Edge singularities and structure of the 3-D Williams expansion

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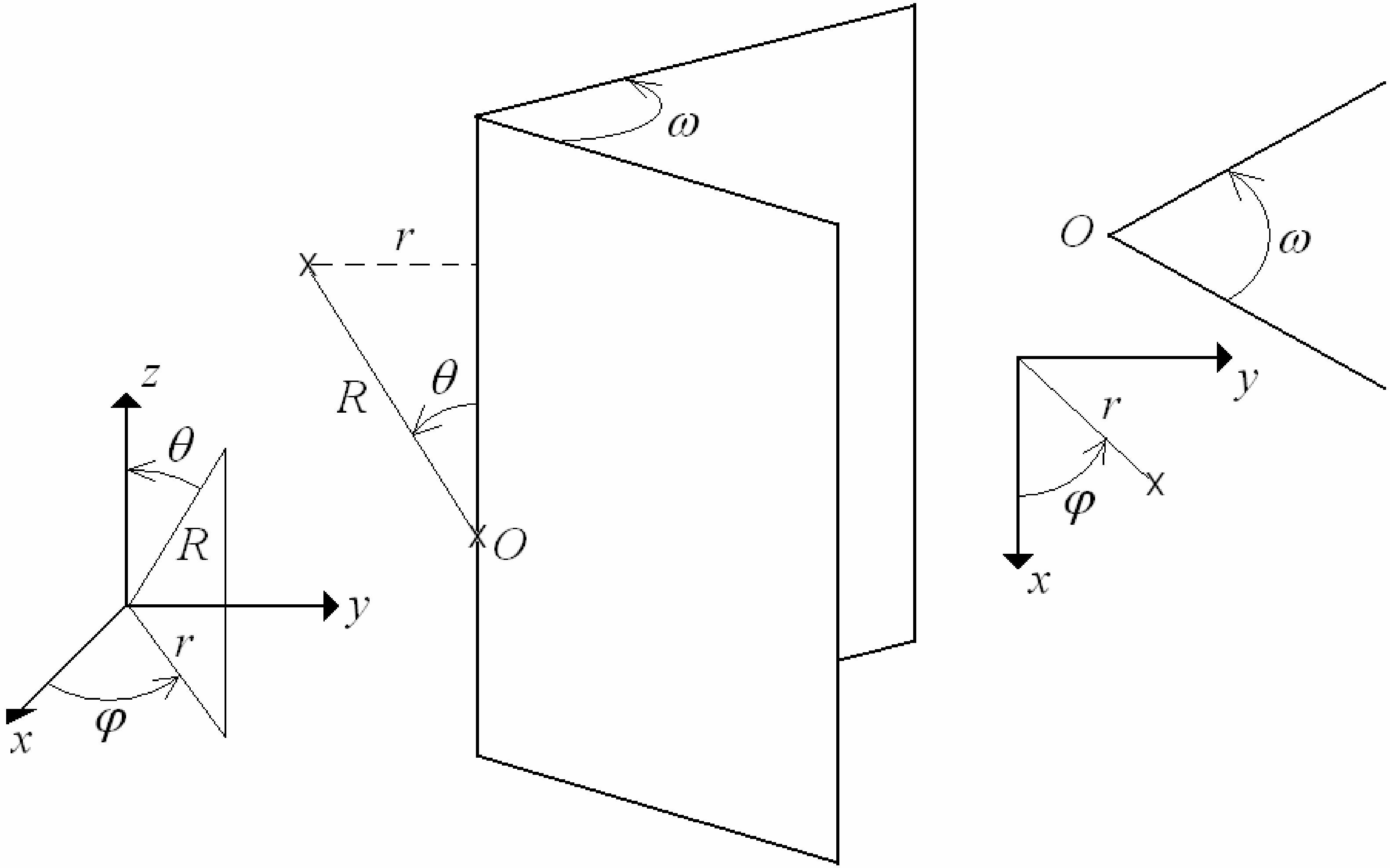
The edge geometry with Cartesian, polar (2D), cylindrical and spherical (3D) coordinates (1/1)



The expansion proposed by Costabel et al. (1/1)

In 2004 an expansion of the elastic 3-D solution $\underline{U}$ in terms of 2-D solutions with additional terms so-called “shadow functions” was proposed in the form

$$\underline{U}(r,\varphi,z)=\sum_{i\geq 1}\sum_{j\geq 0}\partial_z^j A_i(z)\ \underline{\Phi}_j^i(r,\varphi)$$

By Costabel, Dauge and Yosibash (2004, 2005) using cylindrical coordinates. It is derived from the splitting of the partial differential Navier operator in derivatives with respect to z on the one hand and with respect to r and φ on the other hand, allowing a separation of variables.

This expansion holds at any interior point of the edge, i.e. at any point except the ends.

The primary modes (1/1)

The function $\underline{\Phi}_0^i$ is the i-th mode of the 2-D corner problem, so called primary mode of the actual 3-D problem

$$\underline{\Phi}_0^i(r,\varphi) = r^{\alpha_i}\,\underline{\phi}_0^i(\varphi)$$

$\alpha_i \geq 0$ and $\underline{\phi}_0^i(\varphi)$ are respectively the eigenvalues and eigenfunctions of the elastic operator defined on the 2-D domain with a re-entrant corner with opening ω (Leguillon, Sanchez-Palencia 1987).

The shadow functions (1/1)

The $\underline{\Phi}_j^i$'s ($j \geq 1$) are the "shadow functions" to $\underline{\Phi}_0^i$, following the terminology of Costabel et al (2004, 2005)

$$\underline{\Phi}_j^i(r,\varphi) = r^{\alpha_i + j} \underline{\phi}_j^i(\varphi)$$

They are solutions to a partial differential system with a right hand side member depending on $\underline{\Phi}_{j-1}^i$.

ZZZ → But they are not solutions to the Navier operator neither in 2-D nor in 3-D.

The intensity factors (1/1)

The terms $\partial_z^j A_i(z)$ denote the weights of the corresponding primary or shadow functions. They depend on z and derive from a single edge stress intensity function (ESIF) $A_i(z)$. Costabel et al. (2005) proposed the so called "Quasidual Function Method" to extract these ESIF.

The 3-D Williams expansion (1/2)

On the other hand, the 3-D solution in the vicinity of any given point O on the edge can be represented by a Williams (1956) like expansion

$$\underline{U}(R,\varphi,\theta) = \sum_{i\geq 0} K_i R^{\beta_i} \underline{v}^i(\varphi,\theta)$$

Using spherical coordinates with origin at O.

The constant coefficients K_i are the generalized stress intensity factors (GSIF) associated with the different modes characterized by an eigenvalue $\beta_i \geq -0.5$ and an eigenfunction $\underline{v}^i(\varphi,\theta)$ (Leguillon 1995).

The 3-D Williams expansion (2/2)

The expansion holds true for any point O along the edge including the ends.

The β_i's and $\underline{v}^i$'s are independent of the choice of O if it is an interior point but differ at the two ends (Leguillon 1995, 2003).

The 3-D terms and the GSIF's are generally known numerically using appropriate algorithms dedicated to general 3-D edge and corner geometries (Leguillon 1995, Dimitrov, Andrä, Schnack 2002, Apel, Mehrmann, Watkins 2002, Leguillon 2003, Pester 2005).

4 useful properties (1/2)

1. If α_i is an eigenvalue of the 2-D problem, then $-\alpha_i$ is also an eigenvalue with its own eigenfunction $\underline{\xi}_0^i(\varphi)$. If β_i is an eigenvalue of the 3-D problem, then $-\beta_i - 1$ is also an eigenvalue with its own eigenfunction $\underline{w}^i(\varphi,\theta)$. These pairs are baptised dual modes (Leguillon 1995, Pester 2005).

2. The strip $]-1,0[$ is free of 3-D eigenvalues (Kozlov, Maz'ya, Rossman 2001).

4 useful properties (2/2)

3. The 2-D eigenpairs with $\alpha_i \geq 0$ are also 3-D eigenpairs

$$r^{\alpha_i}\,\underline{\phi}_0^i(\varphi) = R^{\alpha_i}\sin^{\alpha_i}\theta\;\underline{\phi}_0^i(\varphi) = R^{\alpha_i}\,\underline{v}^i(\varphi,\theta) \quad \text{with} \quad \underline{v}^i(\varphi,\theta) = \sin^{\alpha_i}\theta\;\underline{\phi}_0^i(\varphi)$$

ZZZ → It must be pointed out that a dual 2-D mode (involving a negative exponent α_i) is not, since $\sin\theta$ vanishes at some points, which is contrary to the H^1 smoothness of $\underline{v}^i$ if $\alpha_i \leq -1$.

4. Let $\underline{V}(x,y,z)$ be any solution to the 3-D problem in the infinite domain (unbounded in r and z, i.e. in R), then $\partial_z^k \underline{V}$ is also a solution for any $k \geq 0$. The problem is invariant under a translation parallel to z.

The structure of the Williams expansion (1/4)

Let us start from a 3-D eigenfunction $R^{\alpha_j}\underline{v}^j(\varphi,\theta)$

• Property 4 → $\partial_z^k R^{\alpha_j}\underline{v}^j(\varphi,\theta)$ are eigenfunctions for $k>0$

• Property 2 → it exists $p\geq 1$ such that the p+1-th derivative with respect to z vanishes and then the p-th one is a 2-D eigenfunction

$$\partial_z^p\left(R^{\alpha_j}\underline{v}^j(\varphi,\theta)\right)=R^{\alpha_i}\underline{v}^i(\varphi,\theta)=r^{\alpha_i}\underline{\phi}_0^i(\varphi)\quad\text{with}\quad\alpha_j=\alpha_i+p$$

• Properties 1 and 4 → the dual mode $R^{-\alpha_i-1}\underline{w}^i(\varphi,\theta)$ and $\partial_z^k\left(R^{-\alpha_i-1}\underline{w}^i(\varphi,\theta)\right)=R^{-\alpha_i-1-k}\underline{w}_k^i(\varphi,\theta)$, are 3-D eigenfunctions for any $k>0$.

The structure of the Williams expansion (2/4)

- Property 1 → the dual functions are $R^{\alpha_i+k}\underline{v}_k^i(\varphi,\theta)$ and by integration

$$\begin{cases} R^{\alpha_i+1}\underline{v}_1^i(\varphi,\theta) = z\ r^{\alpha_i}\,\underline{\phi}_0^i(\varphi) + \underline{c}_1^i(r,\varphi) \\ R^{\alpha_i+2}\underline{v}_2^i(\varphi,\theta) = \dfrac{z^2}{2}\ r^{\alpha_i}\,\underline{\phi}_0^i(\varphi) + z\ \underline{c}_1^i(r,\varphi) + \underline{c}_2^i(r,\varphi) \\ \qquad\qquad \dots \\ R^{\alpha_i+k}\underline{v}_k^i(\varphi,\theta) = \dfrac{z^k}{k!}\ r^{\alpha_i}\,\underline{\phi}_0^i(\varphi) + \dfrac{z^{k-1}}{(k-1)!}\ \underline{c}_1^i(r,\varphi) + \dots + \underline{c}_k^i(r,\varphi) \\ \qquad\qquad \dots \end{cases}$$

ZZZ → each of the expressions is solution to the 3-D problem.

The structure of the Williams expansion (3/4)

To be consistent with the Williams expansion

$$\underline{c}_k^i(r,\varphi) = r^{\alpha_i+k}\,\underline{\psi}_k^i(\varphi)$$

$$\begin{cases} \underline{U}(r,\varphi,z) = K_i r^{\alpha_i}\,\underline{\phi}_0^i(\varphi) + K_i^1 r^{\alpha_i}\left(z\underline{\phi}_0^i(\varphi) + r\,\underline{\psi}_1^i(\varphi)\right) \\ \qquad + K_i^2 r^{\alpha_i}\left(\dfrac{z^2}{2}\underline{\phi}_0^i(\varphi) + z\,r\,\underline{\psi}_1^i(\varphi) + r^2\underline{\psi}_2^i(\varphi)\right) + \ldots \\ \qquad + K_i^k r^{\alpha_i}\left(\dfrac{z^k}{k!}\underline{\phi}_0^i(\varphi) + \dfrac{z^{k-1}}{(k-1)!} r\,\underline{\psi}_1^i(\varphi) + \ldots + r^k\underline{\psi}_k^i(\varphi)\right) + \ldots \end{cases}$$

The structure of the Williams expansion (4/4)

To summarize, the exponents involved in the Williams series are the exponents of the 2-D corner problem and these exponents plus integers.

Complex and multiple eigenvalues do not alter the reasoning since both primal and dual eigenvalues enjoy the same properties.

The 3-D vs. 2-D analysis (1/5)

Rearranging the expansion yields

$$\begin{cases} \underline{U}(r,\varphi,z) = \left(K_i + z\,K_i^1 + \dfrac{z^2}{2} K_i^2 + \ldots + \dfrac{z^k}{k!} K_i^k + \ldots \right) r^{\alpha_i} \underline{\phi}_0^i(\varphi) \\ \qquad + \left(K_i^1 + z\,K_i^2 + \ldots + \dfrac{z^{k-1}}{(k-1)!} K_i^k + \ldots \right) r^{\alpha_i+1} \underline{\psi}_1^i(\varphi) \\ \qquad + \left(K_i^2 + \ldots + \dfrac{z^{k-2}}{(k-2)!} K_i^k + \ldots \right) r^{\alpha_i+2} \underline{\psi}_2^i(\varphi) + \ldots + K_i^k r^{\alpha_i+k} \underline{\psi}_k^i(\varphi) + \ldots \end{cases}$$

The 3-D vs. 2-D analysis (2/5)

By identification the ESIF can be written

$$A_i(z) = K_i + z\, K_i^1 + \frac{z^2}{2}\, K_i^2 + \ldots + \frac{z^k}{k!} K_i^k + \ldots$$

A truncation of this series leads to a polynomial approximation of the ESIF comparable to that employed by Costabel et al. (2004, 2005)

The 3-D vs. 2-D analysis (3/5)

Continuing the identification leads to write the shadow functions as

$$\underline{\Phi}_1^i(r,\varphi) = r^{\alpha_i+1}\underline{\psi}_1^i(\varphi);\ \ldots\ \underline{\Phi}_k^i(r,\varphi) = r^{\alpha_i+k}\underline{\psi}_k^i(\varphi)\ \ldots\ \text{i.e. } \underline{\psi}_k^i(\varphi) = \underline{\phi}_k^i(\varphi) \text{ for } k \geq 1$$

It is clear that the corresponding weights are successive derivatives of $A_i(z)$.

The 3-D vs. 2-D analysis (4/5)

The shadow functions and the 3-D eigenfunctions are linked together according to the following system

$$\begin{cases} \underline{v}_1^i(\varphi,\theta) = \sin^{\alpha_i}\theta\left(\sin\theta\ \underline{\phi}_1^i(\varphi) + \cos\theta\ \underline{\phi}_0^i(\varphi)\right) \\ \underline{v}_2^i(\varphi,\theta) = \sin^{\alpha_i}\theta\left(\sin^2\theta\ \underline{\phi}_2^i(\varphi) + \cos\theta\sin\theta\ \underline{\phi}_1^i(\varphi) + \dfrac{1}{2}\cos^2\theta\ \underline{\phi}_0^i(\varphi)\right) \\ \qquad\qquad \ldots \\ \underline{v}_k^i(\varphi,\theta) = \sin^{\alpha_i}\theta\left(\sin^k\theta\ \underline{\phi}_k^i(\varphi) + \ldots + \dfrac{1}{(k-1)!}\cos^{k-1}\theta\sin\theta\ \underline{\phi}_1^i(\varphi) + \dfrac{1}{k!}\cos^k\theta\ \underline{\phi}_0^i(\varphi)\right) \end{cases}$$

The 3-D vs. 2-D analysis (5/5)

The GSIF's can be computed using a FE approximation $\underline{U}^{FE}$ of $\underline{U}$ and a path independent integral making use of the already mentioned dual modes (Leguillon, Sanchez-Palencia 1987, Labossiere, Dunn 1998)

$$K_i = \frac{H\left(\underline{U}^{FE}, R^{-\alpha_i - 1}\underline{w}^i(\varphi,\theta)\right)}{H\left(R^{\alpha_i}\underline{v}^i(\varphi,\theta), R^{-\alpha_i - 1}\underline{w}^i(\varphi,\theta)\right)} \; ; \; H(\underline{X},\underline{Y}) = \int_\Gamma \left(\underline{T}(\underline{X}).\underline{Y} - \underline{T}(\underline{Y}).\underline{X}\right) \mathrm{d}s$$

The integral H is independent of the surface Γ surrounding O.

$\underline{T}(\underline{X})$ is the traction vector, derived from the displacement field $\underline{X}$ and acting on this surface.

A numerical example (1/2)

Opening $\omega = \pi / 2$, traction-free faces of the wedge.

Computations are carried out solving the eigenvalue problem using a p-version FEM and an algorithm based on a structured eigenvalue method (Apel, Mehrmann, Watkins 2002) .

A numerical example (2/2)

n°	α_i	mult.	n°	α_i	mult.	n°	α_i	mult.
1	**0.**	3	9	**1.629 ± 0.231**	2	17	2.629 ± 0.231	2
2	**0.545**	1	10	1.667	1	18	2.667	1
3	**0.667**	1	11	1.909	1	19	**2.667**	1
4	**0.909**	1	12	2.	4	20	2.909	1
5	1.	3	13	**2.**	1	21	**2.972 ± 0.374**	2
6	**1.**	1	14	**2.301 ± 0.316**	2	22	3.	5
7	**1.333**	1	15	2.333	1	23	3.301 ± 0.316	2
8	1.545	1	16	2.546	1	24	3.333	1

Bold-face numbers are the primary eigenvalues.

Conclusion (1/1)

The analysis extends to non isotropic materials and non homogeneous structures, composite laminates for instance, if the interfaces fulfil the cylindrical geometry (i.e. the interfaces are vertical planes emanating at the edge).

The present results are valid at any interior point of the edge. Of course, it is a completely different situation at the two ends where the problem is a full 3-D problem, no local 2-D solutions exist.