Perturbations singulières complexifiantes. Exemples et applications

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Plan of the lecture

- a) General framework (3 - 4)
- b) Numerical results on shells (5 - 8)
- c) The approximate logarithmic counter (9 - 16)
- d) The Shapiro – Lopatinskii condition, the elliptic Cauchy problems and (non-distributional) singularities (17 -30)
- e) Another analogous system (31 – 34)

Introduction

This paper deals with singular perturbation problems for variational problems of the form

$$\begin{cases} \text{Find } u^\varepsilon \in V \text{ such that, } \forall v \in V \\ a(u^\varepsilon, v) + \varepsilon^2 b(u^\varepsilon, v) = \langle f, v \rangle, \end{cases} \tag{1.1}$$

where a and b are symmetric bilinear forms in the energy space V (a "classical Sobolev like space") for $\varepsilon > 0$, whereas the energy space V_a for $\varepsilon = 0$ is a very large space going out the distribution space. The asymptotics $\varepsilon \to 0$ implies a "complexification process".

The limit problem is ill-posed.
This kind of problems appears in thin shell theory
(elliptic middle surface, fixed by a part of the boundary and free by the rest)

Theorem 2.3. *Let F be in V', u^ε and u be the solutions of*

$$Au^\varepsilon + \varepsilon^2 Bu^\varepsilon = \quad \mathsf{F} \tag{2.15}$$

and

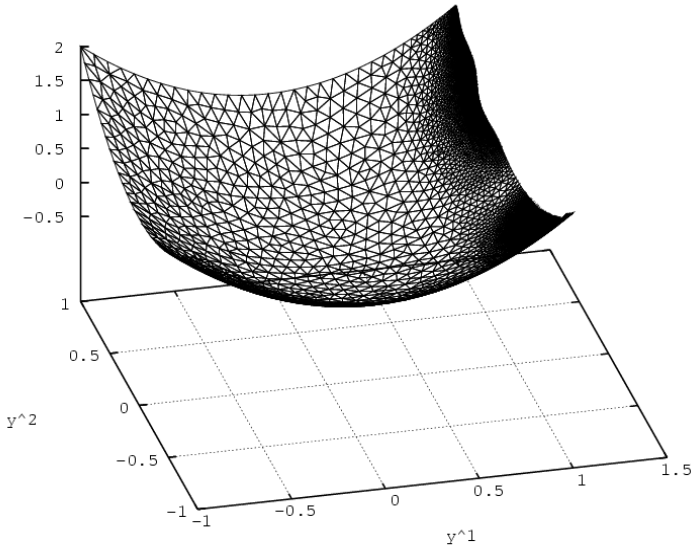
$$Au = F. \tag{2.16}$$

respectively. Then the strong convergence in V_A of u^ε to u^0 holds true.

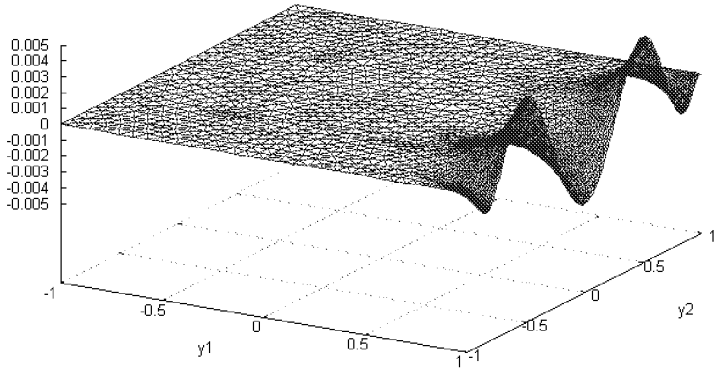
Proposition 2.4. *The convergence in V_A does not imply convergence in the distribution sense, hence V_A is not contained in $\mathcal{D}'(\Gamma)$. Moreover, when F belongs to $H^{-m}(\Gamma)$ but not to $C^\infty(\Gamma)$ (or more generally when F is not in the range of A), the solution u of the limit problem is not a distribution.*

Thin elliptic shells

when a part of the boundary is is free

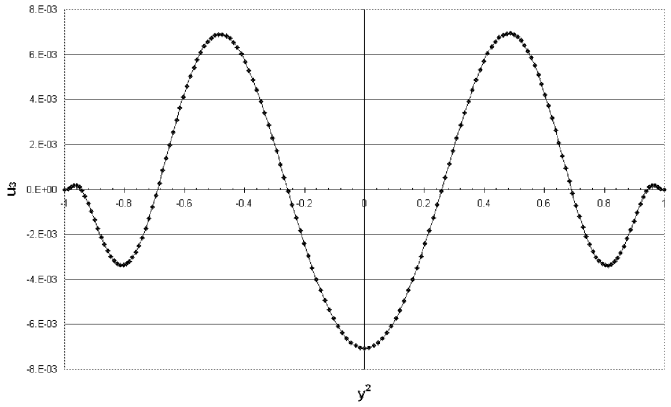


Deformed shape for $\varepsilon=10^{-4}$

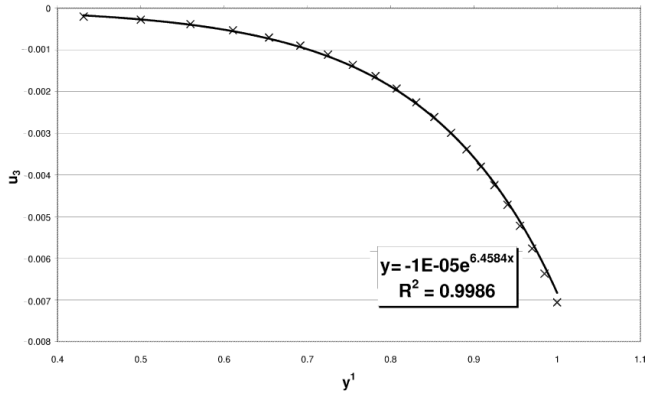


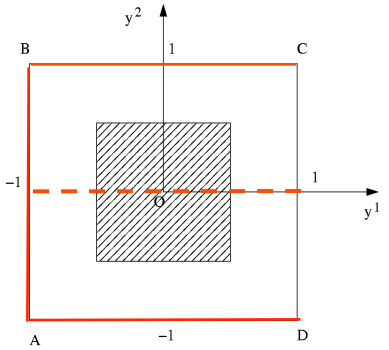
Displacement u_3 for $\varepsilon=10^{-4}$

- The Shapiro-Lopatinskii condition is not satisfied
- Large oscillations along the free edge



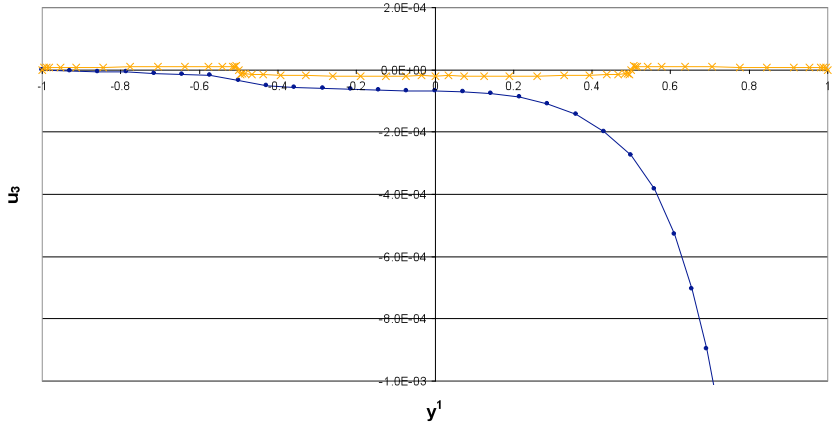
Displacement u_3 for $y^1=1$ and $\varepsilon=10^{-4}$





⇨ Oscillations exponentially decreasing towards the interior of the domain

- Comparison with the classical case



u_3 on the line $y^2=0$

- The singularities presented in the well-inhibited case are still present
- They are hidden by the large instability appearing near the free edge

The main asymptotics is non-local, near the free bondary, whatever the loading

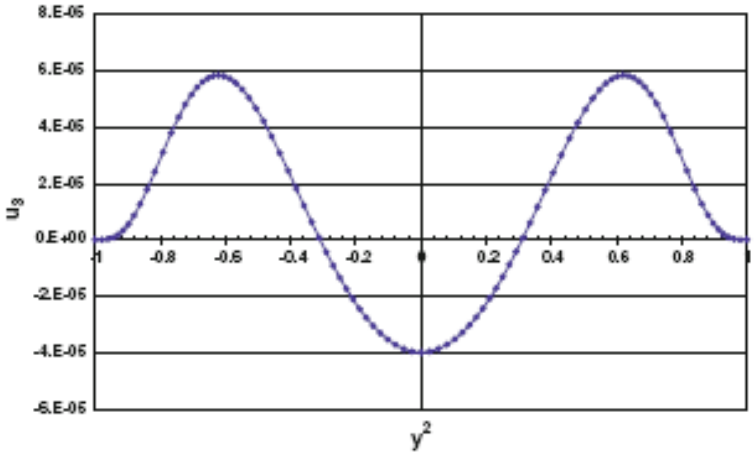


Fig. 15 Displacement u_3 on the line $y^1 = 1$ for $\varepsilon = 10^{-2}$

E = 1/100

E = 1/10000

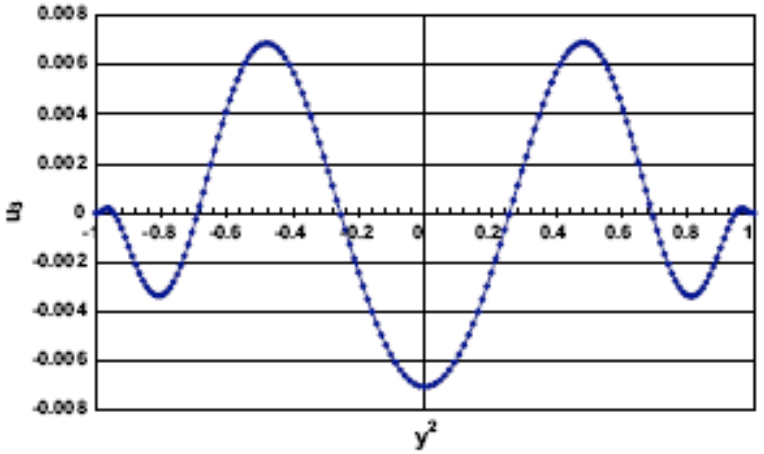


Fig. 17 Displacement u_3 on the line $y^1 = 1$ for $\varepsilon = 10^{-4}$

The sequence consists in some kind of very large oscillations with moderately small wave length.

The limit is well defined in Z'. It amounts to a global (non local) singularity.

The “logarithmic counter” effect

- We consider an equation of the form
- (1) $(S + \varepsilon D)\, u_\varepsilon = f$

- containing the small parameter ε. The unknown is u_ε, a function of the variable x, and f is a given function of x. D is a differential (or pseudodiff) operator (of order 2,for instance), whereas S is a smoothing operator (transforming any function or distribution in a smooth function). S behaves oppositely to a differential operator (which degrades the regularity). For a given non-smooth f, we see that the solution u_ε peut may exist (and do exist under appropriate hypotheses) for $\varepsilon > 0$, but cannot exist for $\varepsilon = 0$. It appears that u_ε becomes more and more crumpled as ε decreases and desappearas a distribution (becomes « infinitely crumpled ») for $\varepsilon = 0$.
-

Tipically, $\log(1/\varepsilon)$ merges as a measure of the number of oscillations describing the « crumpling » of u_ε. For instance, using (when it is possible) a Fourier decomposition of the functions without interaction between different components. If D is -the second order derivative, it multiplies the components by n^2 and S operates by multiplying the components n by e^{-n}, equation (1) becomes:

(1) $$(e^{-n} + \varepsilon\, n^2)\, u^{\varepsilon n} = f^n \,.$$

The parenthesis takes the smallest values when both terms are of the same order, i. e. for n of order $-\log \varepsilon$, so, for these n, $u_{\varepsilon n}$ is very large with respect to f_n.

This amounts to saying that fourier components with
n of order – log ε are very enhanced.
Provided they are present in f
(for instance, if f is a white noise = Dirac mass),
the solution is mostly done with components ot that order.

Tipically, the solution exhibites a number of maxima and minima proportional to – log ε whatever the loading.

Asgggg

As an example, the plots of the deformation of a boundary of the above considered shell:

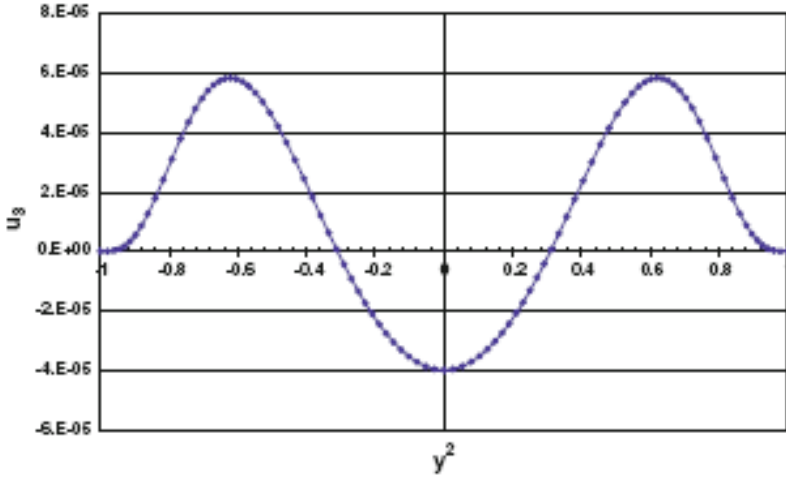


Fig. 15 Displacement u_3 on the line $y^1 = 1$ for $\varepsilon = 10^{-2}$

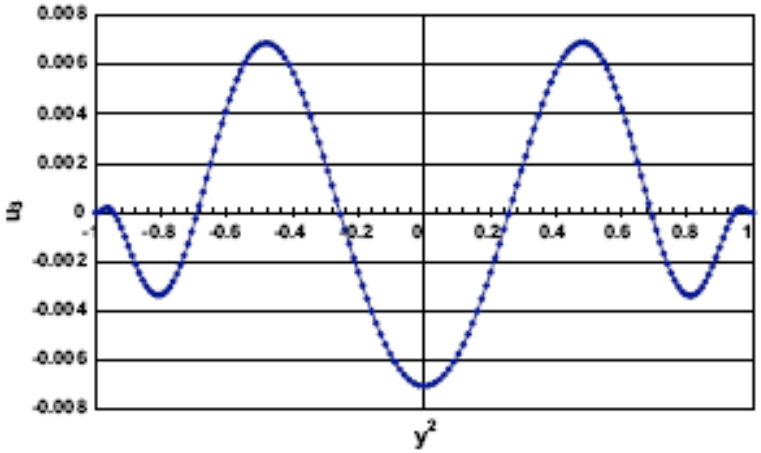


Fig. 17 Displacement u_3 on the line $y^1 = 1$ for $\varepsilon = 10^{-4}$

We observe that for $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-4}$, the number N of maxima and minima are 3 and 5 respectively,
precisely $N = \log(10/\varepsilon)$.

More general cases are analogous

$$(A+\varepsilon^2 B)\theta(x) = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{i\xi x}[a(x,\xi)+\varepsilon^2 b(x,\xi)]\tilde{\theta}(\xi)\,\mathrm{d}\xi.$$

With a ”exponentially decreasing” and “b algebraically increasing” symbols for large ξ
For fixed small ε, the solution uses A for bounded ξ and B for large ξ.

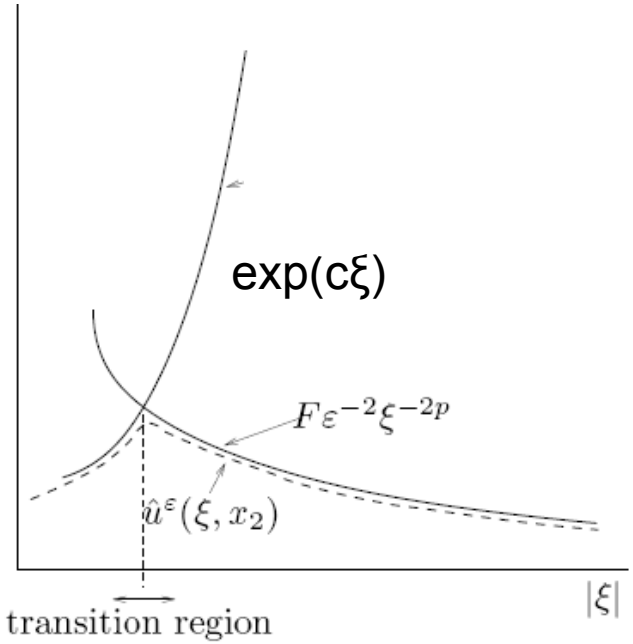


Figure 5.1: $\hat{u}^\varepsilon(\xi, x_2)$ for fixed $\varepsilon > 0$

In fact, most of the contribution
Is concerned with the

transition region
ξ = O[log(1/ε)]

The asymptotics is governed by the new « moderately large » parameter log(1/ε) which emerges naturally.

--The unusual features of these problems come from the fact that in usual PDE theory, (for propagation of singularities or other) smoothing operators are neglected, whereas here the limit behavior is described by such an operator.

--As a matter of fact, a "roughly quantitative" account for the value of $-\log \varepsilon$ ***is exhibited by any (or almost any) solution. In this sense, the system operates a translation from a continuous variable*** ε ***to a discrete (and physically explicit) account of its logarithm.***

Comments and related properties in biology

- ***Obviously, a «logarithmic counter» may be constructed by taking the inverse of any exponential process. The originality of this one relies on static linear processes (nothing to do with an evolution exponential in time).***
- I my opinion, the celebrated Weber-Fechner

(experimental and subject dependent) law

concerning stimulus and sensation has

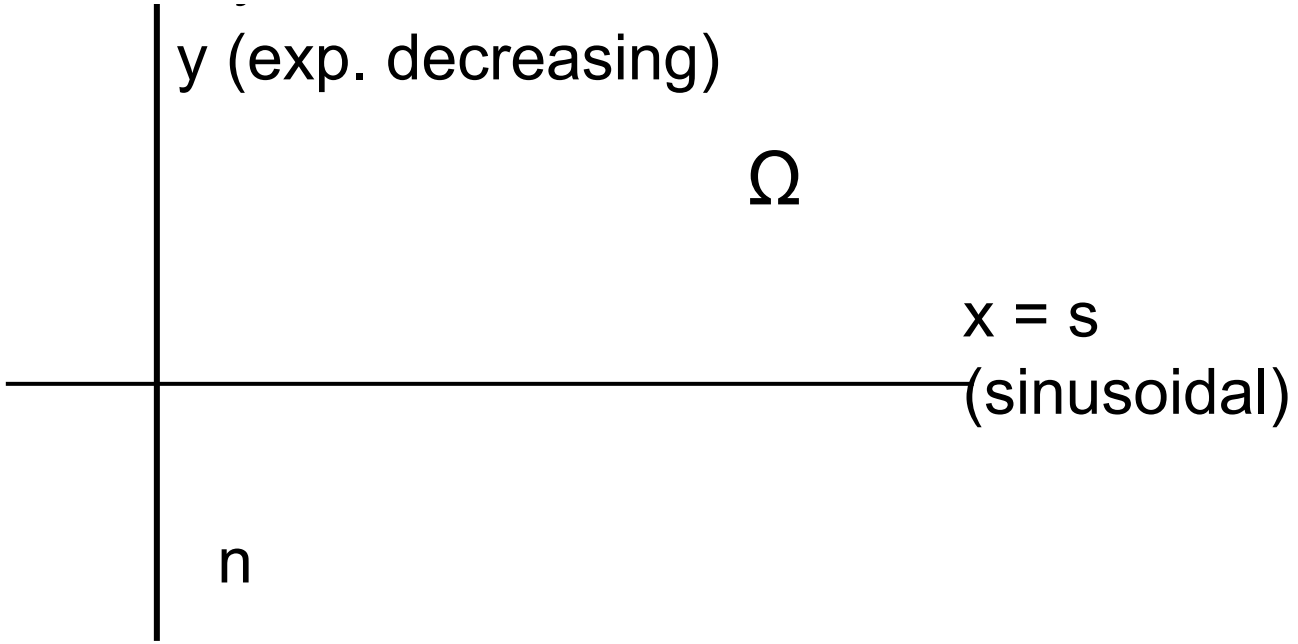
little relation with our “logarithmic counter”.

This "logarithmic counter" should perhaps be used as a very abstract (even allegoric model) accounting for:

- *Biology of developpement. M*athématical model for numerical instructions.Namely, a thickness ε of the shell (or other items) defines (roughly defined) in some sense a «well defined integer», the number of maxima or minima of its deformation under any ulterior imput. For instance the number of petals of a flower, roughly defined by a thickness or other physical parameter, which may vary fairly without modifying the integer part of its logarthm.
- Analogies with the « logarithmic counters» which likely define the mnesic traces of large numbers, as it seems that our conepts of large numbers follow a logarithmic law (S. Dehaene). In addition, long range memory seems rely on chemically fixed patterns, oppositely to short range memory, where dynamics likely plays the main role.

The shapiro – Lopatinskii condition for boundary conditions of elliptic equations (or systems)

The SL condition amounts to the adequation of the BC to the equation. More precisely, considering the problem in a half-space (with 0 data, for the equation and the BC) if a non-zero solution exponentially decreasing inwards the domain exists(resp. does not exist) it is said that the SL condition is not (resp. is) satisfied. SL impeaches the existence of such solutions with 0 data.



Let us consider the limit (membrane) system in the elliptic case (with a free portion Γ_1)

$$\begin{cases} -D_1 T^{11} - D_2 T^{12} = f^1 \\ -D_1 T^{12} - D_2 T^{22} = f^2 \\ -b_{11} T^{11} - 2b_{12} T^{12} + b_{22} T^{22} = f_3 \end{cases}$$

Where the membrane tensions are linked to the displacements by relations which may be written:

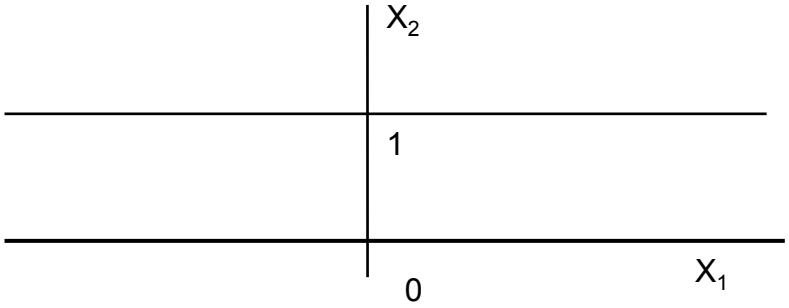
$$\begin{cases} D_1 u_1 - b_{11} u_3 = C_{11\alpha\beta} T^{\alpha\beta} \\ (1/2)[D_2 u_1 + D_1 u_2] - b_{12} u_3 = C_{12\alpha\beta} T^{\alpha\beta} \\ D_2 u_2 - b_{22} u_3 = C_{22\alpha\beta} T^{\alpha\beta} \end{cases}$$

With the boundary conditions on Γ:

$$\begin{cases} T^{11}n_1 + T_{12}n_2 = 0 \\ T^{12}n_1 + T_{22}n_2 = 0 \end{cases}$$

It appears that the existence of a solution implies in particular the solution of a Cauchy elliptic problem (in T^{11} and T^{12} for instance) which is ill-posed and has no solution in general.
This is apparent using the first part of the system with the boundary conditions and eliminating T^{22};
this forms an elliptic system
of first order, (total order 2) with Cauchy's conditions on Γ_1.
If this problem is supposed to have a solution, the second subsystem (in u) is again an elliptic pb with Cauchy data on Γ_0

Model Cauchy problem for an elliptic equation



$$\Delta v = 0$$

With the Cauchy's conditions

$$\begin{cases} v(x_1, 0) = \psi(x_1) = \delta(x_1) \\ \partial_2 v(x_1, 0) = 0 \end{cases}$$

The Fourier transform from x into ξ gives

$$(\partial_2^2 - \xi^2)\hat{v}(\xi, x_2) = 0$$

With

$$\begin{cases} \hat{v}(\xi, 0) = 1 \\ \partial_2 \hat{v}(\xi, 0) = 0 \end{cases}$$

The solution is

$$\hat{v}(\xi, x_2) = cosh(\xi x_2)$$

But the inverse Fourier transform is impossible
in function theory and even
in distribution theory, as the growth at infinity is not tempered
(high frequency components too much important,
then the function is too much singular).

The space of (direct or inverse) Fourier transforms of (non tempered) distributions is denoted by *Z'*.

It is a space of *analytic functionals*
(the test functions are analytic).
Localization is impossible as the support of the test- functions is the whole R.

Heuristic asymptotics

- In order to have an approximate description of the asymptotics we use several ideas
- -First, minimization heuristics shows that the solution should have very small membrane energy. It is described by inextensional displacement fields slightly modified by a boundary layer on Γ_0 accounting for membrane energy. This layer is narrow for high frequency Fourier components. Another boundary layer on Γ_1 accounts for bending energy (according to the "two elliptic cauchy pbs" on Γ_1 and Γ_0).
- -Solving the system of inextensionality, these displacements may be described in terms of their traces on Γ1 (in particular this induces non – locality).
- -As the inextensionality system is elliptic, regularity theory for elliptic systems shows that the operator describing the functions in a neighbourhood of Γ_0 in terms of traces on Γ_1 is smoothing (= sending any function to a smooth function).

The asymptotics is then described by a system of the form:

$$\begin{cases} \text{Find } u^\varepsilon \in V \text{ such that, } \forall v \in V \\ \alpha(u^\varepsilon, v) + \varepsilon^2 \beta(u^\varepsilon, v) = \langle F, v \rangle, \end{cases}$$

with

$$\alpha(u,v) = \int_\Gamma Su\ \overline{Sv}\ \mathrm{d}x,$$

$$\beta(u,v) = \int_\Gamma p(x,D)u\ \overline{p(x,D)v}\ \mathrm{d}x.$$

On the free boundary, S being smoothing
and p pseudodiff of order 3/2.
There are no boundaries, and the problem
is equivalent to an equation of the previous form.

Comments on singularities, functions, distributions and analytic functionals.

Let us comment on functions and their generalizations (distributions, analytical functionals), on $\mathbb{R}$ of the variable x, as well as their corresponding Fourier transforms of the variable ξ.

Let us consider the distributions formed by the Dirac mass and its derivatives:

$$\delta(x),\ \delta'(x),\ \delta''(x), ... \tag{2.1}$$

which may be considered as singularities of increasing order. They belong to the Sobolev spaces of negative order $H^{-s}(\mathbb{R})$ with $s > 1/2, > 3/2, > 5/2, ...$respectively (we may as well consider the H spaces on any domain containing 0) which satisfy

$$H^{-s}(\mathbb{R}) \subset H^{-r}(\mathbb{R}) \quad \text{for} \quad s < r \tag{2.2}$$

We then may define a function (or distribution) f to be more singular than another g when g belongs to some H^{-s} which does not contain f. We may consider singularity either on $\mathbb{R}$ or on any open set Ω, taking the $H^{-s}(\mathbb{R})$ or $H^{-s}(\Omega)$, respectively. All these spaces are contained in the space of distributions $D'(\mathbb{R})$ or $D'(\Omega)$, which obviously contain functions or distributions of any (finite) order s (according to the above definition, i. e., belonging to $H^{-r}(\Omega)$ for $r > s$, not to $H^{-s}(\Omega)$).

We may imagine an expression of the form

$$u(x) = \sum_{n=1}^{+\infty} c_n \delta^{(n)}(x) \tag{2.3}$$

which is (provided it makes sense) of order infinity. But, the spaces of distributions does not contain elements of such kind.

There is another way of defining higher and lower degrees of singularity using the Fourier transform. It is classical that the rapid decreasing of $\hat{f}(\xi)$ at infinity measures the smoothness of $f(x)$. Equivalently, the growing of $\hat{f}(\xi)$ at infinity measures the singularity of $f(x)$. An example is given by the Fourier transforms of (2.1):

$$1_\xi, \quad i\xi, \quad (i\xi)^2, ... \tag{2.4}$$

This allows to define "very singular entities" as the (inverse) Fourier transforms of functions (or distributions) "very rapid increasing" at infinity. For instance, the function

$$\cosh \xi = \hat{u}(\xi) = \sum_{n=1}^{+\infty} \frac{\xi^{2n}}{(2n)!} \tag{2.5}$$

which is obviously a distribution on $\mathbb{R}_\xi$, i. e. an element of $D'(\mathbb{R}_\xi)$, has formally the inverse Fourier transform

$$u(x) = \sum_{n=1}^{+\infty} \frac{1}{(2n)!} (-i)^{2n} \delta^{(2n)}(x) \tag{2.6}$$

which, as we know, is not an element of $D'(\mathbb{R}_x)$.

Elements of Z' have no support (as the test functions in Z are analytic = with support the whole line). An expression as

$$\sum_{0}^{+\infty} a_n \delta^{(n)}(x) \tag{1}$$

may also be written as

$$\sum_{0}^{+\infty} b_n \delta^{(n)}(x-c) \tag{2}$$

with any given c and appropriate b_n. Properties described by elements of Z' are essentially non - local.

Example

$$\hat{u}^{\lambda}(\xi) = \begin{cases} \cosh\xi & \text{for} \quad |\xi| < \lambda \\ 0 & \text{for} \quad |\xi| > \lambda \end{cases} \tag{3.1}$$

so that as $\lambda \to \infty$

$$\hat{u}^{\lambda}(\xi) \to \hat{u}(\xi) \quad \text{in} \quad D'(\mathbb{R}_{\xi}). \tag{3.2}$$

Consequently the inverse Fourier transforms satisfy

$$u(x)^{\lambda} \to u(x) \quad \text{in} \quad Z'(\mathbb{R}_x) \tag{3.3}$$

where

$$\begin{cases} u^{\lambda}(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\lambda}^{\lambda} \cosh\xi.\cos(\xi x)d\xi = \\ \dfrac{1}{2\pi(x^2+1)}\left\{e^{\lambda}\left[\cos(\lambda x) + x\sin(\lambda x)\right] - e^{-\lambda}\left[\cos(\lambda x) - x\sin(\lambda x)\right]\right\}. \end{cases} \tag{3.4}$$

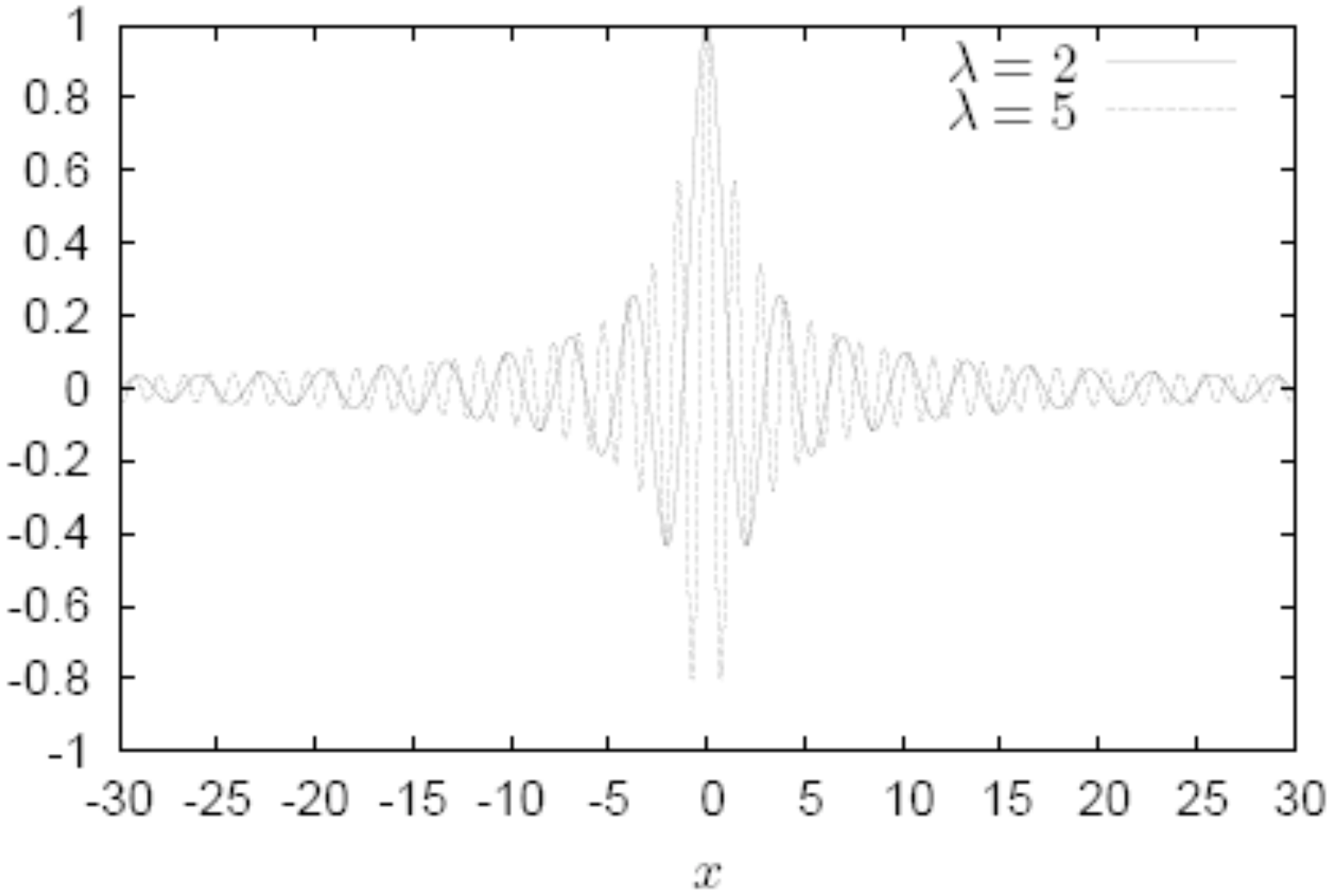
As we are interested by the behavior for $\lambda \nearrow \infty$, we may discard the term with factor $e^{-\lambda}$, so that

$$u^{\lambda}(x) \simeq \frac{e^{\lambda}}{2\pi}\psi^{\lambda}(x) \tag{3.5}$$

with

$$\psi^{\lambda}(x) = \frac{1}{(x^2+1)}[\cos(\lambda x) + x\sin(\lambda x)]. \tag{3.6}$$

The function ψ_λ



Another mechanical system with analogous properties

We consider strictly two-dimensional isotropic elasticity with
$0<K$ =modulus of uniform compressibility
$0<\mu$ =shear modulus.
When K tends to 0 (infinitely soft for uniform compression),
the limit problem is singular in the same way
that (elliptic partially clamped) shells.
The small parameter is $K=\varepsilon$.
This limit amounts to Poisson ratio v= -1(when pulling in a direction,
the material dilates in both directions in the same amount.
« Natural » materials have positive Poisson ratio, but v= -1
may be « constructed » with an appropriate microstructure
(and homogenization).

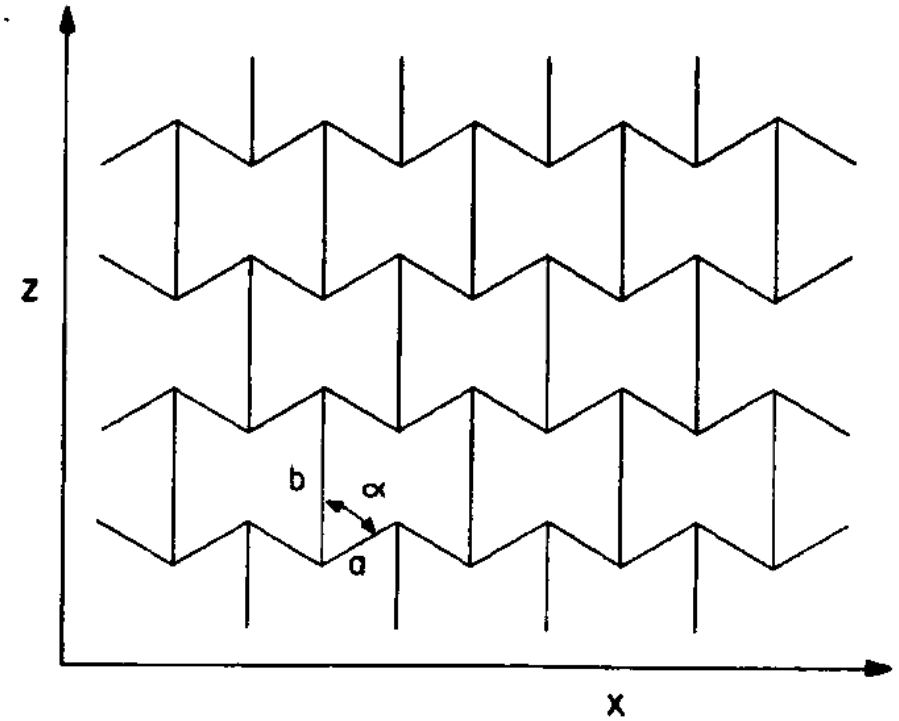
The mathematical reason of this singular behavior is that for v= -1 there is a relation between the components of the stress tensor $(\sigma_{11} + \sigma_{22})= 0$ as the material is infinitely soft. Namely

$\sigma_{11} = \mu(e_{11} - e_{22})$
$\sigma_{22} = \mu(e_{22} - e_{11})$
$\sigma_{12} = 2\mu\, e_{12}$

As a matter of fact, they are only two effective components of σ, and Neumann b. c. amount to Cauchy data for the (elliptic!) system of elasticity with the (two!) unknowns σ_{11} and σ_{12} :
$-\sigma_{11,1} - \sigma_{12,2} = f_1$
$-\sigma_{12,1} + \sigma_{11,2} = f_2$
so that the Neumann b. c. is pathological in the same way as in shells, as it involves a Cauchy elliptic problem.
Asymptotics follows the same trends.

In fact, the very reason of this special behavior is the same as in shells, as the normal component of the displacement acts locally as some kind of weakness to compression

The basic ingredient for constructing materials with Poisson ratio = -1 is:



Basic structure—two dimensions

(Rigid bars with elastic deformation of angles and $\alpha_0 = \pi/4$).

Such kind of micro-structures are known even in 3-dimensional elasticity, and used for modeling (almost infinitely compressible) foams.