The effective behaviour of (heterogeneous) elastic bodies containing defects localized on a surface

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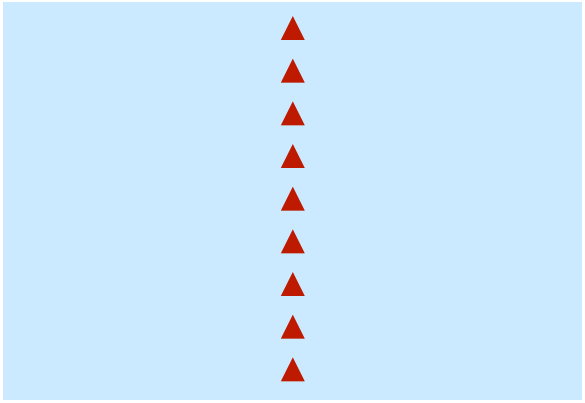
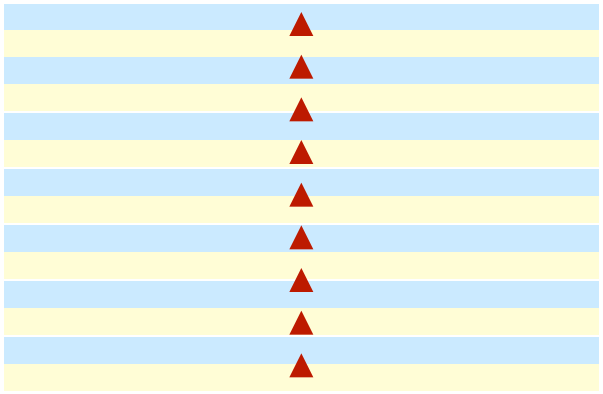
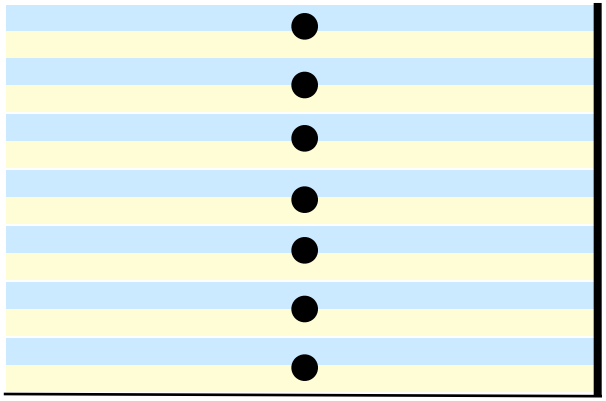
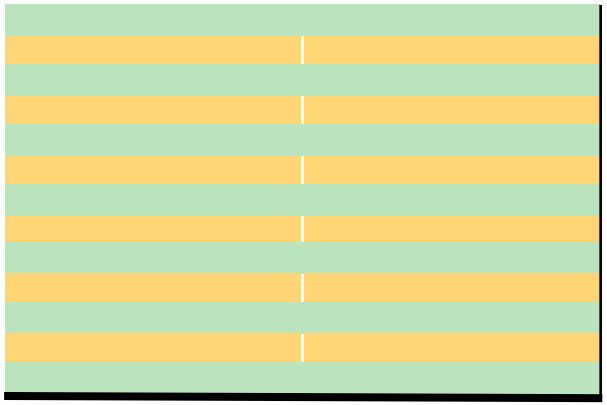
- Matched asymptotic expansions

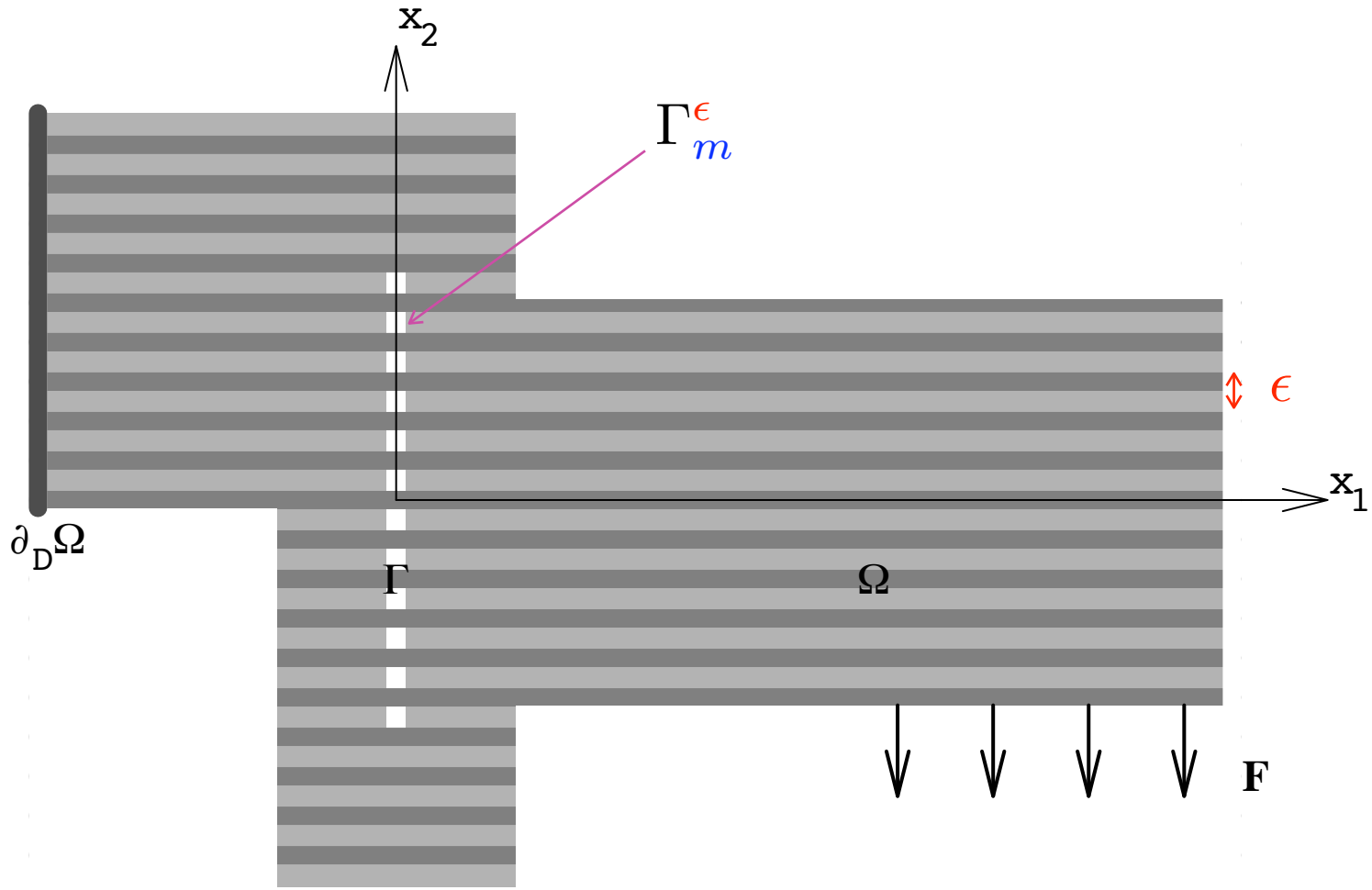
Nguetseng, Sanchez-Palencia (1985)
R. Abdelmoula, JJM (JMPS, 2000)







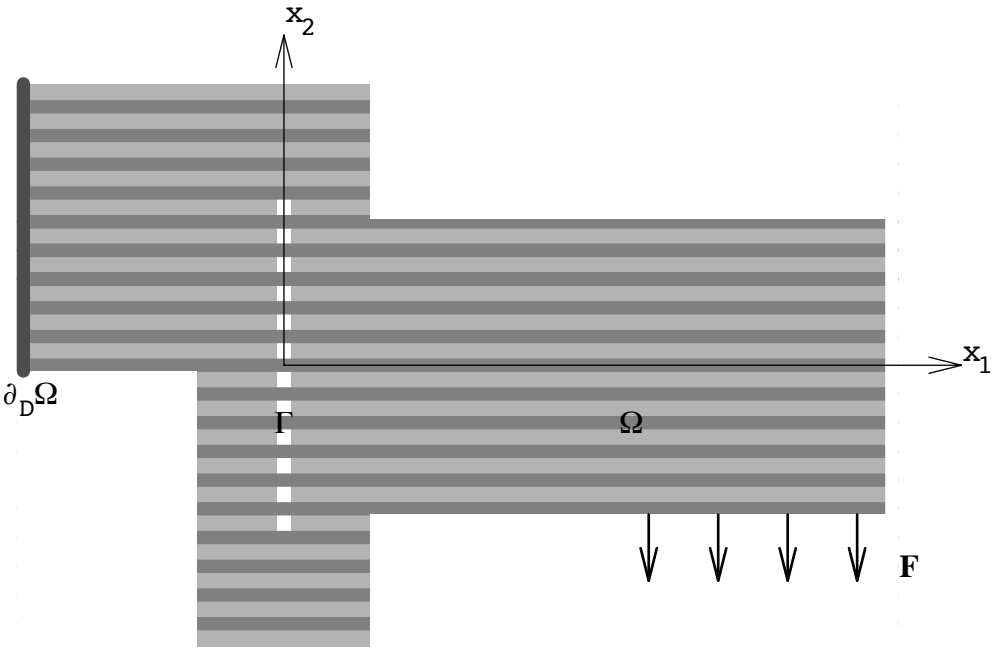




bridged crack = finely periodic surface defect = what effective behaviour?

boundary layer effects → second order effect

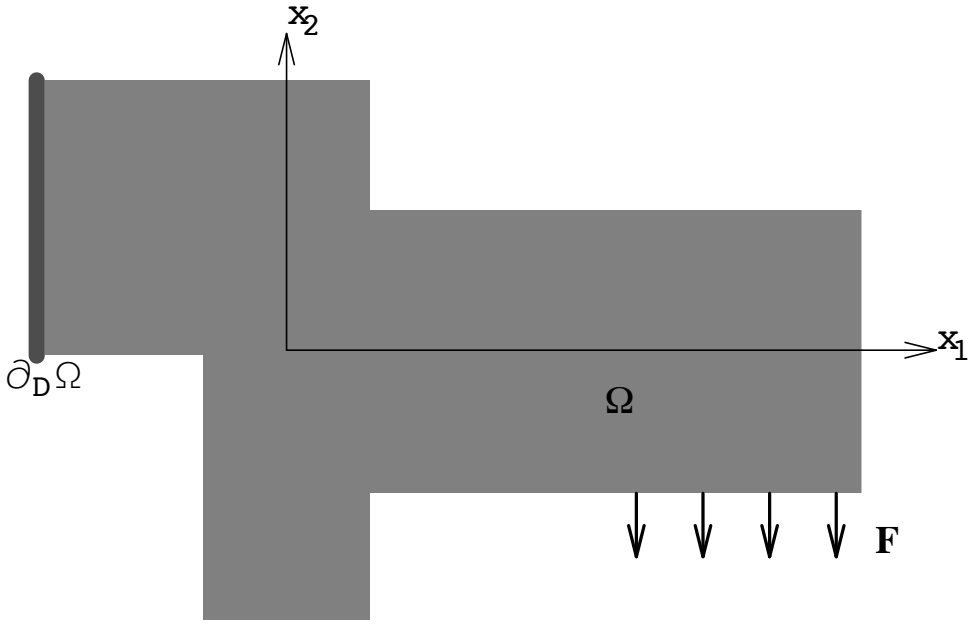
Heterogeneous body



First order homogenization

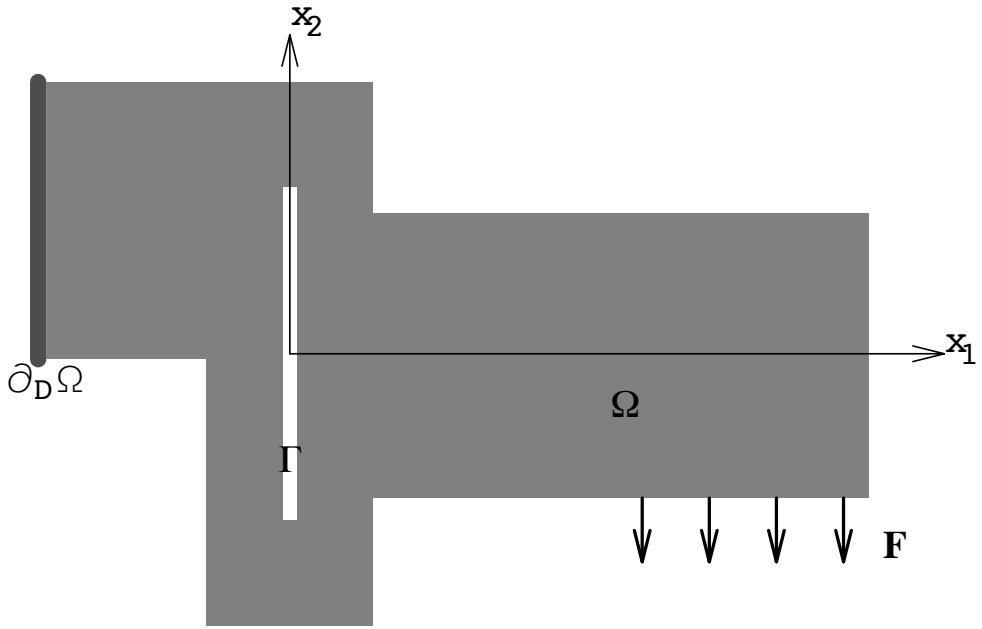
$$\sigma = A^{\mathrm{hom}}\varepsilon \quad \text{in} \quad \Omega$$

$$[\![u]\!] = 0, \quad [\![\sigma]\!]n = 0 \quad \text{on} \quad \Gamma$$



no crack!

Engineer heuristic approach



line spring model

$$[\![\sigma_{i1}]\!] = 0$$

$$\sigma_{i1} = k_i [\![u_i]\!]$$

Issues

- $k_i = ?$
- justification?

Second order asymptotic method

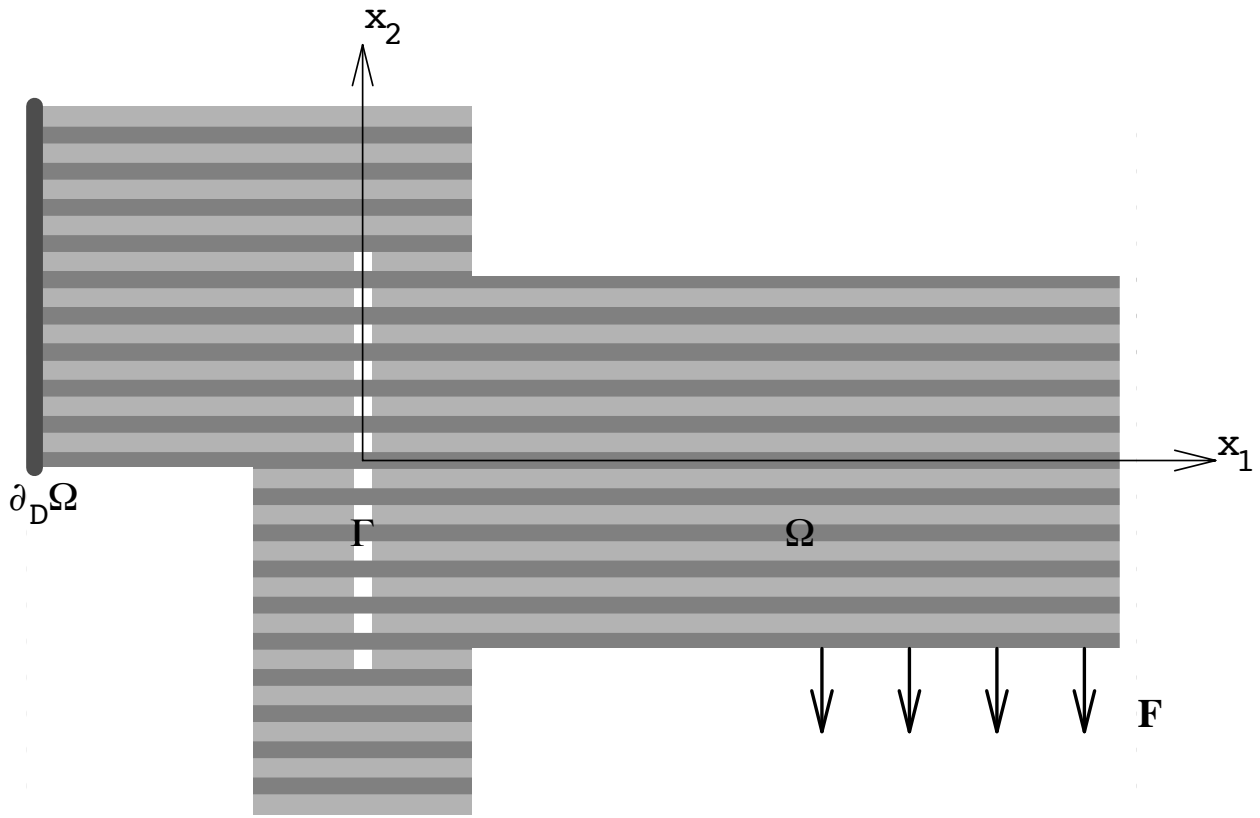
$$[\![\sigma_{i1}]\!] \neq 0$$

$$\frac{1}{2}\int_\Gamma \left(\frac{E_m}{\epsilon\alpha^2 K_1}[\![v_1]\!]^2 + \frac{E_m}{\epsilon K_2}[\![v_2]\!]^2 - 2\frac{\beta}{\alpha}E_m[\![v_1]\!]\frac{\partial \overline{v_2}}{\partial x_2}\right) dx_2$$

$$\overline{\sigma}_{i1} = k_i[\![u_i]\!] + \text{extra terms}$$

line spring model = bad model

Asymptotic method



$$
\begin{array}{rcrcl}
\text{Equilibrium} & in & \Omega \setminus \Gamma_m^\epsilon & : & div\sigma^\epsilon = 0 \\
\text{Constitutive equation} & in & \Omega \setminus \Gamma_m^\epsilon & : & \sigma^\epsilon = A^\epsilon \varepsilon(u^\epsilon) \\
\text{Boundary condition} & on & \Gamma_m^\epsilon & : & \sigma^\epsilon n = 0
\end{array}
$$

Matched asymptotic expansions

Outer expansions $(\text{far from } \Gamma)$

$$u^\epsilon(x_1,x_2)=\sum_{i=0}^{\infty}\epsilon^i u^i(x_1,x_2,\frac{x_2}{\epsilon}) \qquad \sigma^\epsilon(x_1,x_2)=\sum_{i=0}^{\infty}\epsilon^i \sigma^i(x_1,x_2,\frac{x_2}{\epsilon})$$

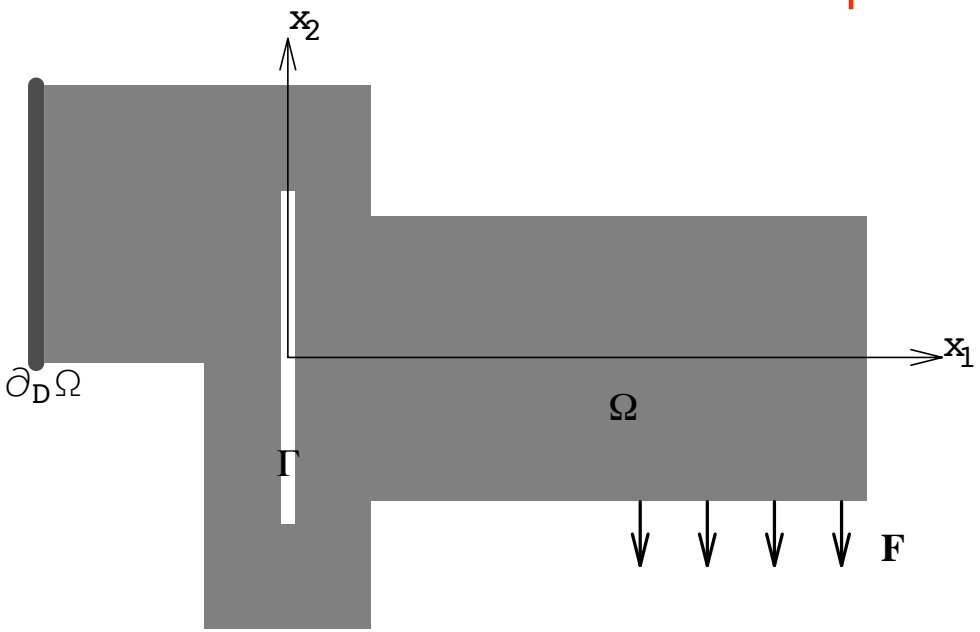
Inner expansions $(\text{near } \Gamma)$

$$u^\epsilon(x_1,x_2)=\sum_{i=0}^{\infty}\epsilon^i v^i(x_2,\frac{x_1}{\epsilon},\frac{x_2}{\epsilon}) \qquad \sigma^\epsilon(x_1,x_2)=\sum_{i=0}^{\infty}\epsilon^i \tau^i(x_2,\frac{x_1}{\epsilon},\frac{x_2}{\epsilon})$$

Matching conditions

$$\lim_{y_1\to\pm\infty}\left(v^i(x_2,y_1,y_2)-\sum_{j=0}^{i}\frac{y_1^{i-j}}{(i-j)!}\frac{\partial^{i-j}u^j}{\partial x_1^{i-j}}(0\pm,x_2,y_2)\right)=0$$

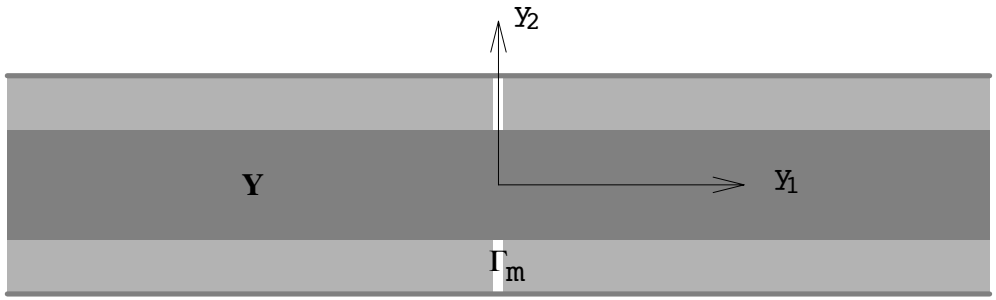
Outer problems



Boundary conditions on the "crack"

Matching conditions

Inner problems



Boundary conditions at infinity

Equilibrium

Order	*Outer*	*Inner*
-1	$0 = div_y \sigma^0$	$0 = div_y \tau^0$
$i \geq 0$	$0 = div_y \sigma^{i+1} + div_x \sigma^i$	$0 = div_y \tau^{i+1} + div_x \tau^i$

Constitutive equations

Order	*Outer*	*Inner*
-1	$0 = A\varepsilon_y(u^0)$	$0 = A\varepsilon_y(v^0)$
$i \geq 0$	$\sigma^i = A\varepsilon_y(u^{i+1}) + A\varepsilon_x(u^i)$	$\tau^i = A\varepsilon_y(v^{i+1}) + A\varepsilon_x(v^i)$

Boundary + Periodic + Matching conditions

Outer first order

$$u^0(x, y_2) = u(x)$$

$$\sigma(x) \equiv \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma^0(x, y_2) dy_2$$

effective behaviour $\quad \sigma = A^{\text{hom}} \varepsilon(u) \quad \text{in} \quad \Omega$

dressmaker principle! $\quad [\![u]\!] = 0, \quad [\![\sigma]\!] n = 0 \quad \text{on} \quad \Gamma$

matrix axial stress

$$\sigma_m^0 = \alpha \sigma_{11} + \beta E_m u_{2,2}$$

$$\alpha = \frac{1 - V_f(1 - \nu_f \nu_m) + V_f(1 - \nu_f^2) E_m / E_f}{1 + V_f V_m \left((1 - \nu_m^2) E_f / E_m + (1 - \nu_f^2) E_m / E_f - 2(1 - \nu_f \nu_m) \right)}$$

$$\beta = \frac{V_f(\nu_m E_f / E_m - \nu_f)}{1 + V_f V_m \left((1 - \nu_m^2) E_f / E_m + (1 - \nu_f^2) E_m / E_f - 2(1 - \nu_f \nu_m) \right)}$$

Equilibrium

Order	*Outer*	*Inner*
-1	$0 = div_y \sigma^0$	$0 = div_y \tau^0$
$i \geq 0$	$0 = div_y \sigma^{i+1} + div_x \sigma^i$	$0 = div_y \tau^{i+1} + div_x \tau^i$

Constitutive equations

Order	*Outer*	*Inner*
-1	$0 = A\varepsilon_y(u^0)$	$0 = A\varepsilon_y(v^0)$
$i \geq 0$	$\sigma^i = A\varepsilon_y(u^{i+1}) + A\varepsilon_x(u^i)$	$\tau^i = A\varepsilon_y(v^{i+1}) + A\varepsilon_x(v^i)$

Boundary + Periodic + Matching conditions

$$\varepsilon_y(v^0) = 0$$

periodicity $\quad v^0(x_2, y) = v^0(x_2)$

matching $\quad u^0(0+, x_2) = v^0(x_2) = u^0(0-, x_2)$

$$div_y \tau^0 = 0$$

integration+ periodicity $\quad \int_{-1/2}^{1/2} \tau_{i1}^0(x_2, +\infty, y_2) dy_2 = \int_{-1/2}^{1/2} \tau_{i1}^0(x_2, -\infty, y_2) dy_2$

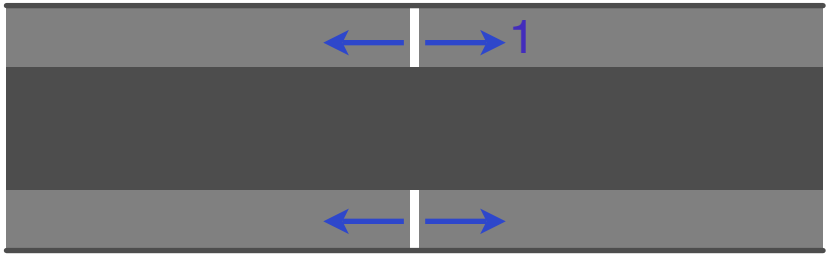
matching $\quad \sigma_{i1}(0+, x_2) = \sigma_{i1}(0-, x_2)$

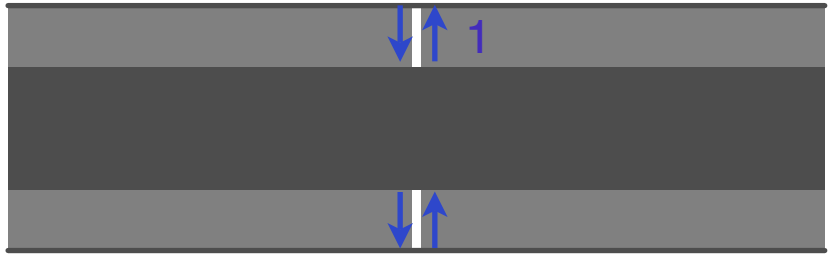
Inner first order

$$v^0(x_2, y) = u(0, x_2)$$

$$v^1(x_2, y) = \sigma_{ij}(0, x_2)\chi^{ij}(y_2) + y_1 u_{,1}(0, x_2) + \underbrace{\sigma^0_m(0, x_2)V^1(y) + \sigma_{21}(0, x_2)V^2(y)}_{\hat{v}} + \check{v}(x_2)$$

$$\tau^0(x_2, y) = \sigma^0(0, x_2, y_2) + \underbrace{\sigma^0_m(0, x_2)T^1(y) + \sigma_{21}(0, x_2)T^2(y)}_{\hat{\tau}}$$





1 2

2 inner "universal" problems

$$K_{ij} \equiv E_m \int_{Y \setminus \Gamma_m} A\varepsilon(V^i) \cdot \varepsilon(V^j)\, dy$$

$$K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$$

Effective "surface" coefficients

Outer second order

$$u^1(x,y_2) = \sigma_{ij}(x)\chi^{ij}(y_2) + \hat{u}(x) \qquad \hat{\sigma}(x) \equiv \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma^1(x,y_2)dy_2$$

second order macroscopic fields

$$In \quad \Omega\backslash\Gamma \quad : \qquad div\hat{\sigma} = 0 \qquad\qquad \hat{\sigma} = A^{\hom}\varepsilon(\hat{u})$$

$$On \quad \Gamma \quad :$$

$$[\![\hat{\sigma}_{11}]\!] = 0 \qquad [\![\hat{\sigma}_{21}]\!] = \beta K_1 \frac{\partial \sigma^0_m}{\partial x_2}$$

$$[\![\hat{u}_1]\!] = \alpha K_1 \frac{\sigma^0_m}{E_m} \qquad [\![\hat{u}_2]\!] = K_2 \frac{\sigma_{21}}{E_m}$$

$$At \quad \partial\Gamma \quad : \quad \text{ponctual forces}$$

singular problem (not finite energy}

$$
\begin{aligned}
u^1(0\pm, x_2, y_2) &= \lim_{y_1 \to \pm\infty} \Big(v^1(x_2, y_1) - y_1 \frac{\partial u^0}{\partial x_1}(0, x_2) \Big) \\
&= \sigma_{ij}(0, x_2)\chi^{ij}(y_2) + \sigma^0_m(0, x_2) V^1(\pm\infty) + \sigma_{21}(0, x_2) V^2(\pm\infty) + \check{v}(x_2)
\end{aligned}
$$

$$
[\![\hat{u}]\!](x_2) = \sigma^0_m(0, x_2)\Big(V^1(+\infty) - V^1(-\infty)\Big) + \sigma_{21}(0, x_2)\Big(V^2(+\infty) - V^2(-\infty)\Big)
$$

symmetry $\qquad V_2^1(+\infty) = V_2^1(-\infty), \qquad V_1^2(+\infty) = V_1^2(-\infty)$

$$
\begin{aligned}
0 &= \frac{\partial \tau^1_{i1}}{\partial y_1} + \frac{\partial \tau^1_{i2}}{\partial y_2} + \frac{\partial \tau^0_{i2}}{\partial x_2} \\
&= \frac{\partial \tau^1_{i1}}{\partial y_1} + \frac{\partial \tau^1_{i2}}{\partial y_2} + \frac{\partial \sigma^0_{i2}}{\partial x_2} + \frac{\partial \hat{\tau}_{i2}}{\partial x_2} \\
&= \frac{\partial \tau^1_{i1}}{\partial y_1} + \frac{\partial \tau^1_{i2}}{\partial y_2} - \frac{\partial \sigma^0_{i1}}{\partial x_1} + \frac{\partial \hat{\tau}_{i2}}{\partial x_2}
\end{aligned}
$$

equilibrium

$$
\int_{\mathbf{Y}} \hat{\tau}_{i1} dy = 0
$$

$$
[\![\hat{\sigma}_{11}]\!] = 0
$$

integration over Y+periodicity+matching

$$
\begin{aligned}
[\![\hat{\sigma}_{i1}]\!](x_2) &= -\frac{\partial}{\partial x_2} \int_{\mathbf{Y}} \hat{\tau}_{i2}(x_2, y) dy \\
&= -\frac{\partial \sigma^0_m}{\partial x_2}(0, x_2) \int_{\mathbf{Y}} T^1_{i2} dy - \frac{\partial \sigma_{21}}{\partial x_2}(0, x_2) \int_{\mathbf{Y}} T^2_{i2} dy
\end{aligned}
$$

symmetry

$$
\int_{\mathbf{Y}} T^2_{22} dy = 0
$$

Up to the second order effective model

$$\Sigma^\epsilon = A^{\text{hom}}\varepsilon(U^\epsilon)$$

$$[\![\Sigma^\epsilon_{11}]\!] = 0 \qquad [\![\Sigma^\epsilon_{21}]\!] = \frac{\beta}{\alpha} E_m \frac{\partial [\![U^\epsilon_1]\!]}{\partial x_2}$$

$$[\![U^\epsilon_1]\!] = \epsilon\alpha K_1 \left(\alpha \frac{\overline{\Sigma^\epsilon_{11}}}{E_m} + \beta \frac{\partial \overline{U^\epsilon_2}}{\partial x_2} \right) \qquad [\![U^\epsilon_2]\!] = \epsilon K_2 \frac{\overline{\Sigma^\epsilon_{21}}}{E_m}$$

$$\overline{f} = (f^+ + f^-)/2$$

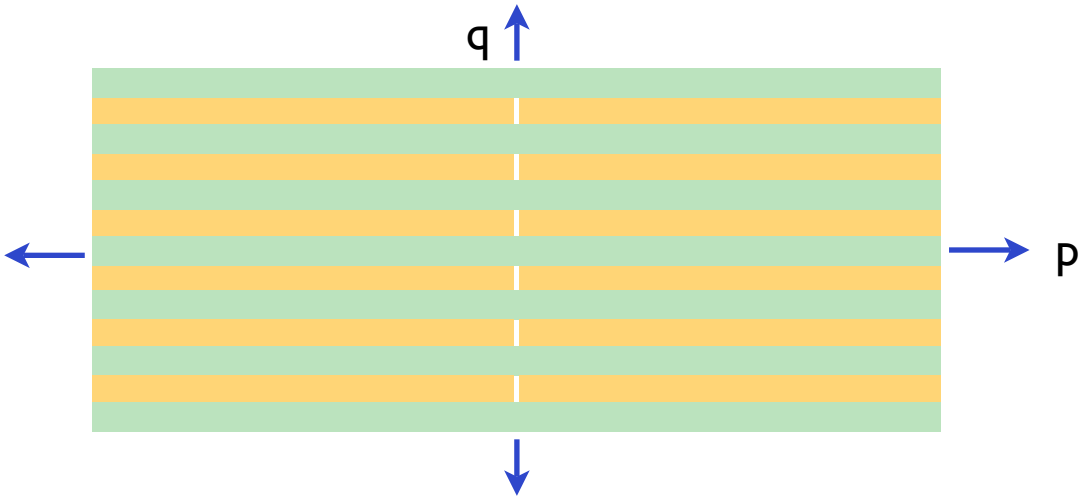
Variational formulation

$$\min_v \left\{ \frac{1}{2} \int_{\Omega\setminus\Gamma} A^{\text{hom}}\varepsilon(v)\cdot\varepsilon(v)dx + \frac{1}{2}\int_\Gamma \left(\frac{E_m}{\epsilon\alpha^2 K_1}[\![v_1]\!]^2 + \frac{E_m}{\epsilon K_2}[\![v_2]\!]^2 - 2\frac{\beta}{\alpha}E_m[\![v_1]\!]\frac{\partial \overline{v_2}}{\partial x_2} \right) dx_2 - f(v) \right\}$$

quadratic surface energy

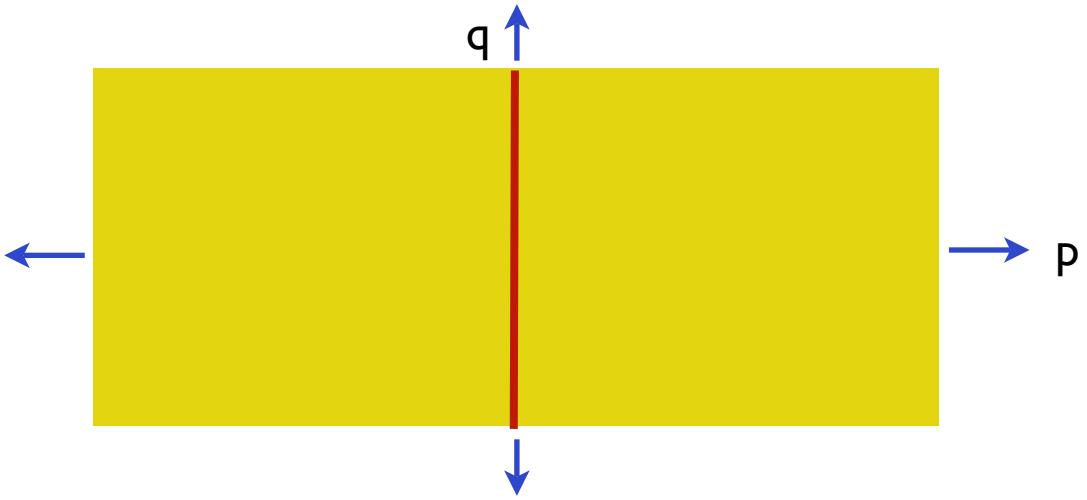
Extra terms

Comparison with the line-spring model



$$\nu_m = \nu_f = \nu$$
$$V_m = V_f = 0.5$$
$$E_m \ll E_f$$

Comparison with the line-spring model



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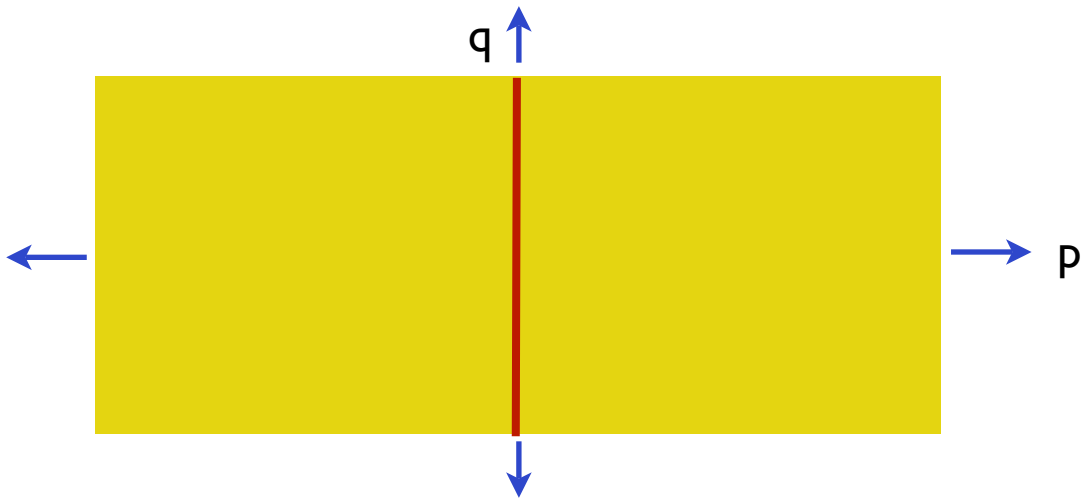


$$\nu_m = \nu_f = \nu$$
$$V_m = V_f = 0.5$$
$$E_m \ll E_f$$

$$\Sigma^\epsilon = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \qquad \varepsilon_{22} \approx \frac{1-\nu^2}{2}\frac{q}{E_m} \qquad \alpha \approx 2\frac{1+\nu^2}{1-\nu^2}\frac{E_m}{E_f} \qquad \beta \approx \frac{2\nu}{1-\nu^2} \qquad K_1 \approx K_1^0$$

line-spring model

$$[\![U_1^\epsilon]\!] = \epsilon\alpha^2 K_1 \frac{p}{E_m}$$

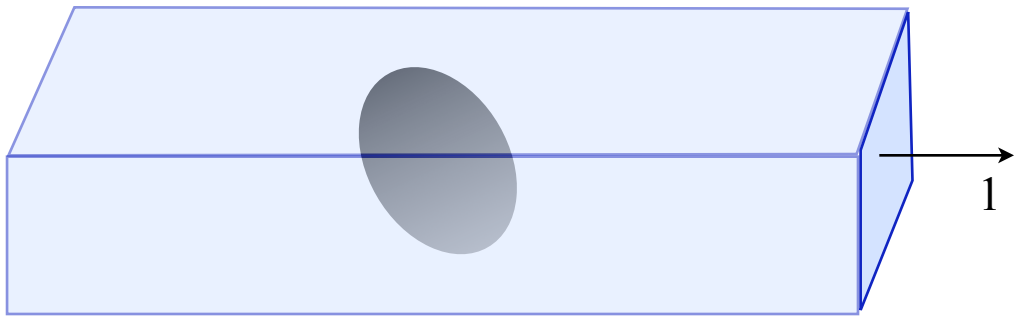
$$[\![U_1^\epsilon]\!] \approx 4\epsilon K_1^0 \frac{(1+\nu^2)^2}{(1-\nu^2)^2}\left(\frac{E_m}{E_f}\right)^2 \frac{p}{E_m}$$

up to second order effective model

$$[\![U_1^\epsilon]\!] = \epsilon\alpha^2 K_1 \frac{p}{E_m} + \epsilon\alpha\beta K_1 \varepsilon_{22}$$

$$[\![U_1^\epsilon]\!] \approx 2\epsilon K_1^0 \frac{1+\nu^2}{1-\nu^2}\frac{E_m}{E_f}\frac{\nu q}{E_m}$$

Generalization



- 3D
- holes or inclusions

- 6 elementary problems in 3D (3 in 2D)
- the surface energy contains the jump of the displacement and the tangential strains

$$\frac{1}{2}\int_\Gamma \left(\frac{B_{ij}}{\epsilon}[\![v_i]\!][\![v_j]\!] + C_{i\alpha\beta}[\![v_i]\!]\varepsilon_{\alpha\beta}(v) + \epsilon D_{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta}(v)\varepsilon_{\gamma\delta}(v) \right) d\Gamma \qquad \begin{array}{l} i,\, j \in \{1,2,3\} \\ \alpha,\, \beta,\, \gamma,\, \delta \in \{2,3\} \end{array}$$

Perspectives : Properties of the effective coefficients B, C, D

Conclusion

- ✓ The boundary layer effects can be represented by an effective surface energy richer than that given by the line spring model
- ✓ The effective coefficients are obtained by solving "universal" problems (characteristic of the local defect)