Skin-Effect Description in Electromagnetism with a Scaled Asymptotic Expansion

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References

GABRIEL CALOZ, MONIQUE DAUGE, VICTOR PÉRON (2009)
Uniform Estimates for Transmission Problems with High Contrast in Heat Conduction and Electromagnetism

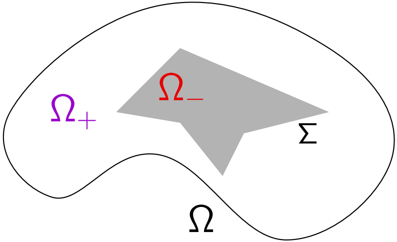
References

Gabriel Caloz, Monique Dauge, Victor Péron (2009)
Uniform Estimates for Transmission Problems with High Contrast in Heat Conduction and Electromagnetism

Monique Dauge, Erwan Faou, Victor Péron (2009)
Asymptotic Behavior at High Conductivity of Skin Depth in Electromagnetism

Transmission Problems with High Contrast Media

ELECTROMAGNETISM

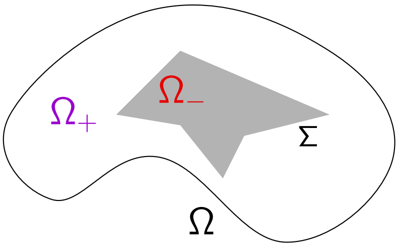


- Ω_- Highly conducting material $\subset\subset \Omega$: Conductivity $\sigma_- \equiv \sigma \gg 1$
- $\Sigma = \partial\Omega_-$: Interface
- Ω_+ Insulating or dielectric body: Conductivity $\sigma_+ = 0$

Aim: Understand the electromagnetic phenomenon as $\sigma \to \infty$

Transmission Problems with High Contrast Media

HEAT CONDUCTION



- $\Omega_- \subset\subset \Omega$: heat conductivity a_-
- $\Sigma = \partial\Omega_-$: Interface
- Ω_+: heat conductivity a_+

Aim: Describe the transmission phenomenon as $\left| a_- / a_+ \right| \gg 1$

Outline

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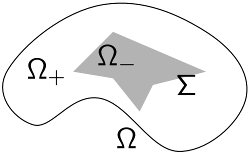
1. **Uniform Estimates for Transmission Problems**
2. 3D Multiscaled Asymptotic Expansion
3. Numerical Simulations

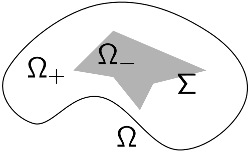
Transmission Problems in High Contrast Media



- Heat transfer equation $(\mathbf{P}_{\underline{a}})$: $\quad \operatorname{div} \underline{a}\, \mathbf{grad}\, \varphi = f$

$$\underline{a} = (a_+, a_-) \quad \text{s.t.} \quad |a_-|/|a_+| \to \infty$$

Transmission Problems in High Contrast Media



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$$\underline{a} = (a_+, a_-) \quad \text{s.t.} \quad |a_-|/|a_+| \to \infty$$

- Maxwell equations $(\mathbf{P}_{\underline{\sigma}})$:
  $\operatorname{curl} \mathbf{E} - i\omega\mu_0 \mathbf{H} = 0$ and $\operatorname{curl} \mathbf{H} + (i\omega\varepsilon_0 - \underline{\sigma})\mathbf{E} = \mathbf{j}$

$$\underline{\sigma} = (\sigma_+, \sigma_-) \quad \text{s.t.} \quad \sigma_+ = 0 \quad \text{and} \quad \sigma_- \equiv \sigma \to \infty$$

Issues

1. *Uniform piecewise estimates and regularity* in Sobolev norms for solutions φ of $\left(\mathbf{P}_{\underline{a}}\right)$
2. *Uniform* L^2 *estimates* for solutions $\left(\mathbf{E},\mathbf{H}\right)$ of $\left(\mathbf{P}_{\underline{\sigma}}\right)$

Issues

1. *Uniform piecewise estimates and regularity* in Sobolev norms for solutions φ of $\left(\mathbf{P}_{\underline{a}}\right)$
2. *Uniform* L^2 *estimates* for solutions $\left(\mathbf{E},\mathbf{H}\right)$ of $\left(\mathbf{P}_{\underline{\sigma}}\right)$

Hypothesis

Σ *is a bounded Lipschitz surface in* $\mathbb{R}^3$

Scalar case:

- H.P. Huy, E. Sanchez-Palencia (1974)
  Limit problem and Strong convergence results as $a_-\big/a_+ \to \infty$
- S. Hassani, S. Nicaise, A. Maghnouji (2009)

Maxwell case:

- H. Haddar, P. Joly, H.N. Nguyen (2008)
  Uniform estimates under a stronger regularity assumption on Σ

Scalar Problem

Variationnal Problem

$$V_\mathrm{D} = \mathrm{H}^1_0(\Omega) \quad \text{or} \quad V_\mathrm{N} = \{\varphi \in \mathrm{H}^1(\Omega) \mid \int_\Omega \varphi \,\mathrm{d}\mathbf{x} = 0\}$$

Scalar Problem

Variationnal Problem

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$(\mathbf{VP}_{\underline{a}})$: Find $\varphi \in V = V_{\mathrm{D}}$ or $V = V_{\mathrm{N}}$ such that

$$\forall \psi \in V, \quad \int_{\Omega_+} a_+ \nabla\varphi^+ \cdot \nabla\overline{\psi}^+ \, \mathrm{d}\mathbf{x} + \int_{\Omega_-} a_- \nabla\varphi^- \cdot \nabla\overline{\psi}^- \, \mathrm{d}\mathbf{x} = - \int_\Omega f \, \overline{\psi} \, \mathrm{d}\mathbf{x} + (a_+ - a_-) \int_\Sigma g \, \overline{\psi} \, \mathrm{d}s$$

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Hypothesis $(\mathrm{H}_{\underline{a}})$

1. $f \in \mathrm{L}^2(\Omega)$ *and* $g \in \mathrm{L}^2(\Sigma)$
2. $\int_\Omega f \, \mathrm{d}\mathbf{x} = \int_\Sigma g \, \mathrm{d}s = 0$ *if* $V = V_{\mathrm{N}}$ *and* $\int_\Sigma g \, \mathrm{d}s = 0$ *if* $V = V_{\mathrm{D}}$

Uniform Estimates

Theorem

Let us assume that $a_+ \neq 0$*. There exist* $\rho_0 > 0$ *independent of* a_+ *such that for all* $a_- \in \{z \in \mathbb{C} \mid |z| \geqslant \rho_0 |a_+|\}$*, and for all data* (f, g) *satisfying* $(\mathbf{H}_{\underline{a}})$*,* $(\mathbf{VP}_{\underline{a}})$ *has a unique solution* $\varphi \in V$*, which is piecewise* $\mathrm{H}^{3/2}$ *and*

$$\|\varphi^+\|_{\frac{3}{2},\Omega_+} + \|\varphi^-\|_{\frac{3}{2},\Omega_-} \leqslant C_{\rho_0}\left(|a_+|^{-1}\|f\|_{0,\Omega} + \|g\|_{0,\Sigma}\right)$$

with $C_{\rho_0} > 0$*, independent of* a_+*,* a_-*,* f*, and* g*.*

Gabriel Caloz, Monique Dauge, V. Péron (2009)

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$$\|\varphi^+\|_{\frac{3}{2},\Omega_+} + \|\varphi^-\|_{\frac{3}{2},\Omega_-} \leqslant C_{\rho_0}\left(|a_+|^{-1}\|f\|_{0,\Omega} + \|g\|_{0,\Sigma}\right)$$

with $C_{\rho_0} > 0$, *independent of* a_+, a_-, f, *and* g.

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Key of the proof: asymptotic expansion for φ with respect to the powers of $\rho^{-1} := a_+(a_-)^{-1}$ and *convergence* of these series in the piecewise $\mathrm{H}^{3/2}$-norm

Remarks

1. Similar estimate when the roles of a_+ and a_- are exchanged. More precise estimate where a_+ and a_- play symmetric roles
2. The assumption that Σ is Lipschitz is necessary
3. For *polyhedral* Lipschitz interface Σ, uniform piecewise H^s estimates:

$$\|\varphi^+\|_{s,\Omega_+} + \|\varphi^-\|_{s,\Omega_-} \leqslant C_{\rho_0}\big(|a_+|^{-1}\|f\|_{s-2,\Omega} + \|g\|_{s-\frac{3}{2},\Sigma}\big)$$

for $s \leqslant s_\Sigma$ with some $\frac{3}{2} < s_\Sigma \leqslant 2$

4. For *smooth* interface Σ, uniform piecewise H^s estimates for any $s \geqslant 2$

Maxwell Problem

Framework and Boundary Conditions

$$(\mathbf{P}_{\underline{\sigma}}) \qquad \operatorname{curl}\mathbf{E} - i\omega\mu_0\mathbf{H} = 0 \quad \text{and} \quad \operatorname{curl}\mathbf{H} + (i\omega\varepsilon_0 - \underline{\sigma})\mathbf{E} = \mathbf{j} \qquad \underline{\sigma} = (0, \sigma)$$

1. $\mathbf{E} \times \mathbf{n} = 0$ and $\mathbf{H} \cdot \mathbf{n} = 0$ on $\partial\Omega$
2. $\mathbf{E} \cdot \mathbf{n} = 0$ and $\mathbf{H} \times \mathbf{n} = 0$ on $\partial\Omega$

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1. B. C. 1: $\mathbf{j} \in \mathbf{H}(\mathrm{div},\Omega) = \{\mathbf{u}\in\mathbf{L}^2(\Omega) \mid \operatorname{div}\mathbf{u}\in \mathrm{L}^2(\Omega)\}$
2. B. C. 2: $\mathbf{j} \in \mathbf{H}_0(\mathrm{div},\Omega) = \{\mathbf{u}\in\mathbf{H}(\mathrm{div},\Omega) \mid \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}$

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Hypothesis (*SH*)

The angular frequency ω is not an eigenfrequency of the problem

$$\begin{cases} \operatorname{curl}\mathbf{E} - i\omega\mu_0\mathbf{H} = 0 \quad \textit{and} \quad \operatorname{curl}\mathbf{H} + i\omega\varepsilon_0\mathbf{E} = 0 & \textit{in} \quad \Omega_+ \\ \mathbf{E} \times \mathbf{n} = 0 \quad \textit{and} \quad \mathbf{H} \cdot \mathbf{n} = 0 & \textit{on} \quad \Sigma \\ B.C.1 \quad \textit{or} \quad B.C.2 & \textit{on} \quad \partial\Omega \end{cases}$$

Maxwell Problem

Uniform $\mathbf{L}^2(\Omega)$ Estimate

Theorem

Under Hypothesis (SH), there are σ_0 and $C > 0$, such that for all $\sigma \geqslant \sigma_0$, the Maxwell problem $(\mathbf{P}_{\underline{\sigma}})$ with B.C.2 and $\mathbf{j} \in \mathbf{H}_0(\mathrm{div}, \Omega)$ has a unique solution $(\mathbf{E}, \mathbf{H})$ in $\mathbf{L}^2(\Omega)^2$, which satisfies:

$$\|\mathbf{E}\|_{0,\Omega} + \|\mathbf{H}\|_{0,\Omega} + \sqrt{\sigma}\, \|\mathbf{E}\|_{0,\Omega_-} \leqslant C \|\mathbf{j}\|_{\mathbf{H}(\mathrm{div},\Omega)}$$

Similar result for B.C.1 and $\mathbf{j} \in \mathbf{H}(\mathrm{div}, \Omega)$

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Application: Convergence of asymptotic expansion at high conductivity

Maxwell Problem

Proof of Maxwell Uniform Estimate

Lemma

Under Hypothesis (SH)*, there are* σ_0 *and* $C_0 > 0$ *such that if* $\sigma \geqslant \sigma_0$ *any solution* $(\mathbf{E}, \mathbf{H}) \in \mathbf{L}^2(\Omega)^2$ *of problem* $(\mathbf{P}_{\underline{\sigma}})$ *with B.C.2 and data* $\mathbf{j} \in \mathbf{H}_0(\mathrm{div}, \Omega)$ *satisfies the estimate*

$$\|\mathbf{E}\|_{0,\Omega} \leqslant C_0 \|\mathbf{j}\|_{\mathbf{H}(\mathrm{div},\Omega)}$$

*Similar statement for B.C.*1 *and* $\mathbf{j} \in \mathbf{H}(\mathrm{div}, \Omega)$*.*

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*Similar statement for B.C.*1 *and* $\mathbf{j} \in \mathbf{H}(\mathrm{div}, \Omega)$*.*

Keys of the proof:

- Electrical Field Decomposition into a regular field and a gradient field
  - C. Amrouche, C. Bernardi, M. Dauge, V. Girault (1998)
- Scalar Estimates used for the gradient field $\nabla \varphi$

Outline

References and Notations

Asymptotic expansion as $\sigma \longrightarrow \infty$ of solutions of $\left(\mathbf{P}_{\underline{\sigma}}\right)$ *when* Σ *is smooth*:

- E. Stephan, R.C. McCamy (1983-84-85)
  Plane Interface and Eddy Current Approximation
- H. Haddar, P. Joly, H.N. Nguyen (2008)
  Smooth Interface and Impedance boundary conditions
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  Smooth Interface and Perfectly Conducting Electric or Magnetic Wall B.C.

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Hypothesis

1. Σ *is a smooth surface*
2. *Perfectly Conducting Magnetic Wall b.c.*
3. $\mathbf{j} \in \mathbf{H}_0(\mathrm{div}, \Omega)$ *and* $\mathbf{j} = 0$ *in* Ω_-
4. *Hypothesis (SH) on* ω

Asymptotic Expansion

$$\delta := \sqrt{\frac{\omega\varepsilon_0}{\sigma}} \rightarrow 0 \quad \text{as} \quad \sigma \rightarrow \infty$$

By Theorem there exists δ_0 s.t. for all $\delta \leqslant \delta_0$, there exists a unique solution $\big(\mathbf{E}_{(\delta)}, \mathbf{H}_{(\delta)}\big)$ to $\big(\mathbf{P}_{\underline{\sigma}}\big)$

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Then it is possible to construct series expansions in powers of δ:

$$\mathbf{E}^{+}_{(\delta)}(\mathbf{x}) \approx \sum_{j\geqslant 0} \delta^j \mathbf{E}^{+}_j(\mathbf{x}), \qquad \mathbf{E}^{-}_{(\delta)}(\mathbf{x}) \approx \chi(y_3) \sum_{j\geqslant 0} \delta^j \mathbf{W}_j(y_\beta, \frac{y_3}{\delta})$$

$$\mathbf{E}^{+}_j \in \mathbf{H}(\mathrm{curl}, \Omega_+) \quad , \quad \mathbf{W}_j \in \mathbf{H}(\mathrm{curl}, \Sigma \times \mathbb{R}_+) \quad \textit{profiles}$$

(y_β, y_3): "*normal coordinates*" to Σ in a tubular neighborhood $\mathcal{U}_-$ of Σ in Ω_-

Asymptotic Expansion

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Similar result for the magnetic field.

Validation of the asymptotic expansion

Remainders

Aim: proving estimates for remainders $\mathbf{R}_{m;\,\delta}$

$$\mathbf{R}_{m;\,\delta} := \mathbf{E}_{(\delta)} - \sum_{j=0}^{m} \delta^j \mathbf{E}_j \quad \text{in} \quad \Omega$$

Evaluation of the right hand side when the Maxwell operator is applied to $\mathbf{R}_{m;\,\delta}$:

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{R}^+_{m;\,\delta} - \kappa^2\alpha_+\mathbf{R}^+_{m;\,\delta} & = & 0 & \text{in} & \Omega_+ \\ \operatorname{curl}\operatorname{curl}\mathbf{R}^-_{m;\,\delta} - \kappa^2\alpha_-\mathbf{R}^-_{m;\,\delta} & = & \mathbf{j}^-_{m;\,\delta} & \text{in} & \Omega_- \\ \left[\mathbf{R}_{m;\,\delta}\times\mathbf{n}\right]_\Sigma & = & 0 & \text{on} & \Sigma \\ \left[\operatorname{curl}\mathbf{R}_{m;\,\delta}\times\mathbf{n}\right]_\Sigma & = & \mathbf{g}_{m;\,\delta} & \text{on} & \Sigma \\ \operatorname{curl}\mathbf{R}^+_{m;\,\delta}\times\mathbf{n} & = & 0 & \text{on} & \partial\Omega \end{cases}$$

with $\alpha_+ = 1$ and $\alpha_- = 1 + i/\delta^2$

Validation of the asymptotic expansion

Estimates for Remainders

$$\left\|\mathbf{j}^{-}_{m;\,\delta}\right\|_{2,\Omega_-} + \left\|\mathbf{g}_{m;\,\delta}\right\|_{\frac{1}{2},\Sigma} + \left\|\operatorname{curl}_\Sigma \mathbf{g}_{m;\,\delta}\right\|_{\frac{3}{2},\Sigma} \leqslant C_m \delta^{m-1}$$

with $C_m > 0$ independent of δ.

Validation of the asymptotic expansion

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with $C_m > 0$ independent of δ.

Theorem

Under Hypothesis in the framework above, for all $m \in \mathbb{N}$ and $\delta \in (0, \delta_0]$, the remainders $\mathbf{R}_{m;\,\delta}$ satisfy the optimal estimates

$$\|\mathbf{R}^{+}_{m;\,\delta}\|_{0,\Omega_+} + \|\operatorname{curl}\mathbf{R}^{+}_{m;\,\delta}\|_{0,\Omega_+} + \delta^{-\frac{1}{2}}\|\mathbf{R}^{-}_{m;\,\delta}\|_{0,\Omega_-} + \delta^{\frac{1}{2}}\|\operatorname{curl}\mathbf{R}^{-}_{m;\,\delta}\|_{0,\Omega_-} \leqslant C'_m \delta^{m+1}$$

Electrical Profiles

$\mathbf{w}_j =: (\mathcal{W}_{\alpha,j}\ ;\ w_j)$ in coordinates (y_β, Y_3) with $Y_3 = \frac{y_3}{\delta}$

Electrical Profiles

$\mathbf{W}_j =: (\mathcal{W}_{\alpha,j}\ ;\ w_j)$ in coordinates (y_β, Y_3) with $Y_3 = \frac{y_3}{\delta}$

$$\mathbf{W}_0 = 0$$

$$\mathcal{W}_{\alpha,1} = -j_{\alpha,0}(y_\beta)\,\mathrm{e}^{-\lambda Y_3} \quad \text{and} \quad w_1 = 0$$

$$\mathcal{W}_{\alpha,2} = \Big[\lambda^{-1}\big\{b_\alpha^\sigma\, j_{\sigma,0} - \mathcal{H}\, j_{\alpha,0}\big\} - j_{\alpha,1} + Y_3\big\{b_\alpha^\sigma\, j_{\sigma,0} - \mathcal{H}\, j_{\alpha,0}\big\}\Big](y_\beta)\,\mathrm{e}^{-\lambda Y_3}$$

$$w_2 = -\lambda^{-1} D_\alpha\, j_0^\alpha(y_\beta)\,\mathrm{e}^{-\lambda Y_3}$$

Electrical Profiles

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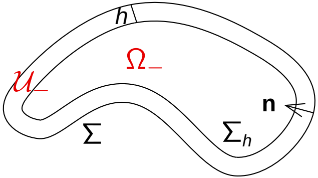
$$w_2 = -\lambda^{-1} D_\alpha\, j_0^\alpha(y_\beta)\,\mathrm{e}^{-\lambda Y_3}$$

with

$$\mathcal{H} \quad \text{mean curvature of } \Sigma$$

$$j_{\alpha,k}(y_\beta) := \lambda^{-1}(\operatorname{curl}\mathbf{E}_k^+ \times \mathbf{n})_\alpha(y_\beta, 0) \quad \text{and} \quad \lambda = \kappa\,\mathrm{e}^{-i\pi/4}$$

Curl Operator and Levi-Civita Tensor



Levi-Civita Tensor ϵ:

$$\epsilon^{ijk} = \epsilon_0(i,j,k) \Big/ \sqrt{\det\left(a_{\alpha\beta}(h)\right)}$$

$a_{\alpha\beta}(h)$: metric on Σ_h and $\epsilon_0(i,j,k) =$ sign of (i,j,k)

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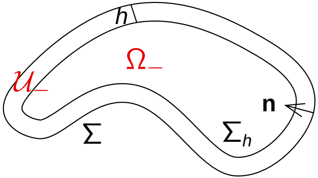

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Curl Operator in Normal Parameterization:

$$\begin{cases} (\nabla \times \mathbf{E})^\alpha = \epsilon^{3\beta\alpha}(\partial_3^h E_\beta - \partial_\beta E_3) & \text{on} \quad \Sigma_h \\ (\nabla \times \mathbf{E})^3 = \epsilon^{3\alpha\beta} D_\alpha^h E_\beta & \text{on} \quad \Sigma_h \end{cases}$$

D_α^h : Covariant Derivative on Σ_h

Magnetical Profiles

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Magnetical Profiles

$\mathbf{V}_j =: (\mathcal{V}_j^\alpha \,;\, v_j)$ in (y_β, Y_3) coordinates

$$\mathbf{V}_0(y_\beta, Y_3) = H_0(y_\beta)\,\mathrm{e}^{-\lambda Y_3}\,,$$

$$\mathcal{V}_1^\alpha(y_\beta, Y_3) = \Big[H_1^\alpha(y_\beta) + Y_3\Big(\mathcal{H}\,H_0^\alpha + b_\sigma^\alpha\,H_0^\sigma\Big)(y_\beta)\Big]\,\mathrm{e}^{-\lambda Y_3}\,,$$

$$v_1(y_\beta, Y_3) = \lambda^{-1}\mathbf{D}_\alpha\,H_0^\alpha(y_\beta)\,\mathrm{e}^{-\lambda Y_3}$$

with $H_j^\alpha(y_\beta) := (\mathbf{n} \times \mathbf{H}_j^+) \times \mathbf{n}^\alpha(y_\beta, 0)$

Magnetical Profiles

$\mathbf{V}_j =: \left(\mathcal{V}_j^\alpha\ ;\ v_j\right)$ in $\left(y_\beta, Y_3\right)$ coordinates

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Comparison with

H. HADDAR, P. JOLY, H.N. NGUYEN (2008)

Skin Effect and Skin Depth

1D model of *Skin Effect* : $\{z > 0\}$ = half-space conductor body

$$\|\mathbf{E}_\sigma(z)\| = E_0\, \mathrm{e}^{-\frac{z}{\ell(\sigma)}}\,, \quad \ell(\sigma) := \sqrt{\frac{2}{\omega\mu_0\sigma}}$$ *Skin Depth*

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$\ell(\sigma) = \delta / \operatorname{Re}\lambda\,.$

3D model of *Skin Effect*, using *profiles* $\mathbf{V}_0$, $\mathbf{V}_1$,

$$\mathbf{V}(y_\alpha, y_3) := \mathbf{V}_0(y_\alpha, \frac{y_3}{\delta}) + \delta\mathbf{V}_1(y_\alpha, \frac{y_3}{\delta})$$

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Definition

Suppose that for all y_α, $\|\mathbf{V}(y_\alpha, 0)\| \neq 0$. The *Skin Depth* in a conductor body at conductivity σ and y_α on a Regular Surface Σ is the positive real $\mathcal{L}(\sigma, y_\alpha)$ defined on Σ taking the smallest value s.t.

$$\|\mathbf{V}(y_\alpha, \mathcal{L}(\sigma, y_\alpha))\| = \frac{1}{\mathrm{e}}\|\mathbf{V}(y_\alpha, 0)\|$$

Asymptotic Behavior for Skin Depth

Theorem

Let Σ be a Regular Surface with mean curvature $\mathcal{H}$. Suppose that $H_0(y_\alpha) \neq 0$. The Skin Depth $\mathcal{L}(\sigma, y_\alpha)$ has the asymptotic expansion

$$\mathcal{L}(\sigma, y_\alpha) = \ell(\sigma)\big(1 + \mathcal{H}(y_\alpha)\,\ell(\sigma) + \mathcal{O}(\sigma^{-1})\big)$$

MONIQUE DAUGE, ERWAN FAOU, V. PÉRON (2009)

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Monique Dauge, Erwan Faou, V. Péron (2009)

Key of the proof:

$$\|\mathbf{V}(y_\alpha, y_3)\|^2 = \Big[\|H_0\|^2 + 2\delta\,\mathrm{Re}\langle H_0, H_1\rangle + 2y_3\mathcal{H}\,\|H_0\|^2 + \mathcal{O}(\delta, y_3)\Big]\,\mathrm{e}^{-2y_3/\ell(\sigma)}$$

Outline

Numerical Simulations

FEM Discretization

- Magnetic Field

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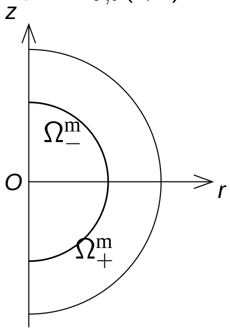

Numerical Simulations

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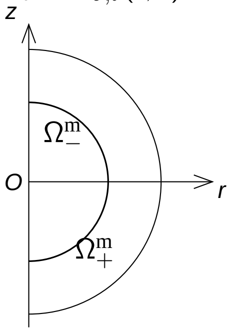
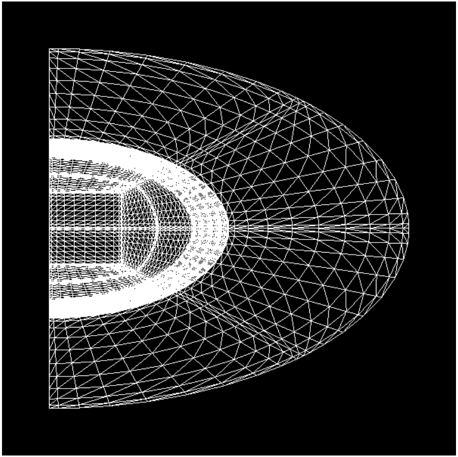


- Finite Element Library Melina

Numerical simulations

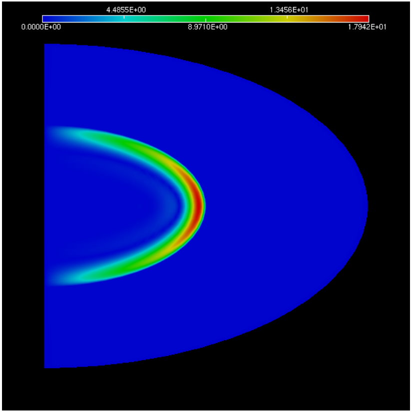
Mesh Refinement

$\color{red}{\Omega_-}$ & Ω coaxial ellipsoidal domains $\longrightarrow$ $\color{red}{\Omega^{\mathrm{m}}_-}$ & Ω^{m} coaxial half-ellipsis

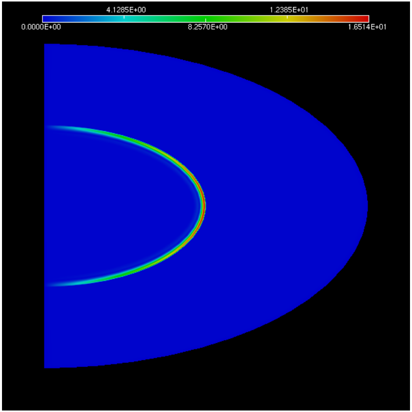


Numerical Simulations

Skin Effect in ellipsoidal geometry



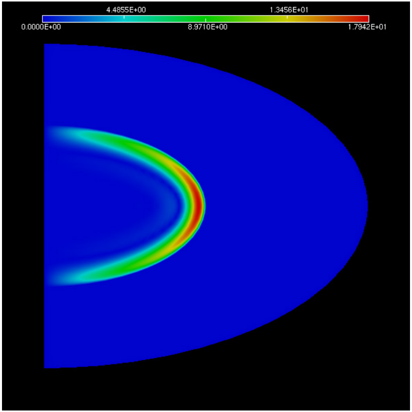
$\left|\operatorname{Im} H_\sigma\right|$ for $\sigma = 5S.m^{-1}$



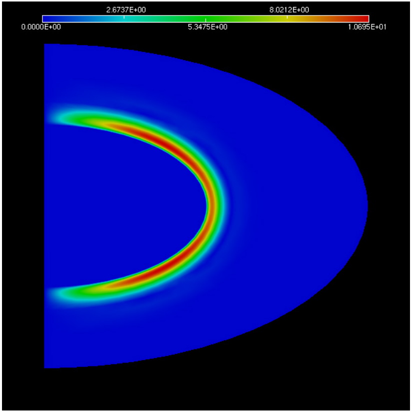
$\left|\operatorname{Im} H_\sigma\right|$ for $\sigma = 80S.m^{-1}$

Asymptotic Behavior for Skin Depth in Ellipsoidal Geometry

$\left| \operatorname{Im} H_\sigma \right|$ for $\sigma = 5S.m^{-1}$



Conductor $= \Omega_-^{\mathrm{m}}$ $(\mathcal{H} > 0)$



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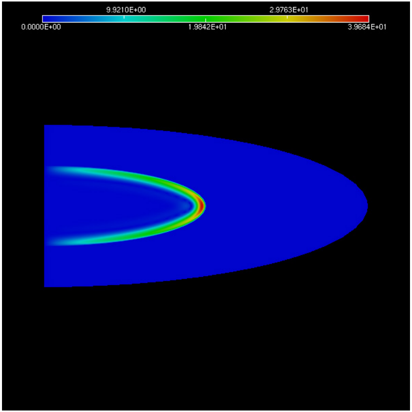
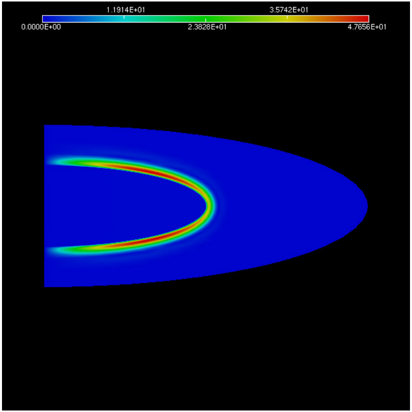


Conductor $= \Omega^{\mathrm{m}}_{-}$



Conductor $= \Omega^{\mathrm{m}}_{+}$

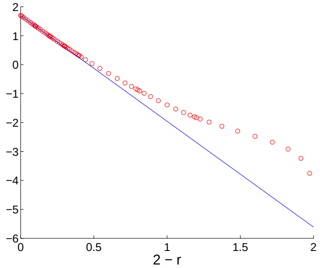
Numerical Post-Treatment

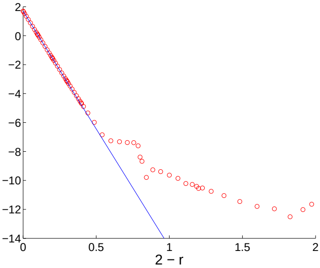
Linear Regression

Extract $\left|H_\sigma(h)\right|$ along edges of the mesh of Ω^{m}_{-} when $z=0$: $h=2-r$.

Linear Regression in $skin(\sigma) \longrightarrow n(\sigma)$ points $\longrightarrow$ "numerical" slope $s(\sigma)$

$$\log_{10}\left|H_\sigma(h)\right| = s(\sigma)h + b$$





○ : $\log_{10}\left|H_\sigma(2-r)\right|$ when $\sigma=5$

$n(\sigma)=7\,,\, s(\sigma)=-3,65542$

○ : $\log_{10}\left|H_\sigma(2-r)\right|$ when $\sigma=80$

$n(\sigma)=3\,,\, s(\sigma)=-16,279162$

Post-Treatment

Accurary of Asymptotics

- Troncated Asymptotic Expansion : $\mathrm{H}^{-}_{\theta,1} := \mathcal{H}^{-}_{\theta,0} + \delta\mathcal{H}^{-}_{\theta,1}$

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- "Theoretical" slope $\beta(\sigma,\Sigma) := \frac{1}{\ln 10}\left(\mathcal{H} - \frac{1}{\ell(\sigma)}\right)$
- Relative Error between slopes: $\mathit{error}(\sigma) := \left|\frac{\beta(\sigma,\Sigma) - s(\sigma)}{\beta(\sigma,\Sigma)}\right|$

σ	5	20	80
$\mathit{skin}(\sigma)$ (cm)	10.3	5.15	2.58
$n(\sigma)$	7	5	3
$s(\sigma)$	$-3,65542$	$-7,883903$	$-16,279162$
$\beta(\sigma,\Sigma)$	$-3,673319$	$-7,889506$	$-16,32188$
$\mathit{error}(\sigma)$ (%)	0, 48	0, 07	0, 26