varepsilon](t)\phi^\varepsilon_{{\rm int},N-1}(s)\right) v \,\mathrm{d}s\,\mathrm{d}t$$

$$= \varepsilon^N \int_{\Omega^\varepsilon_{\rm int}} \frac{1}{(1+s\kappa)^{N+1}} \left(-\partial_s^2 - \frac{\kappa}{1+s\kappa}\partial_s + c(\varepsilon)\right) \underbrace{(-\partial_t^2)^{\frac{N}{2}} \left(\{u^\varepsilon\}(t)\phi^\varepsilon_{{\rm int},N}(s) + \ldots\right)}_{r^\varepsilon_N(s,t)(1+s\kappa)}$$

Scaling of $\phi^\varepsilon_{{\rm int},j}(s)$ with $O(\sqrt{\varepsilon})$ in the $L^2([-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}])$-norm and Cauchy-Schwarz

Thin sheet basis

Extension to higher orders – for the circular sheet

Error of candidate function in $H^1(\Omega^\varepsilon_{\rm int})$-seminorm

$$|u^\varepsilon - w^\varepsilon_N|^2_{H^1(\Omega^\varepsilon_{\rm int})} \le |\mathsf{a}_{\rm int}(u^\varepsilon - w^\varepsilon_N, u^\varepsilon - w^\varepsilon_N)| = |\mathsf{a}_{\rm int}(w^\varepsilon_N, u^\varepsilon - w^\varepsilon_N)| \to O(\varepsilon^{2N-1})$$

Bilinear form $(u \in H^2(\Omega^\varepsilon_{\rm int}), v \in H^1_0(\Omega^\varepsilon_{\rm int}))$

$$\mathsf{a}_{\rm int}(u,v) := \int_{\Omega^\varepsilon_{\rm int}} (1+s\kappa)\underbrace{\left(-\partial_s^2 - \frac{\kappa}{1+s\kappa}\partial_s + c(\varepsilon)\right)}_{\text{scales with } \varepsilon} uv - \frac{1}{1+s\kappa}\partial_t^2 uv \,\mathrm{d}s\,\mathrm{d}t$$

Choose basis functions

$$\left(-\partial_s^2 - \frac{\kappa}{1+s\kappa} + c(\varepsilon)\right)\phi^\varepsilon_{{\rm int},j}(s) = 0 \qquad j = 0, 1$$

$$\left(-\partial_s^2 - \frac{\kappa}{1+s\kappa} + c(\varepsilon)\right)\phi^\varepsilon_{{\rm int},j}(s) = \varepsilon^2\phi^\varepsilon_{{\rm int},j-2}(s), \qquad j = 2, \ldots, N-1$$

it remains

$$\mathsf{a}_{\rm int}(w^\varepsilon_N, v) = \varepsilon^{N-2}\int_{\Omega^\varepsilon_{\rm int}} \frac{(-\partial_t^2)^{\frac{N}{2}}}{(1+s\kappa)^{N+1}}\Big(\{u^\varepsilon\}(t)\phi^\varepsilon_{{\rm int},N-2}(s) + [u^\varepsilon](t)\phi^\varepsilon_{{\rm int},N-1}(s)\Big)\, v\,\mathrm{d}s\,\mathrm{d}t$$

$$= \varepsilon^N \int_{\Omega^\varepsilon_{\rm int}} \frac{1}{(1+s\kappa)^{N+1}}\left(-\partial_s^2 - \frac{\kappa}{1+s\kappa}\partial_s + c(\varepsilon)\right)\underbrace{(-\partial_t^2)^{\frac{N}{2}}\Big(\{u^\varepsilon\}(t)\phi^\varepsilon_{{\rm int},N}(s) + \ldots\Big)}_{r^\varepsilon_N(s,t)(1+s\kappa)}$$

Scaling of $\phi^\varepsilon_{{\rm int},j}(s)$ with $O(\sqrt{\varepsilon})$ in the $L^2([-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}])$-norm and Cauchy-Schwarz

Theorem (The modelling error of a candidate function)

Let N be an even integer. There exists a function $w_N^\varepsilon \in \widetilde{W}_{\mathrm{int},N}^\varepsilon$ such that

$$|w_N^\varepsilon - u_{\mathrm{int}}^\varepsilon|_{H^1(\Omega_{\mathrm{int}}^\varepsilon)} \leq C_N \varepsilon^{N-1/2}, \qquad \|w_N^\varepsilon - u_{\mathrm{int}}^\varepsilon\|_{L^2(\Omega_{\mathrm{int}}^\varepsilon)} \leq C_N \varepsilon^{N+1/2},$$

with a constant C_N independent of ε and the choice of $c(\varepsilon)$.

This result transfers to the overall modelling error due to Cea's lemma.

$$\|u^\varepsilon - u_N^\varepsilon\|_{H^1(\Omega)} \leq C \inf_{u_N \in \widetilde{W}_N^\varepsilon} \|u^\varepsilon - u_N\|_{H^1(\Omega)} = C \inf_{u_N \in \widetilde{W}_{\mathrm{int},N}^\varepsilon} \|u^\varepsilon - u_N\|_{H^1(\Omega_{\mathrm{int}}^\varepsilon)}.$$

Corollary (The modelling error for the circular sheet)

Let N be an even integer, $\kappa'(t) = 0$ and $u_N^\varepsilon \in W_N^\varepsilon$ the solution of

$$\int_{\Omega_{\mathrm{ext}}^\varepsilon} \nabla u_N^\varepsilon \cdot \nabla v \,\mathrm{d}x + \int_{\Omega_{\mathrm{int}}^\varepsilon} \nabla u_N^\varepsilon \cdot \nabla v + c(\varepsilon) u_N^\varepsilon v \,\mathrm{d}x = \int_{\Omega_{\mathrm{ext}}^\varepsilon} f v \,\mathrm{d}x,$$

Then, there exists a constant C_N independent of ε such that

$$\|u^\varepsilon - u_N^\varepsilon\|_{H^1(\Omega)} \leq C_N \varepsilon^{N-\frac{1}{2}}.$$

Thin sheet basis

Extension to higher orders – curved sheet

Circular sheet $-\Delta + c(\varepsilon) = \underbrace{\left(-\partial_s^2 - \frac{\kappa}{1+s\kappa}\partial_s + c(\varepsilon)\right)}_{\text{scales with } \varepsilon} - \frac{1}{1+s\kappa}\partial_t\left(\frac{1}{1+s\kappa}\partial_t\right)$

Choose basis functions ϕ^ε_{2j}, ϕ^ε_{2j+1} in the kernel of $(-\partial_s^2 - \frac{\kappa}{1+s\kappa}\partial_s + c(\varepsilon))^{j+1}$, $j \in \mathbb{N}_0$

Basis $I_0\left(\sqrt{c(\varepsilon)}\frac{1+s\kappa}{|\kappa|}\right)p(s)$, $K_0\left(\sqrt{c(\varepsilon)}\frac{1+s\kappa}{|\kappa|}\right)p(s)$ instable for small curvatures

Thin sheet basis

Extension to higher orders – curved sheet

Circular sheet $-\Delta + c(\varepsilon) = \underbrace{\left(-\partial_s^2 - \frac{\kappa}{1+s\kappa}\partial_s + c(\varepsilon)\right)}_{\text{scales with } \varepsilon} - \frac{1}{1+s\kappa}\partial_t\left(\frac{1}{1+s\kappa}\partial_t\right)$

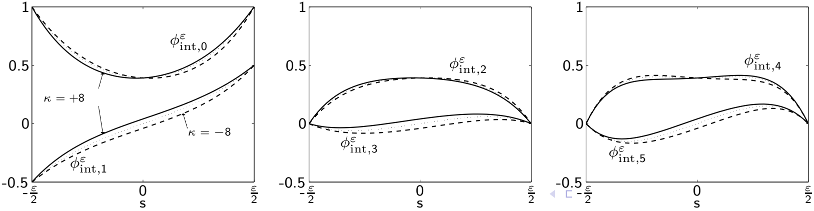
Choose basis functions ϕ_{2j}^ε, ϕ_{2j+1}^ε in the kernel of $\left(-\partial_s^2 - \frac{\kappa}{1+s\kappa}\partial_s + c(\varepsilon)\right)^{j+1}$, $j \in \mathbb{N}_0$

Basis $I_0\left(\sqrt{c(\varepsilon)}\frac{1+s\kappa}{|\kappa|}\right)p(s)$, $K_0\left(\sqrt{c(\varepsilon)}\frac{1+s\kappa}{|\kappa|}\right)p(s)$ instable for small curvatures

$$-\Delta + c(\varepsilon) = \underbrace{\left(-\partial_s^2 - \frac{\kappa}{1+s\kappa}\partial_s + c(\varepsilon) + \left(\frac{\kappa}{1+s\kappa}\right)^2\right)}_{\text{scales with } \varepsilon} - \frac{1}{1+s\kappa}\partial_t\left(\frac{1}{1+s\kappa}\partial_t\right) - \left(\frac{\kappa}{1+s\kappa}\right)^2$$

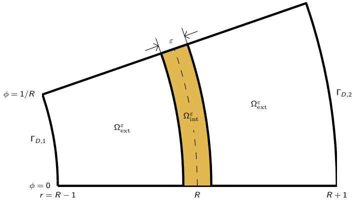
Choose basis functions ϕ_{2j}^ε, ϕ_{2j+1}^ε in the kernel of $\left(-\partial_s^2 - \frac{\kappa(t)}{1+s\kappa(t)}\partial_s + c(\varepsilon) + \left(\frac{\kappa}{1+s\kappa}\right)^2\right)^{j+1}$

Basis $\frac{e^{\pm\sqrt{c(\varepsilon)}\,s}}{\sqrt{1+s\kappa}}p(s)$ stable for small curvatures

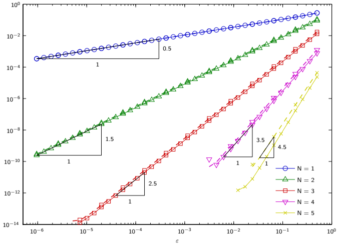


Thin sheet basis

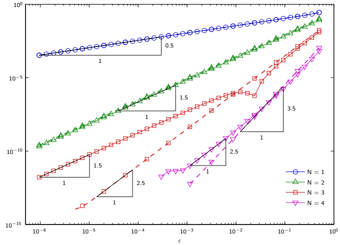
Numerical results for a circular sheet



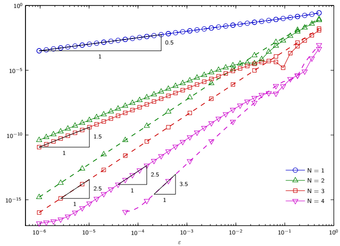
Semianalytical study with $f = 0$,
$g = \sin(\pi t)$ on $\Gamma_{D,1} \cup \Gamma_{D,2}$,
otherwise $g = 0$
Error in H^1-seminorm inside the sheet.



(a) Constant conductivity $c = 1$. $\kappa = \frac{1}{2}$.



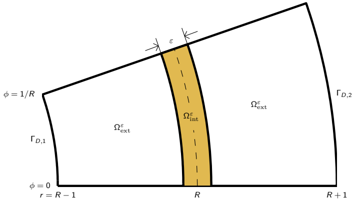
(b) Conductivity $c(\varepsilon) = \frac{1}{\varepsilon}$. $\kappa = \frac{1}{2}$.



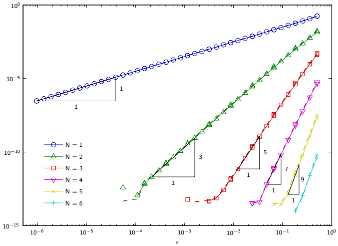
(c) Conductivity $c(\varepsilon) = \frac{1}{\varepsilon^2}$. $\kappa = \frac{1}{2}$.

Thin sheet basis

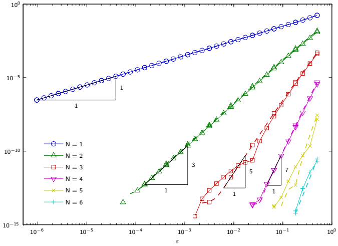
Numerical results for a circular sheet



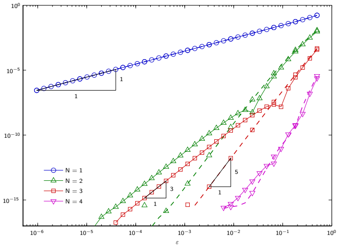
Semianalytical study with $f = 0$,
$g = \sin(\pi t)$ on $\Gamma_{D,1} \cup \Gamma_{D,2}$,
otherwise $g = 0$
Error in H^1-seminorm outside the sheet.



(a) Constant conductivity $c = 1$. $\kappa = \frac{1}{2}$.



(b) Conductivity $c(\varepsilon) = \frac{1}{\varepsilon}$. $\kappa = \frac{1}{2}$.

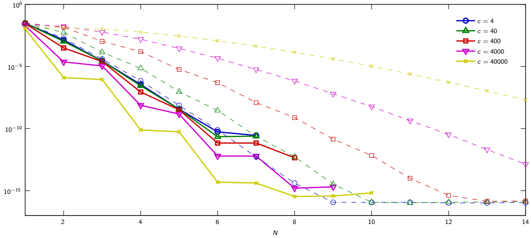


(c) Conductivity $c(\varepsilon) = \frac{1}{\varepsilon^2}$. $\kappa = \frac{1}{2}$.

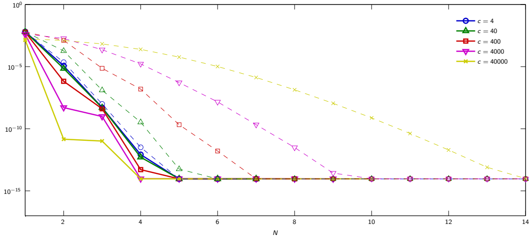
Thin sheet basis

Dependance of the number of basis functions in comparison with polynomial basis

$\varepsilon = 0.05$



(a) Error in the H^1-norm inside the sheet.

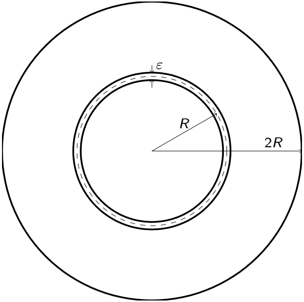


(b) Error in the H^1-norm outside the sheet.

Outline

Asymptotic Expansions

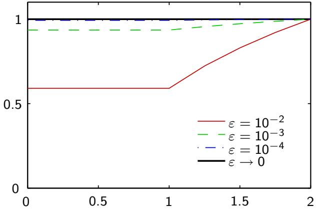
Choice of the asymptotics



$\alpha = 0$

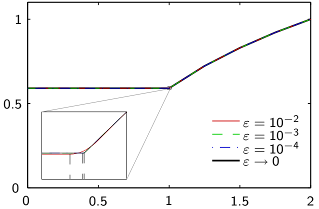
$\alpha = 0$

asymptotically no shielding for $\varepsilon \to 0$



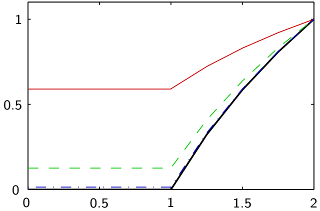
$\alpha = 1$

asymptotically constant shielding for $\varepsilon \to 0$



$\alpha = 2$

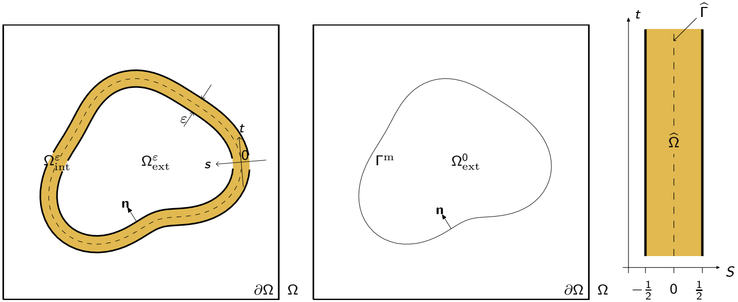
asymptotically impermeable sheet for $\varepsilon \to 0$



Families of problems with

$$c(\varepsilon) = \frac{c_0}{\varepsilon^\alpha}$$

$\Rightarrow$ Choose $\alpha = 1$ (borderline case)



Stretched variable in $\Omega^\varepsilon_{\rm int}$ $\quad S = \varepsilon^{-1}s \quad \Rightarrow$ normalised domain $\widehat{\Omega} := \widehat{\Gamma} \times [-\frac{1}{2}, \frac{1}{2}]$.

Notation $v(\underline{x}) = v(t,s) = V(t,S)$ for a function v in $\Omega^\varepsilon_{\rm int}$. $(t,S) \in \widehat{\Omega}$.

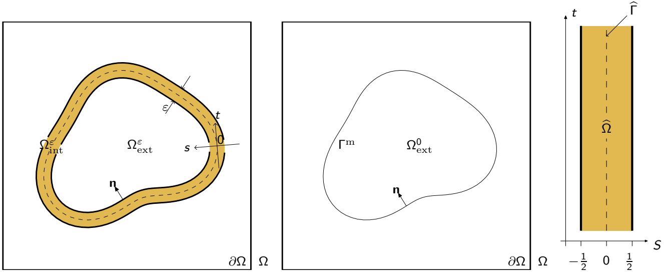
Expansion of $u^\varepsilon_{\rm ext}(\underline{x})$ and $U^\varepsilon_{\rm int}(t,S)$: We seek the exact solution with the form

$$u^\varepsilon_{\rm ext}(\underline{x}) = \sum_{i=0}^{\infty} \varepsilon^i u^i_{\rm ext}(\underline{x}) + \underset{\varepsilon\to 0}{o}(\varepsilon^\infty) \ , \quad U^\varepsilon_{\rm int}(t,S) = \sum_{i=0}^{\infty} \varepsilon^i U^i_{\rm int}(t,S) + \underset{\varepsilon\to 0}{o}(\varepsilon^\infty)$$

where the terms $u^i_{\rm ext}(x)$ and $U^i_{\rm int}(t,S)$ do not depend on ε.

Asymptotic Expansions

Taylor expansion of $u^i(t, \pm\varepsilon/2)$



Dirichlet transmission condition for expansion coefficient

$$0 = u^\varepsilon_{\rm ext}(t, \pm\frac{\varepsilon}{2}) - U^\varepsilon_{\rm int}(t, \pm\frac{1}{2}) = \sum_{i=0}^{\infty} \varepsilon^i \left(\sum_{j=0}^{i} (\pm\tfrac{1}{2})^j \, \frac{1}{j!} \, \partial_s^j u^{i-j}_{\rm ext}(t, \pm 0) - U^i_{\rm int}(t, \pm\tfrac{1}{2}) \right) + \underset{\varepsilon \to 0}{o}(\varepsilon^\infty) \ .$$

Neumann transmission condition

$$0 = \varepsilon \partial_s u^\varepsilon_{\rm ext}(t, \pm\frac{\varepsilon}{2}) - \partial_S U^\varepsilon_{\rm int}(t, \pm\frac{1}{2})$$

$$= \sum_{i=0}^{\infty} \varepsilon^i \left(\sum_{j=0}^{i-1} (\pm\tfrac{1}{2})^j \, \frac{1}{j!} \, \partial_s^{j+1} u^{i-j-1}_{\rm ext}(t, \pm 0) - \partial_S U^i_{\rm int}(t, \pm\tfrac{1}{2}) \right) + \underset{\varepsilon \to 0}{o}(\varepsilon^\infty) \ .$$

$$\Delta = \partial_s^2 + \frac{\kappa(t)}{1+s\kappa(t)}\partial_s + \frac{1}{1+s\kappa(t)}\,\partial_t\left(\frac{1}{1+s\kappa(t)}\partial_t\right), \qquad \text{in } \Omega^\varepsilon_{\text{int}}$$

$$\Delta = \varepsilon^{-2}\partial_S^2 + \frac{\varepsilon^{-1}\kappa(t)}{1+\varepsilon S\kappa(t)}\partial_S + \frac{1}{1+\varepsilon S\kappa(t)}\,\partial_t\left(\frac{1}{1+\varepsilon S\kappa(t)}\partial_t\right). \qquad \text{in } \widehat{\Omega} \qquad (1)$$

Laplacian (1) can be expanded in powers of ε

$$\Delta = \varepsilon^{-2}\left(\partial_S^2 + \sum_{l=1}^{L-1}\varepsilon^l \mathbf{A}_l + \varepsilon^L \mathbf{R}_\varepsilon^L\right), \quad \text{for all } L \geq 1, \qquad (2)$$

where for $l, L \geq 1$ there are defined

$$\widehat{\mathbf{A}}_l^0(t) := (-\kappa(t))^{l-2}(l-1)\left(\partial_t^2 + \frac{l-2}{2}\frac{\kappa'(t)}{\kappa(t)}\partial_t\right), \quad \widehat{\mathbf{A}}_l^1(t) := -(-\kappa(t))^l,$$

$$\mathbf{A}_l(t,S) := \widehat{\mathbf{A}}_l^0(t)S^{l-2} + \widehat{\mathbf{A}}_l^1(t)S^{l-1}\partial_S,$$

Hierarchical definition of external and internal expansion functions.

$$\left\{\begin{aligned}
-\Delta u^i_{\mathrm{ext}} &= f\delta^i_0 && \text{in } \Omega^0_{\mathrm{ext}},\\
u^i_{\mathrm{ext}} &= g\delta^i_0 && \text{on } \partial\Omega,\\
\partial^2_S U^i_{\mathrm{int}}(t,S) &= c_0 U^{i-1}_{\mathrm{int}}(t,S) - \sum_{l=1}^{i} \mathbf{A}_l U^{i-l}_{\mathrm{int}}(t,S) && \text{in } \widehat{\Omega}.\\
U^i_{\mathrm{int}}(t,\pm\tfrac{1}{2}) - u^i_{\mathrm{ext}}(t,0^\pm) &= \sum_{l=1}^{i}\left(\pm\frac{1}{2}\right)^l \frac{1}{l!}\partial^l_s u^{i-l}_{\mathrm{ext}}(t,0^\pm) && \text{on } \Gamma^{\mathrm{m}},\\
\partial_S U^i_{\mathrm{int}}(t,\pm\tfrac{1}{2}) &= \sum_{l=1}^{i}\left(\pm\frac{1}{2}\right)^{l-1}\frac{1}{(l-1)!}\,\partial^l_s u^{i-l}_{\mathrm{ext}}(t,0^\pm) && \text{on } \Gamma^{\mathrm{m}}\\
c_0\int_{-\frac{1}{2}}^{\frac{1}{2}} U^i_{\mathrm{int}}(t,S)\,\mathrm{d}S - k(t)\left[U^i_{\mathrm{int}}\right](t) - \left[\partial_s u^i_{\mathrm{ext}}\right](t) &= \\
\sum_{l=1}^{i}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}}\mathbf{A}_{l+1}U^{i-l}_{\mathrm{int}}(t,S)\,\mathrm{d}S + \left(\frac{1}{4}\right)^{\lfloor l/2\rfloor}\frac{1}{l!}[\partial^{l+1}_s u^{i-l}_{\mathrm{ext}}]^l(t)\right) & && \text{on } \Gamma^{\mathrm{m}}
\end{aligned}\right. \tag{3}$$

Jumps and mean values

$$[V](t) := V(t,1/2) - V(t,-1/2), \quad \{V\}(t) := \frac{1}{2}\left(V(t,1/2) + V(t,-1/2)\right),$$

$$[v](t) := v(t,0^+) - v(t,0^-), \qquad \{v\}(t) := \frac{1}{2}\left(v(t,0^+) + v(t,0^-)\right).$$

$$[v]^n(t) := \begin{cases} [v](t) & n \text{ even} \\ \{v\}(t) & n \text{ odd}. \end{cases}$$

Existence, Uniqueness and Regularity of the hierarchical problem

Theorem

The sequences (u^i_{ext}) *and* (U^i_{int}) *exist and are uniquely defined.* U^i_{int} *are polynomials in* S *of degree* $2i$ *with monomial coefficients* $U^i_{\mathrm{int},j}$, $j = 0, \ldots, 2i-1$. *For any* $k \in \mathbb{N}_0$ *and* $i \in \mathbb{N}_0$ *there exists a constant* $C_{i,k} < +\infty$ *such that*

$$\|u^i_{\mathrm{ext}}(\underline{x})\|_{H^k(\widetilde{\Omega}^0_{\mathrm{ext}})} \leq C_{i,k}, \qquad \|U^i_{\mathrm{int},j}(t)\|_{H^{k+1/2}(\Gamma^{\mathrm{m}})} \leq C_{i,k},$$

given that Γ^{m} *is* C^∞.

Convergence of the modelling error

Lemma

The remainder $r^{\varepsilon,N+1}(x) := u^{\varepsilon}_{\mathrm{ext/int}}(\underline{x}) - u^{\varepsilon,N}_{\mathrm{ext/int}}(\underline{x})$ *satisfies*

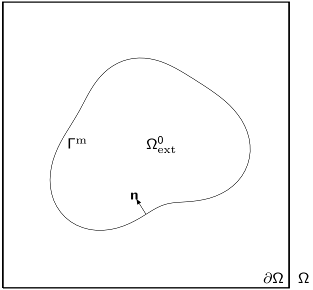
$$\|r^{\varepsilon,N+1}\|_{H^1(\Omega^\varepsilon_{\mathrm{ext}})} + \sqrt{\varepsilon}\|r^{\varepsilon,N+1}\|_{H^1(\Omega^\varepsilon_{\mathrm{int}})} \leq C_N \varepsilon^{N+1}.$$

Are the expansion functions u^i_{ext} and so $u^{\varepsilon,N}_{\mathrm{ext}}$ computable?

Asymptotic Expansions

Uncoupled problems for the first three exterior expansion functions

Solving the equation for $\widetilde{U}^i_{\text{int}}(t,s)$ and $\{U^i_{\text{int}}\}(t)$ interface conditions yields. $\Rightarrow$
Problem in Ω^0_{ext} for $u^i_{\text{ext}}(x)$



Order 0

$$\begin{cases} -\Delta u^0_{\text{ext}}(x) = f & \text{in } \Omega^0_{\text{ext}} \\ u^0_{\text{ext}}(x) = g & \text{on } \partial\Omega \\ \left[u^0_{\text{ext}}\right](t) = 0 & \text{on } \Gamma^m \\ \left[\partial_n u^0_{\text{ext}}\right](t) - c_0\, u^0_{\text{ext}}(t,0) = 0 & \text{on } \Gamma^m \end{cases}$$

Order 1

$$\begin{cases} \Delta u^1_{\text{ext}}(x) = 0 & \text{in } \Omega^0_{\text{ext}} \\ u^1_{\text{ext}}(x) = 0 & \text{on } \partial\Omega \\ \left[u^1_{\text{ext}}\right](t) = 0, & \text{on } \Gamma^m \\ \left[\partial_n u^1_{\text{ext}}\right](t) - c_0 u^1_{\text{ext}}(t) = \dfrac{c_0^2}{6}\, u^0_{\text{ext}}(t,0) & \text{on } \Gamma^m \end{cases}$$

Order 2

$$\begin{cases} \Delta u^2_{\text{ext}}(x) = 0 & \text{in } \Omega^0_{\text{ext}} \\ u^2_{\text{ext}}(x) = 0 & \text{on } \partial\Omega \\ \left[u^2_{\text{ext}}\right](t) = -\dfrac{c_0}{24}\kappa(t) u^0_{\text{ext}}(t,0) - \dfrac{c_0}{12}\left\{\partial_n u^0_{\text{ext}}\right\}(t) & \text{on } \Gamma^m, \\ \left[\partial_n u^2_{\text{ext}}\right](t) - c_0\left\{u^2_{\text{ext}}\right\}(t) = \dfrac{c_0^2}{6}\, u^1_{\text{ext}}(t) + \dfrac{c_0}{24}\kappa(t)\left\{\partial_n u^0_{\text{ext}}\right\}(t) & \\ \qquad + c_0\left(\dfrac{7}{240}c_0^2 - \dfrac{\partial_t^2}{12}\right) u^0_{\text{ext}}(t,0) & \text{on } \Gamma^m \end{cases}$$

C++ class library Concepts

- Started by Christian Lage during his Ph.D. studies (1995).
- Used and improved by Frauenfelder, Matache, Schmidlin, Schmidt, Kauf and severals students.
- Concept Oriented Design using mathematical principles [1].
- Currently two parts: *hp*-FEM (nodal and edge elements), BEM (wavelet and multipole methods).
- most of it is released under GPL, `http://www.concepts.math.ethz.ch`

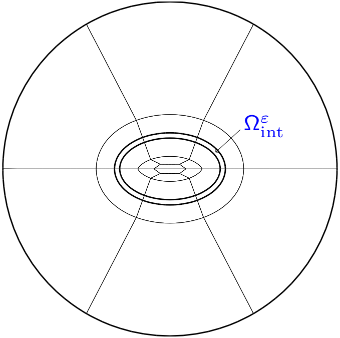
Features used and/or implemented in this project

- quadrilateral elements with high-order tensor product basis functions,
- elements on edges with high-order basis functions,
- bilinear and linear forms on these respective elements,
- exact mapping of curved boundaries,
- various functions of finite element solutions (trace, jump, extension, gradient, ...).

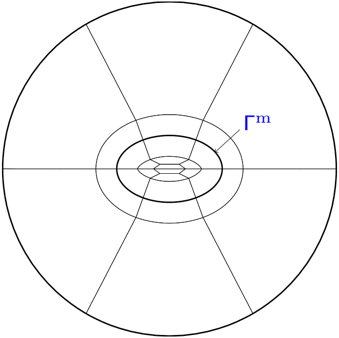
[1] Ph. Frauenfelder and Ch. Lage, "Concepts—An Object Oriented Software Package for Partial Differential Equations", *Math. Model. Numer. Anal.* 36 (5), pp. 937–951 (2002).

Asymptotic Expansions

Numerical Results – computations for an ellipsoidal sheet with Concepts



Meshes $\mathcal{M}^{\varepsilon}$ resolving the thin sheets of thickness ε for the reference solutions $u^{\varepsilon}(\underline{x})$.



Limit mesh $\mathcal{M}^{0}$ for the computation of the asymptotic expansion functions $u^{i}_{\mathrm{ext}}(\underline{x})$.

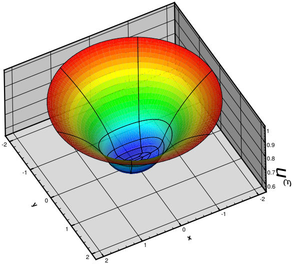
Following

- computation of the polynomials $U^{i}_{\mathrm{int}}(t, S)$,
- representation of $u^{\varepsilon,N}(\underline{x}) = \sum_{i=0}^{N} \varepsilon^{i} u^{i}(\underline{x})$ on meshes $\mathcal{M}^{\varepsilon}$
- to calculate the modelling error in dependence of ε.

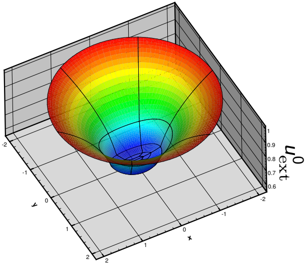
Asymptotic Expansions

Numerical Results – computations for an ellipsoidal sheet with Concepts

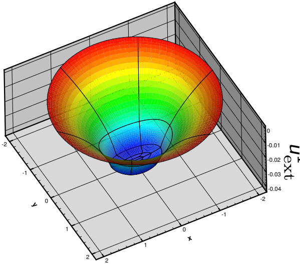
$c_0 = 1$



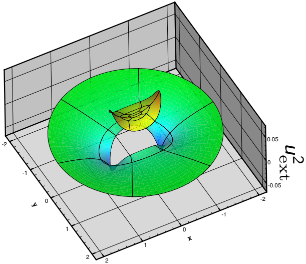
(c) $\varepsilon = 1/8$



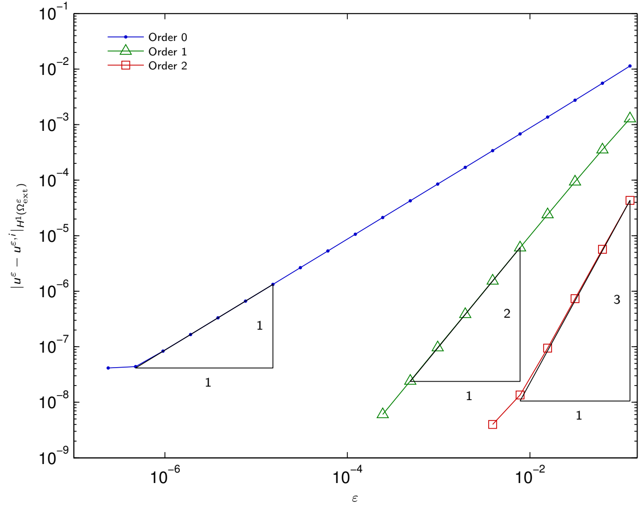
(d) Order 0



(e) Order 1



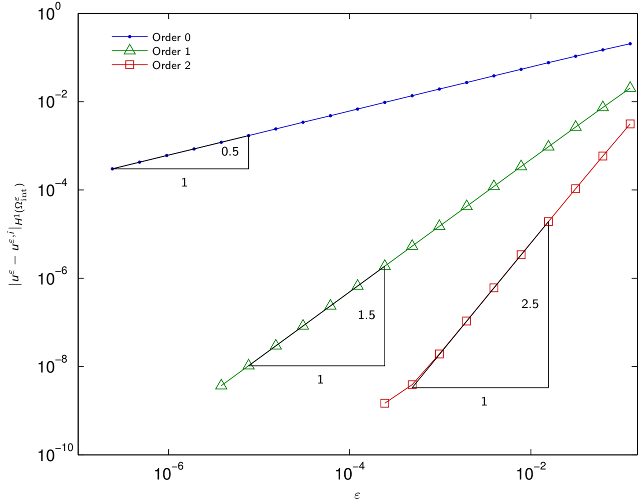
(f) Order 2



H^1-seminorm in the exterior sub-domain $\Omega^\varepsilon_{\mathrm{ext}}$.

Asymptotic Expansions

Numerical Results – computations for an ellipsoidal sheet with Concepts



H^1-seminorm in the sheet $\Omega^\varepsilon_{\mathrm{int}}$.

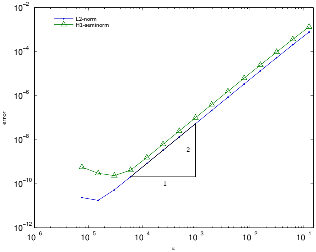
Transmission condition of order $N = 1$

$$\begin{cases} -\Delta \tilde{u}^{\varepsilon,1}_{\mathrm{ext}}(x) = f & \text{in } \Omega^0_{\mathrm{ext}}, \\ \left[\tilde{u}^{\varepsilon,1}_{\mathrm{ext}}\right](t) = 0 & \text{on } \Gamma^m, \\ \left[\partial_n \tilde{u}^{\varepsilon,1}_{\mathrm{ext}}\right](t) - c_0\left(1 + \frac{c_0}{6}\varepsilon\right) \tilde{u}^{\varepsilon,1}_{\mathrm{ext}}(t,0) = 0 & \text{on } \Gamma^m, \\ \tilde{u}^{\varepsilon,1}_{\mathrm{ext}}(x) = g & \text{on } \partial\Omega_{\mathrm{ext}}. \end{cases}$$

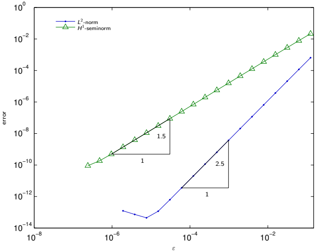
Stability for $\Im c_0 > 0$ and $\Re c_0 \geq 0$

$$\|\tilde{u}^{\varepsilon,1}\|_{H^1(\Omega^0_{\mathrm{ext}})} \leq C \|f\|_{L^2(\Omega^0_{\mathrm{ext}})}.$$

Modelling error $\quad \|\tilde{u}^{\varepsilon,1} - u^{\varepsilon}\|_{H^1(\Omega^{\varepsilon}_{\mathrm{ext}})} \leq C\,\varepsilon^2$



(a) Errors in the exterior sub-domain $\Omega^{\varepsilon}_{\mathrm{ext}}$.



(b) Errors in the sheet $\Omega^{\varepsilon}_{\mathrm{int}}$.

Rewrite the systems in general form

$$-\Delta u^i_{\rm ext}(\underline{x}) = f\delta^i_0, \qquad \text{in } \Omega^0_{\rm ext},$$

$$\left[u^i_{\rm ext}\right](t) = \sum_{\ell=2}^{i}(\gamma_\ell u^{j-\ell}_{\rm ext})(t), \qquad \text{on } \Gamma^{\rm m},$$

$$\left[\partial_n u^i_{\rm ext}\right](t) - c_0\left\{u^i_{\rm ext}\right\}(t) = \sum_{\ell=1}^{i}(\zeta_\ell u^{j-\ell}_{\rm ext})(t), \qquad \text{on } \Gamma^{\rm m},$$

$$u^i_{\rm ext}(x) = g\delta^i_0 \qquad \text{on } \partial\Omega$$

with the operators (up to order 3)

$$(\gamma_2 u)(t) := -\frac{c_0}{24}\big(\kappa(t)\{u\}(t) + 2\{\partial_n u\}(t)\big), \qquad (\gamma_3 u)(t) := \frac{c_0^2}{240}\big(\kappa(t)\{u\}(t) + 2\{\partial_n u\}(t)\big),$$

$$(\zeta_1 u)(t) := \frac{c_0^2}{6}\{u\}(t), \qquad (\zeta_2 u)(t) := \frac{c_0}{12}\left(\frac{7}{20}c_0^2 - \partial_t^2\right)\{u\}(t) + \frac{c_0}{24}\kappa(t)\{\partial_n u\}(t)$$

$$(\zeta_3 u)(t) := \frac{c_0^2}{40}\left(\frac{17}{40}c_0^2 - \frac{7}{10}\kappa^2(t) - \partial_t^2\right)\{u\}(t) - \frac{c_0^2}{240}\kappa(t)\{\partial_n u\}(t).$$

Transmission condition of order $N > 1$

$$
\begin{aligned}
-\Delta \tilde{u}^{\varepsilon,N}_{\mathrm{ext}}(\underline{x}) &= f, && \text{in } \Omega^0_{\mathrm{ext}},\\
\left[u^i_{\mathrm{ext}}\right](t) - \boldsymbol{\gamma}^{\varepsilon,N}_0\{\tilde{u}^{\varepsilon,N}_{\mathrm{ext}}\}(t) - \boldsymbol{\gamma}^{\varepsilon,N}_1\{\partial_n \tilde{u}^{\varepsilon,N}_{\mathrm{ext}}\}(t) &= 0, && \text{on } \Gamma^{\mathrm{m}},\\
\left[\partial_n \tilde{u}^{\varepsilon,N}_{\mathrm{ext}}\right](t) - (c_0 + \boldsymbol{\zeta}^{\varepsilon,N}_0)\left\{\tilde{u}^{\varepsilon,N}_{\mathrm{ext}}\right\}(t) - \boldsymbol{\zeta}^{\varepsilon,N}_1\{\partial_n \tilde{u}^{\varepsilon,N}_{\mathrm{ext}}\}(t) &= 0, && \text{on } \Gamma^{\mathrm{m}},\\
\tilde{u}^{\varepsilon,N}_{\mathrm{ext}}(x) &= g && \text{on } \partial\Omega
\end{aligned}
$$

with $\boldsymbol{\gamma}^{\varepsilon,N}_\ell := \sum_{j=2}^N \varepsilon^j \boldsymbol{\gamma}_{j,\ell}$ and $\boldsymbol{\zeta}^{\varepsilon,N}_\ell := \sum_{j=1}^N \varepsilon^j \boldsymbol{\zeta}_{j,\ell}$ and the operators (up to order 3)

$$
\begin{aligned}
\boldsymbol{\gamma}^{\varepsilon,2}_0 &= -\varepsilon^2 \frac{c_0}{24}\kappa(t), & \boldsymbol{\gamma}^{\varepsilon,3}_0 &= \boldsymbol{\gamma}^{\varepsilon,2}_0 + \varepsilon^3 \frac{c_0^2}{240}\kappa(t)\\
\boldsymbol{\gamma}^{\varepsilon,2}_1 &= -\varepsilon^2 \frac{c_0}{12}, & \boldsymbol{\gamma}^{\varepsilon,3}_1 &= \boldsymbol{\gamma}^{\varepsilon,2}_1 + \varepsilon^3 \frac{c_0^2}{120},\\
\boldsymbol{\zeta}^{\varepsilon,2}_0 &= \varepsilon \frac{c_0^2}{6} + \varepsilon^2 \frac{c_0}{12}\left(\frac{7}{20}c_0^2 - \partial_t^2\right), & \boldsymbol{\zeta}^{\varepsilon,3}_0 &= \boldsymbol{\zeta}^{\varepsilon,2}_0 + \varepsilon^3 \frac{c_0^2}{40}\left(\frac{17}{40}c_0^2 - \frac{7}{10}\kappa^2(t) - \partial_t^2\right)\\
\boldsymbol{\zeta}^{\varepsilon,2}_1 &= \varepsilon^2 \frac{c_0}{24}\kappa(t), & \boldsymbol{\zeta}^{\varepsilon,3}_1 &= \boldsymbol{\zeta}^{\varepsilon,2}_1 - \varepsilon^3 \frac{c_0^2}{240}\kappa(t)
\end{aligned}
$$

Greens formula gives

$$
\int_{\Omega^0_{\mathrm{ext}}} \nabla \tilde{u}^{\varepsilon,N}_{\mathrm{ext}} \cdot \nabla v \,\mathrm{d}x + \int_{\Gamma^{\mathrm{m}}} \underbrace{\{\partial_n \tilde{u}^{\varepsilon,N}_{\mathrm{ext}}\}}_{\lambda}[v] + [\partial_n \tilde{u}^{\varepsilon,N}_{\mathrm{ext}}]\{v\}\,\mathrm{d}t = \int_{\Omega^0_{\mathrm{ext}}} fv \,\mathrm{d}x
$$

Transmission condition of order $N > 1$

$$
\begin{aligned}
-\Delta \tilde{u}^{\varepsilon,N}_{\text{ext}}(\underline{x}) &= f, && \text{in } \Omega^0_{\text{ext}},\\
\left[\tilde{u}^{\varepsilon,N}_{\text{ext}}\right](t) - \gamma^{\varepsilon,N}_0\{\tilde{u}^{\varepsilon,N}_{\text{ext}}\}(t) - \gamma^{\varepsilon,N}_1\{\partial_n \tilde{u}^{\varepsilon,N}_{\text{ext}}\}(t) &= 0, && \text{on } \Gamma^{\mathrm{m}},\\
\left[\partial_n \tilde{u}^{\varepsilon,N}_{\text{ext}}\right](t) - (c_0 + \zeta^{\varepsilon,N}_0)\left\{\tilde{u}^{\varepsilon,N}_{\text{ext}}\right\}(t) - \zeta^{\varepsilon,N}_1\{\partial_n \tilde{u}^{\varepsilon,N}_{\text{ext}}\}(t) &= 0, && \text{on } \Gamma^{\mathrm{m}},\\
\tilde{u}^{\varepsilon,N}_{\text{ext}}(x) &= g && \text{on } \partial\Omega
\end{aligned}
$$

Mixed variational formulation

Seek $(\tilde{u}^{\varepsilon,N}_{\text{ext}}, \lambda) \in HT^{1,1}(\Omega^0_{\text{ext}}, g) \times L^2(\Gamma^{\mathrm{m}})$ such that for all $(v, \lambda') \in HT^{1,1}(\Omega^0_{\text{ext}}, 0) \times L^2(\Gamma^{\mathrm{m}})$

$$
\int_{\Omega^0_{\text{ext}}} \nabla \tilde{u}^{\varepsilon,N}_{\text{ext}} \cdot \nabla v \,\mathrm{d}x + \int_{\Gamma^{\mathrm{m}}} (c_0 + \zeta^{\varepsilon,N}_0)\{\tilde{u}^{\varepsilon,N}_{\text{ext}}\}\{v\} + \lambda[v]\,\mathrm{d}t = \int_{\Omega^0_{\text{ext}}} fv\,\mathrm{d}x
$$

$$
\int_{\Gamma^{\mathrm{m}}} ([\tilde{u}^{\varepsilon,N}_{\text{ext}}](t) - \gamma^{\varepsilon,N}_0\{\tilde{u}^{\varepsilon,N}_{\text{ext}}\}(t))\lambda' - \gamma^{\varepsilon,N}_1 \lambda\lambda'\,\mathrm{d}t = 0.
$$

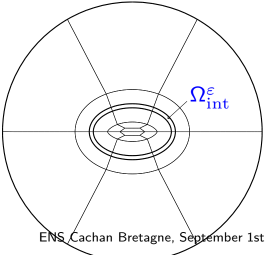
Stability for $N = 2, 3$, $\Im c_0 > 0$ and $\Re c_0 \geq 0$

$$
\|\tilde{u}^{\varepsilon,N}\|_{H^1(\Omega^0_{\text{ext}})} + \varepsilon\|\lambda\|_{L^2(\Gamma^{\mathrm{m}})} + \varepsilon^{-1}\|[\tilde{u}^{\varepsilon,N}]\|_{L^2(\Gamma^{\mathrm{m}})} \leq C\|f\|_{L^2(\Omega^0_{\text{ext}})}.
$$

Modelling error $\quad \|\tilde{u}^{\varepsilon,N} - u^{\varepsilon}\|_{H^1(\Omega^{\varepsilon}_{\text{ext}})} \leq C\,\varepsilon^{N+1}$

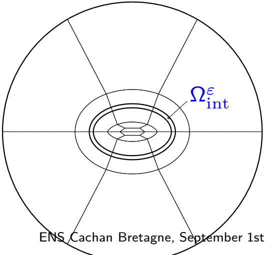
Comparison of thin sheet basis and asymptotic expansions

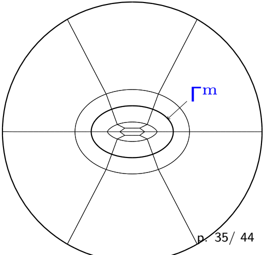
	Thin sheet basis
Order	Any for circular sheets.
Mesh	Sheet has to be resolved.
Usability	Simple definition of basis to any order.
Condition	Local.
Computation	In one step.
Convergence	For $\varepsilon \to 0$ and for $N \to \infty$.
DOF	Additional variable for each function.
Discretisation	H^1-continuous elements.



Comparison of thin sheet basis and asymptotic expansions

	Thin sheet basis	Asymptotic expansions
Order	Any for circular sheets.	Any proven order for C^∞-sheets.
Mesh	Sheet has to be resolved.	Only interface.
Usability	Simple definition of basis to any order.	Practically limited order, more and more terms (derived up to order 4).
Condition	Local.	Local.
Computation	In one step.	Iteratively or in one step.
Convergence	For $\varepsilon \to 0$ and for $N \to \infty$.	For $\varepsilon \to 0$.
DOF	Additional variable for each function.	No additional variable.
Discretisation	H^1-continuous elements.	High order tangential continuity on Γ^{m} needed with increasing order.





Conclusions

Introduction

- Applications like casings or shielding layer.
- Difficult to create meshes for thin sheets and expensive to use.
- Model problem considering shielding and skin effects.

Thin sheet basis

- First order impedance boundary conditions based on fundamental solutions of an ODE in thickness direction
- Resolving the sheet for higher order approximations.
- Derivation of the basis for circular sheet in kernel of $\left(-\partial_s^2 - \frac{\kappa}{1+s\kappa}\partial_s + c(\varepsilon)\right)^j$.
- Proof of convergence: Best-approximation, candidate function, cancelling of low order terms.
- Validation by semi-analytical studies. Convergence for $\varepsilon \to 0$ and $N \to \infty$

Asymptotic Expansions

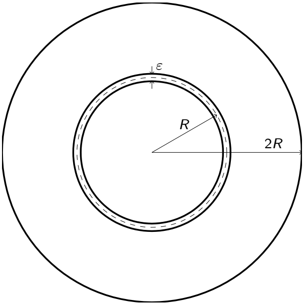
- Expansion for $c(\varepsilon) = \frac{c_0}{\varepsilon}$ (asymptotically constant shielding).
- Derived (coupled) model by asymptotic expansions of arbitrary order.
- Showed existence, uniqueness, regularity and consistency.
- Models only for exterior field for order 0, 1 and 2.
- Validation by numerical results with high-order FEM (Concepts).
- Collectively computed model of order $N = 1$. Same formulation.
- Collectively computed models of order $N > 1$. Mixed formulation.
- Showed existence, uniqueness and convergence up to order 3.

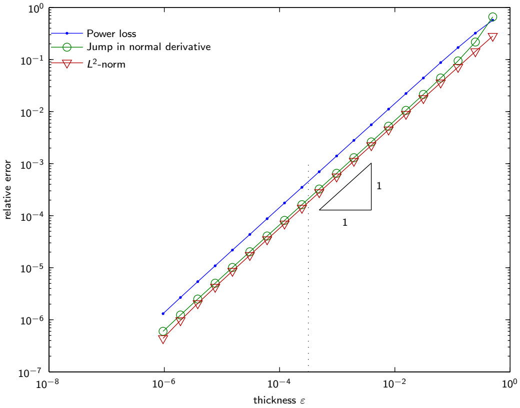
Comparison

Thin sheet basis

Extension to higher orders

- Dirichlet-to-Neumann-map of high-order,
- Extension of the outer solution up to the mid-line





$\Rightarrow$ Consider resolved sheets for methods of higher order

Problem with this structure

$$
\begin{aligned}
-\Delta u(\underline{x}) &= f, && \text{in } \Omega^0_{\mathrm{ext}},\\
[u]\,(t) + \varepsilon^2 c_0 \{\partial_n u\}(t) &= 0, && \text{on } \Gamma^{\mathrm{m}},\\
[\partial_n u]\,(t) - c_0\{u\}\,(t) &= 0, && \text{on } \Gamma^{\mathrm{m}},\\
u &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

and the associated variational formulation

$$
\int_{\Omega^0_{\mathrm{ext}}} \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\Gamma^{\mathrm{m}}} c_0\{u\}\{v\} + \lambda[v]\,\mathrm{d}t = \int_{\Omega^0_{\mathrm{ext}}} fv\,\mathrm{d}x
$$

$$
\int_{\Gamma^{\mathrm{m}}} [u]\lambda' + \varepsilon^2 c_0 \lambda\lambda'\,\mathrm{d}t = 0.
$$

Ellipticity ?? Testing with $v = \overline{u}$ and $\lambda' = \overline{\lambda}$ and substracting second from first eq.

$$
\int_{\Omega^0_{\mathrm{ext}}} |\nabla u|^2\,\mathrm{d}x + \int_{\Gamma^{\mathrm{m}}} c_0\left(|\{u\}|^2 - \varepsilon^2|\lambda|^2\right)\,\mathrm{d}t
$$

Lemma (Stability)

Let $N = 2, 3$, $\Im c_0 > 0$ and $\Re c_0 \geq 0$. It holds

$$\|\tilde{u}^{\varepsilon,N}\|_{H^1(\Omega^0_{\mathrm{ext}})} + \varepsilon\|\lambda\|_{L^2(\Gamma^{\mathrm{m}})} + \varepsilon^{-1}\|[\tilde{u}^{\varepsilon,N}]\|_{L^2(\Gamma^{\mathrm{m}})} \leq C\|f\|_{L^2(\Omega^0_{\mathrm{ext}})}.$$

Proof. $$\int_{\Omega^0_{\mathrm{ext}}} \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\Gamma^{\mathrm{m}}} c_0\{u\}\{v\} + \lambda[v] + [u]\lambda' + \varepsilon^2 c_0 \lambda\lambda' \,\mathrm{d}t = \int_{\Omega^0_{\mathrm{ext}}} fv \,\mathrm{d}x.$$

1. Testing with $v = \overline{u}$ and λ' such that $\varepsilon^2 c_0 \lambda' = -[\overline{u}]$.

$$\int_{\Omega^0_{\mathrm{ext}}} |\nabla u|^2 \,\mathrm{d}x + \int_{\Gamma^{\mathrm{m}}} c_0|\{u\}|^2 - \varepsilon^2\overline{c_0}|\lambda'|^2 \,\mathrm{d}t = \int_{\Omega^0_{\mathrm{ext}}} f\overline{u} \,\mathrm{d}x,$$

and taking imaginary part

$$\Im c_0 \int_{\Gamma^{\mathrm{m}}} |\{u\}|^2 + \varepsilon^2|\lambda'|^2 \,\mathrm{d}t = \Im \int_{\Omega^0_{\mathrm{ext}}} f\overline{u} \,\mathrm{d}x$$

and we have $\|\{u\}\|_{L^2(\Gamma^{\mathrm{m}})} \leq C\|f\|_{L^2(\Omega^0_{\mathrm{ext}})}$, $\|[u]\|_{L^2(\Gamma^{\mathrm{m}})} \leq C\varepsilon\|f\|_{L^2(\Omega^0_{\mathrm{ext}})}$.

Lemma (Stability)

Let $N = 2, 3$, $\Im c_0 > 0$ *and* $\Re c_0 \geq 0$. *It holds*

$$\|\tilde{u}^{\varepsilon,N}\|_{H^1(\Omega^0_{\mathrm{ext}})} + \varepsilon\|\lambda\|_{L^2(\Gamma^{\mathrm{m}})} + \varepsilon^{-1}\|[\tilde{u}^{\varepsilon,N}]\|_{L^2(\Gamma^{\mathrm{m}})} \leq C\|f\|_{L^2(\Omega^0_{\mathrm{ext}})}.$$

Proof. $$\int_{\Omega^0_{\mathrm{ext}}} \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\Gamma^{\mathrm{m}}} c_0\{u\}\{v\} + \lambda[v] + [u]\lambda' + \varepsilon^2 c_0 \lambda\lambda' \,\mathrm{d}t = \int_{\Omega^0_{\mathrm{ext}}} fv \,\mathrm{d}x.$$

1. We have $\|\{u\}\|_{L^2(\Gamma^{\mathrm{m}})} \leq C\|f\|_{L^2(\Omega^0_{\mathrm{ext}})}$, $\|[u]\|_{L^2(\Gamma^{\mathrm{m}})} \leq C\varepsilon\|f\|_{L^2(\Omega^0_{\mathrm{ext}})}$.
2. Testing with $v = 0$ and λ' such that $\varepsilon^2 c_0 \lambda' = [\overline{u}]$

$$\left|\int_{\Gamma^{\mathrm{m}}} \lambda[\overline{u}] \,\mathrm{d}t\right| \leq \frac{1}{|c_0|}\varepsilon^{-2}\|[u]\|^2_{L^2(\Gamma^{\mathrm{m}})} \leq C\|f\|^2_{L^2(\Omega^0_{\mathrm{ext}})}.$$

3. Testing with $v = \overline{u}$ and $\lambda' = 0$, multiplying with $(1+i)$ and taking real part

$$\gamma\|u\|^2_{H^1(\Omega^0_{\mathrm{ext}})} \leq \int_{\Omega^0_{\mathrm{ext}}} |\nabla u|^2 \,\mathrm{d}x + \int_{\Gamma^{\mathrm{m}}} \Re\left((1+i)c_0\right)|\{u\}|^2$$
$$= \Re\left(\left((1+i)\int_{\Omega^0_{\mathrm{ext}}} f\overline{u} \,\mathrm{d}x - \int_{\Gamma^{\mathrm{m}}} \lambda[\overline{u}] \,\mathrm{d}t\right)\right) \leq C\|f\|^2_{L^2(\Omega^0_{\mathrm{ext}})}.$$

Lemma (Stability)

Let $N = 2, 3$, $\Im c_0 > 0$ *and* $\Re c_0 \geq 0$. *It holds*

$$\|\tilde{u}^{\varepsilon,N}\|_{H^1(\Omega^0_{\mathrm{ext}})} + \varepsilon\|\lambda\|_{L^2(\Gamma^{\mathrm{m}})} + \varepsilon^{-1}\|[\tilde{u}^{\varepsilon,N}]\|_{L^2(\Gamma^{\mathrm{m}})} \leq C\|f\|_{L^2(\Omega^0_{\mathrm{ext}})}.$$

Proof. $$\int_{\Omega^0_{\mathrm{ext}}} \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\Gamma^{\mathrm{m}}} c_0\{u\}\{v\} + \lambda[v] + [u]\lambda' + \varepsilon^2 c_0 \lambda\lambda' \,\mathrm{d}t = \int_{\Omega^0_{\mathrm{ext}}} fv \,\mathrm{d}x.$$

1. We have $\|\{u\}\|_{L^2(\Gamma^{\mathrm{m}})} \leq C\|f\|_{L^2(\Omega^0_{\mathrm{ext}})}$, $\|[u]\|_{L^2(\Gamma^{\mathrm{m}})} \leq C\varepsilon\|f\|_{L^2(\Omega^0_{\mathrm{ext}})}$.
2. We have $\left|\int_{\Gamma^{\mathrm{m}}} \lambda[\overline{u}] \,\mathrm{d}t\right| \leq C\|f\|^2_{L^2(\Omega^0_{\mathrm{ext}})}$.
3. We have $\|u\|_{H^1(\Omega^0_{\mathrm{ext}})} \leq C\|f\|_{L^2(\Omega^0_{\mathrm{ext}})}$.
4. Testing with $v = 0$ and λ' such that $\varepsilon^2 c_0 \lambda' = \overline{\lambda}$

$$\int_{\Gamma^{\mathrm{m}}} |\lambda|^2 \,\mathrm{d}t \leq \varepsilon^{-2}\frac{1}{|c_0|}\left|\int_{\Gamma^{\mathrm{m}}} \lambda[\overline{u}] \,\mathrm{d}t\right| \leq C\varepsilon^{-2}\|f\|^2_{L^2(\Omega^0_{\mathrm{ext}})}.$$

$\square$

System for order $N = 2, 3$ and general right hand sides.

$$
\begin{aligned}
-\Delta\tilde{u}^{\varepsilon,N}_{\mathrm{ext}}(\underline{x}) &= f, && \text{in } \Omega^0_{\mathrm{ext}},\\
\left[\tilde{u}^{\varepsilon,N}_{\mathrm{ext}}\right](t) - \gamma^{\varepsilon,N}_0\{\tilde{u}^{\varepsilon,N}_{\mathrm{ext}}\}(t) - \gamma^{\varepsilon,N}_1\{\partial_n\tilde{u}^{\varepsilon,N}_{\mathrm{ext}}\}(t) &= g_1(t), && \text{on } \Gamma^{\mathrm{m}},\\
\left[\partial_n\tilde{u}^{\varepsilon,N}_{\mathrm{ext}}\right](t) - (c_0 + \zeta^{\varepsilon,N}_0)\left\{\tilde{u}^{\varepsilon,N}_{\mathrm{ext}}\right\}(t) - \zeta^{\varepsilon,N}_1\{\partial_n\tilde{u}^{\varepsilon,N}_{\mathrm{ext}}\}(t) &= g_2(t), && \text{on } \Gamma^{\mathrm{m}},\\
\tilde{u}^{\varepsilon,N}_{\mathrm{ext}}(x) &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

Lemma (Stability)

Let $N = 2, 3$, $\Im c_0 > 0$ and $\Re c_0 \geq 0$. It holds

$$
\|\tilde{u}^{\varepsilon,N}\|_{H^1(\Omega^0_{\mathrm{ext}})} + \varepsilon\|\lambda\|_{L^2(\Gamma^{\mathrm{m}})} \leq C\left(\|f\|_{L^2(\Omega^0_{\mathrm{ext}})} + \varepsilon^{-2}\|g_1\|_{L^2(\Gamma^{\mathrm{m}})} + \|g_2\|_{H^{-1/2}(\Gamma^{\mathrm{m}})}\right).
$$

Error $\tilde{r}^{\varepsilon,N+1}_{\mathrm{ext}} = \tilde{u}^{\varepsilon,N}_{\mathrm{ext}} - u^{\varepsilon,N}_{\mathrm{ext}}$ solves system of the same structure with $f = 0$,
$g_1 = O(\varepsilon^{N+1})$ and $g_2 = O(\varepsilon^{N+1})$.

Attention: Error only $O(\varepsilon^{N-1})$??

System for order $N = 2, 3$ and general right hand sides.

$$
\begin{aligned}
-\Delta \tilde{u}^{\varepsilon,N}_{\text{ext}}(\underline{x}) &= f, && \text{in } \Omega^0_{\text{ext}},\\
\left[\tilde{u}^{\varepsilon,N}_{\text{ext}}\right](t) - \gamma^{\varepsilon,N}_0\{\tilde{u}^{\varepsilon,N}_{\text{ext}}\}(t) - \gamma^{\varepsilon,N}_1\{\partial_n \tilde{u}^{\varepsilon,N}_{\text{ext}}\}(t) &= g_1(t), && \text{on } \Gamma^{\mathrm{m}},\\
\left[\partial_n \tilde{u}^{\varepsilon,N}_{\text{ext}}\right](t) - (c_0 + \zeta^{\varepsilon,N}_0)\left\{\tilde{u}^{\varepsilon,N}_{\text{ext}}\right\}(t) - \zeta^{\varepsilon,N}_1\{\partial_n \tilde{u}^{\varepsilon,N}_{\text{ext}}\}(t) &= g_2(t), && \text{on } \Gamma^{\mathrm{m}},\\
\tilde{u}^{\varepsilon,N}_{\text{ext}}(x) &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

Lemma (Stability)

Let $N = 2, 3$, $\Im c_0 > 0$ and $\Re c_0 \geq 0$. It holds

$$
\|\tilde{u}^{\varepsilon,N}\|_{H^1(\Omega^0_{\text{ext}})} + \varepsilon\|\lambda\|_{L^2(\Gamma^{\mathrm{m}})} \leq C\left(\|f\|_{L^2(\Omega^0_{\text{ext}})} + \varepsilon^{-2}\|g_1\|_{L^2(\Gamma^{\mathrm{m}})} + \|g_2\|_{H^{-1/2}(\Gamma^{\mathrm{m}})}\right).
$$

However, we can find functions $v^{N,j}_{\text{ext}}$ solving similar system than u^N_{ext} and $\tilde{r}^{\varepsilon,N+1,k}_{\text{ext}} = \tilde{u}^{\varepsilon,N}_{\text{ext}} - u^{\varepsilon,N}_{\text{ext}} - \sum_{j=N+1}^{N+k} \varepsilon^j \underbrace{v^{N,j}}_{\mathcal{O}(1)}$ solves system of the above structure with

$$
f = 0,\ g_1 = O(\varepsilon^{N+1+k}),\ g_2 = O(\varepsilon^{N+1+k}).
$$

Result: $\tilde{r}^{\varepsilon,N+1,k}_{\text{ext}}$ is $O(\varepsilon^{N+k-1})$ and so with $k = 2$

$$
\begin{aligned}
\|\tilde{u}^{\varepsilon,N}_{\text{ext}} - u^{\varepsilon,N}_{\text{ext}}\|_{H^1(\Omega^0_{\text{ext}})} &\leq \|\tilde{r}^{\varepsilon,N+1,2}_{\text{ext}}\|_{H^1(\Omega^0_{\text{ext}})} + \varepsilon^{N+1}\|v^{N,N+1}_{\text{ext}}\|_{H^1(\Omega^0_{\text{ext}})} + \varepsilon^{N+2}\|v^{N,N+2}_{\text{ext}}\|_{H^1(\Omega^0_{\text{ext}})}\\
&\leq C\varepsilon^{N+1}
\end{aligned}
$$

Asymptotic Expansions

Asymptotic expansion

$$u^\varepsilon_{\rm ext}(\underline{x}) \approx u^{\varepsilon,N}_{\rm ext}(\underline{x}) := \sum_{i=0}^N \varepsilon^i u^i_{\rm ext}(\underline{x}), \qquad u^\varepsilon_{\rm int}(t,s) \approx u^{\varepsilon,N}_{\rm int}(t,s) := \sum_{i=0}^N \varepsilon^i U^i_{\rm int}(t,\frac{s}{\varepsilon})$$

Hierarchical partly uncoupled problem

- Existence, Uniqueness and Regularity for any order
- Convergence of asymptotic expansion

$$\|u^\varepsilon - u^{\varepsilon,N}_{\rm ext}\|_{H^1(\Omega^\varepsilon_{\rm ext})} + \sqrt{\varepsilon}\|u^\varepsilon - u^{\varepsilon,N}_{\rm ext}\|_{H^1(\Omega^\varepsilon_{\rm int})} \leq C_N \varepsilon^{N+1}$$

Hierarchical uncoupled problem for $u^i_{\rm ext}$

- derived up to order 4, numerical experiments up to order 2

Collective computation for $\tilde{u}^{\varepsilon,N}_{\rm ext} \approx u^{\varepsilon,N}_{\rm ext}$

- system for order 1 like for $u^i_{\rm ext}$, well-posed, regularity, convergence
- mixed variational system for order higher than 1
- well-posedness and convergence for $\Im c_0 > 0$ for order 2 and 3