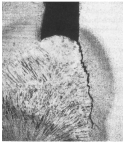
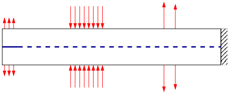
# On the inviscid limit of a model for crack propagation

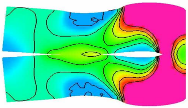
Dorothee Knees[1], Alexander Mielke, Chiara Zanini

[1]Weierstrass Institute for Applied Analysis and Stochastics, Berlin
knees@wias-berlin.de

Workshop Multiscale Asymptotics and Computational Approximation for Surface Defects and Applications in Mechanics, ENS Cachan
August 31 - September 1st, 2009







Goal: Describe a model for quasistatic/rate independent crack evolution on the basis of the Griffith fracture criterion

Problem: It is intrinsic that the crack evolution might be discontinuous. How to model jumps appropriately?

First step, linear elasticity:
problems and basic ideas

Second step, finite strains:
Additional challenge: nonunique minimizers

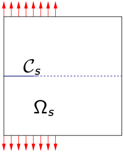
Assumptions:

- Small strain elasticity:

  $u : \Omega_s \to \mathbb{R}^2$ displacements, $\varepsilon(u)$ linearized strains

- Elastic energy density:

  $W(\varepsilon(u)) = \frac{1}{2}\mathbf{C}\varepsilon(u) : \varepsilon(u)$, $\mathbf{C}$ elasticity tensor.




- Non-interpenetration conditions on the crack $\mathcal{C}_s$:

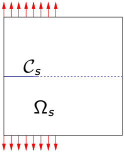
$$V(\Omega_s) = \{\, v \in H^1(\Omega_s)\,;\; v\big|_{\Gamma_D} = 0,\;\; [\![v]\!] \cdot \vec{n} \geq 0 \text{ on } \mathcal{C}_s \,\}$$

Assumptions:

- Small strain elasticity:

  $u : \Omega_s \to \mathbb{R}^2$ displacements, $\varepsilon(u)$ linearized strains
- Elastic energy density:

  $W(\varepsilon(u)) = \frac{1}{2}\mathbf{C}\varepsilon(u) : \varepsilon(u)$, $\mathbf{C}$ elasticity tensor.




- Non-interpenetration conditions on the crack $\mathcal{C}_s$:

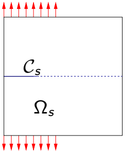
$$V(\Omega_s) = \{\, v \in H^1(\Omega_s)\,;\, v\big|_{\Gamma_D} = 0,\ [\![v]\!] \cdot \vec{n} \geq 0 \text{ on } \mathcal{C}_s \,\}$$

Deformation energy

$$\mathcal{E}(t,s,v) = \int_{\Omega_s} W(\varepsilon(v))\,\mathrm{d}x - \int_{\Gamma_N} h(t)\cdot v\,\mathrm{d}s \quad \text{for } v \in V(\Omega_s).$$

$$u(t,s) = \operatorname{argmin}_{V(\Omega_s)} \mathcal{E}(t,s,\cdot) \qquad \text{unique minimizer.}$$

## Assumptions:




- Small strain elasticity:

  $u : \Omega_s \to \mathbb{R}^2$ displacements, $\varepsilon(u)$ linearized strains
- Elastic energy density:

  $W(\varepsilon(u)) = \frac{1}{2}\mathbf{C}\varepsilon(u) : \varepsilon(u)$, $\mathbf{C}$ elasticity tensor.
- Non-interpenetration conditions on the crack $\mathcal{C}_s$:

$$V(\Omega_s) = \{\, v \in H^1(\Omega_s)\,;\; v\big|_{\Gamma_D} = 0,\;\; [\![v]\!] \cdot \vec{n} \geq 0 \text{ on } \mathcal{C}_s \,\}$$

## Deformation energy

$$\mathcal{E}(t,s,v) = \int_{\Omega_s} W(\varepsilon(v))\,\mathrm{d}x - \int_{\Gamma_N} h(t)\cdot v\,\mathrm{d}s \quad \text{for } v \in V(\Omega_s).$$

$$u(t,s) = \mathrm{argmin}_{V(\Omega_s)}\,\mathcal{E}(t,s,\cdot) \qquad \text{unique minimizer.}$$

## Reduced energy

$$\mathcal{I}(t,s) = \inf_{v\in V(\Omega_s)} \mathcal{E}(t,s,v) = \mathcal{E}(t,s,u(t,s)).$$

## Griffith criterion (1921)

The crack is stationary, if the (locally) released elastic energy is less than the energy dissipated to create the new crack surface.

## Griffith criterion (1921)

The crack is stationary, if the (locally) released elastic energy is less than the energy dissipated to create the new crack surface.

Energy release rate: $\mathcal{G}(t,s) = -\partial_s \mathcal{I}(t,s) = -\partial_s\big(\mathcal{E}(t,s,u(t,s))\big)$

Fracture toughness: $\kappa > 0$

Dissipated energy: $\mathcal{R}(s_{\text{new}} - s_{\text{old}}) = \begin{cases} \kappa(s_{\text{new}} - s_{\text{old}}) & \text{if } s_{\text{new}} \geq s_{\text{old}} \\ \infty & \text{else} \end{cases}.$

## Evolution criterion:

local stability: $\kappa \geq \mathcal{G}(t,s(t))$,

complementarity condition: $\dot{s}(t)\big(\kappa - \mathcal{G}(t,s(t))\big) = 0$.

## Griffith criterion (1921)

The crack is stationary, if the (locally) released elastic energy is less than the energy dissipated to create the new crack surface.

Energy release rate: $\mathcal{G}(t,s) = -\partial_s \mathcal{I}(t,s) = -\partial_s\big(\mathcal{E}(t,s,u(t,s))\big)$

Fracture toughness: $\kappa > 0$

Dissipated energy: $\mathcal{R}(s_{\text{new}} - s_{\text{old}}) = \begin{cases} \kappa(s_{\text{new}} - s_{\text{old}}) & \text{if } s_{\text{new}} \geq s_{\text{old}} \\ \infty & \text{else} \end{cases}.$

## Evolution criterion:

local stability: $\kappa \geq \mathcal{G}(t,s(t))$,

complementarity condition: $\dot{s}(t)\big(\kappa - \mathcal{G}(t,s(t))\big) = 0.$

Shortly: $\boxed{0 \in \partial\mathcal{R}(\dot{s}(t)) + \partial_s \mathcal{I}(t,s(t))}$

Question: What are physically/mathematically suitable formulations for the motion of the crack tip?

**Theorem (ERR for strictly convex elastic energies)**

*It holds for* $\mathcal{I}(t,s) = \min\{\, \mathcal{E}(t,s,v)\,;\; v \in V(\Omega_s)\,\}$

$$\mathcal{I} \in C^1([0,T]\times(0,L)),$$
$$\mathcal{G} = -\partial_s \mathcal{I} \in C^0([0,T]\times(0,L)).$$

*Moreover, the Griffith formula is valid, i.e.*

$$\mathcal{G}(t,s) = \int_{\Omega_s} \mathbb{E}(\nabla u(t,s)) : \nabla \varrho_s \,\mathrm{d}x,$$
$$\mathbb{E}(\nabla u) = \nabla u^\top D_F W(\nabla u) - W(\nabla u),$$

*where* $u(t,s) = \operatorname{argmin} \mathcal{E}(t,s,\cdot)$ *and* $\mathbb{E}(\nabla u)$ *Eshelby tensor associated with* $\nabla u$*;*
$T_{s,\delta} : \Omega_s \to \Omega_{s+\delta}$ *diffeomorphism;* $\quad \varrho_s = \left(\frac{\mathrm{d}}{\mathrm{d}\delta} T_{s,\delta}\right)\Big|_{\delta=0}$.

Remark: Plane strain, isotropic elasticity (Maz'ya, Nazarov '87):

$$\mathcal{G}(t,s) = \tfrac{1-\nu}{4\mu}\left(K_1^2(t,s) + K_2^2(t,s)\right)$$

Formula: Destuynder/Djaoua'81, Khludnev/Sokolowski'00;
Continuity: KMZ'08

$V$ reflexive Banach space, $\Sigma = [-\sigma_0, \sigma_0]$ parameter-interval

Energy functional $E : \Sigma \times V \to \mathbb{R}_\infty$
Reduced energy $I(\sigma) = \inf\{\, E(\sigma, v)\, ;\ v \in V\,\}$
Set of (nonunique) minimizers $U(\sigma) = \{\, v \in V\, ;\ E(\sigma, v) = I(\sigma)\,\}$.

## Assumptions

E1 For all $\sigma$ the sublevels of $E(\sigma, \cdot)$ are compact.

E2 $E(\cdot, v) \in C^1(\Sigma)$ and $|\partial_\sigma E(\sigma, v)| \leq c_1(1 + E(\sigma, v))\ \forall \sigma \in \Sigma$.

E3 Continuity of $\partial_\sigma E$ along weakly convergent sequ. of minimizers $u_n \in U(\sigma_n)$.

$V$ reflexive Banach space, $\Sigma = [-\sigma_0, \sigma_0]$ parameter-interval

| | |
|---|---|
| Energy functional | $E : \Sigma \times V \to \mathbb{R}_\infty$ |
| Reduced energy | $I(\sigma) = \inf\{ E(\sigma, v) ; v \in V \}$ |
| Set of (nonunique) minimizers | $U(\sigma) = \{ v \in V ; E(\sigma, v) = I(\sigma) \}$. |

### Assumptions

**E1** For all $\sigma$ the sublevels of $E(\sigma, \cdot)$ are compact.

**E2** $E(\cdot, v) \in C^1(\Sigma)$ and $|\partial_\sigma E(\sigma, v)| \le c_1(1 + E(\sigma, v))$ $\forall \sigma \in \Sigma$.

**E3** Continuity of $\partial_\sigma E$ along weakly convergent sequ. of minimizers $u_n \in U(\sigma_n)$.

### Theorem (KMZ08)

*$I \in C^{Lip}(\Sigma)$ and the left and right derivatives of $I$ exist and are left and right continuous, respectively. Moreover,*

$$\partial_\sigma^+ I(\sigma) = \min\{ \partial_\sigma E(\sigma, u) ; u \in U(\sigma) \},$$

$$\partial_\sigma^- I(\sigma) = \max\{ \partial_\sigma E(\sigma, u) ; u \in U(\sigma) \}$$

$V$ reflexive Banach space, $\Sigma = [-\sigma_0, \sigma_0]$ parameter-interval

| | |
|---|---|
| Energy functional | $E : \Sigma \times V \to \mathbb{R}_\infty$ |
| Reduced energy | $I(\sigma) = \inf\{ E(\sigma, v)\, ;\ v \in V \}$ |
| Set of (nonunique) minimizers | $U(\sigma) = \{ v \in V\, ;\ E(\sigma, v) = I(\sigma) \}$. |

### Assumptions

**E1** For all $\sigma$ the sublevels of $E(\sigma, \cdot)$ are compact.

**E2** $E(\cdot, v) \in C^1(\Sigma)$ and $|\partial_\sigma E(\sigma, v)| \le c_1(1 + E(\sigma, v))\ \forall \sigma \in \Sigma$.

**E3** Continuity of $\partial_\sigma E$ along weakly convergent sequ. of minimizers $u_n \in U(\sigma_n)$.

### Theorem (KMZ08)

*$I \in C^{Lip}(\Sigma)$ and the left and right derivatives of $I$ exist and are left and right continuous, respectively. Moreover,*

$$\partial_\sigma^+ I(\sigma) = \min\{ \partial_\sigma E(\sigma, u)\, ;\ u \in U(\sigma) \},$$

$$\partial_\sigma^- I(\sigma) = \max\{ \partial_\sigma E(\sigma, u)\, ;\ u \in U(\sigma) \}$$

### Corollary

*If all minimizers are unique, then $I \in C^1(\Sigma)$ and $\partial_\sigma I(\sigma) = \partial_\sigma E(\sigma, u)$, $u \in U(\sigma)$.*

For $h > 0$:

$$\frac{1}{h}\big(E(\sigma+h,u_\sigma)-E(\sigma,u_\sigma)\big)$$

$$\geq \frac{1}{h}\big(I(\sigma+h)-I(\sigma)\big)$$

$$\geq \frac{1}{h}\big(E(\sigma+h,u_{\sigma+h})-E(\sigma,u_{\sigma+h})\big)$$

For $h > 0$:

$$\frac{1}{h}\big(E(\sigma + h, u_\sigma) - E(\sigma, u_\sigma)\big)$$

$$\geq \frac{1}{h}\big(I(\sigma + h) - I(\sigma)\big)$$

$$\geq \frac{1}{h}\big(E(\sigma + h, u_{\sigma+h}) - E(\sigma, u_{\sigma+h})\big)$$

$$\uparrow$$

$$= \int_0^1 \partial_\sigma E(\sigma + rh, u_{\sigma+h})\,\mathrm{d}r$$

by E3: convergence to $\partial_\sigma E(\sigma, u_\sigma)$

$$\Longrightarrow \partial_\sigma^+ I(\sigma) \geq \min\{\, \partial_\sigma E(\sigma, u)\,;\ u \in U(\sigma)\,\}.$$

For $h > 0$:

$$\tfrac{1}{h}\big(E(\sigma+h,u_\sigma)-E(\sigma,u_\sigma)\big)$$

$$\geq \tfrac{1}{h}\big(I(\sigma+h)-I(\sigma)\big)$$

$$\geq \tfrac{1}{h}\big(E(\sigma+h,u_{\sigma+h})-E(\sigma,u_{\sigma+h})\big)$$

$$= \int_0^1 \partial_\sigma E(\sigma+rh,u_{\sigma+h})\,\mathrm{d}r$$

by E3: convergence to $\partial_\sigma E(\sigma,u_\sigma)$

$$\Longrightarrow \partial_\sigma^+ I(\sigma) \geq \min\{\,\partial_\sigma E(\sigma,u)\,;\ u\in U(\sigma)\,\}.$$

by E2: convergence to $\partial_\sigma E(\sigma,u_\sigma)$.

Since $u_\sigma$ was an arbitrary minimizer we may take the infimum:

$\partial_\sigma^+ I(\sigma) \leq \min\{\,\partial_\sigma E(\sigma,u)\,;\ u\in U(\sigma)\,\}$.

For $h > 0$:

$$\tfrac{1}{h}\big(E(\sigma+h,u_\sigma) - E(\sigma,u_\sigma)\big)$$

$$\geq \tfrac{1}{h}\big(I(\sigma+h) - I(\sigma)\big)$$

$$\geq \tfrac{1}{h}\big(E(\sigma+h,u_{\sigma+h}) - E(\sigma,u_{\sigma+h})\big)$$

$$= \int_0^1 \partial_\sigma E(\sigma+rh,u_{\sigma+h})\,\mathrm{d}r$$

by E3: convergence to $\partial_\sigma E(\sigma,u_\sigma)$

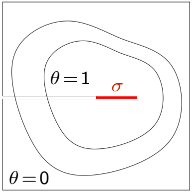
$$\Longrightarrow \partial_\sigma^+ I(\sigma) \geq \min\{\,\partial_\sigma E(\sigma,u)\,;\ u\in U(\sigma)\,\}.$$

by E2: convergence to $\partial_\sigma E(\sigma,u_\sigma)$.
Since $u_\sigma$ was an arbitrary minimizer we may take the infimum:
$\partial_\sigma^+ I(\sigma) \leq \min\{\,\partial_\sigma E(\sigma,u)\,;\ u\in U(\sigma)\,\}$.

Hence,

$$\lim_{h\searrow 0}\tfrac{1}{h}\big(I(\sigma+h) - I(\sigma)\big) = \min\{\,\partial_\sigma E(\sigma,u)\,;\ u\in U(\sigma)\,\}.$$

Fix $s \in (0, L)$.
For $\sigma \in [-\sigma_0, \sigma_0]$ define a family of inner variations via




$$\begin{aligned} T_{s,\sigma} : \Omega_s &\to \Omega_{s+\sigma} \\ x &\mapsto x + \sigma\theta(x) \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \end{aligned}$$

Chose $V = V(\Omega_s)$ and define $E : [-\sigma_0, \sigma_0] \times V \to \mathbb{R}$ via

$$E(\sigma, v) = \int_{\Omega_s} \det \nabla T_{s,\sigma} W(\nabla v (\nabla T_{s,\sigma})^{-1}) \,\mathrm{d}x - \int_{\Gamma_N} h \cdot v \,\mathrm{d}s.$$

Hence,

$$\begin{aligned} \mathcal{I}(s+\sigma) &= \min\{ \mathcal{E}(s+\sigma, v)\,;\, v \in V(\Omega_{s+\sigma}) \} \\ &= \min\{ E(\sigma, w)\,;\, w \in V(\Omega_s) \} = I(\sigma). \end{aligned}$$

Apply abstract theorem to $E \implies$ assertions on the energy release rate.

First attempt: Continuous evolution

Find the crack tip position $s : [0, T] \to [0, L]$ with $s(0) = s_0 > 0$ such that for every $t \in [0, T]$

(a) $s$ nondecreasing, $\partial_t s(t) = \dot{s}(t)$ exists

(b) local stability: $\kappa \geq \mathcal{G}(t, s(t))$,

(c) complementarity condition: $\dot{s}(t)\big(\kappa - \mathcal{G}(t, s(t))\big) = 0$.

Shortly: $0 \in \partial\mathcal{R}(\dot{s}(t)) - \mathcal{G}(t, s(t))$.

First attempt: Continuous evolution

Find the crack tip position $s : [0, T] \to [0, L]$ with $s(0) = s_0 > 0$ such that for every $t \in [0, T]$

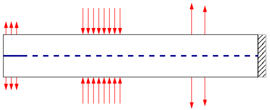
(a) $s$ nondecreasing, $\partial_t s(t) = \dot{s}(t)$ exists

(b) local stability: $\kappa \geq \mathcal{G}(t, s(t))$,

(c) complementarity condition: $\dot{s}(t)\big(\kappa - \mathcal{G}(t, s(t))\big) = 0$.

Shortly: $0 \in \partial\mathcal{R}(\dot{s}(t)) - \mathcal{G}(t, s(t))$.

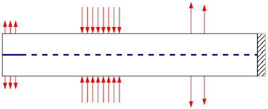
Problem: Continuity assumption for $s$ is too strong: $\to$ Example



Assumption: monotone loading: $h(t) = th_1,\ h_1 \in L^2(\Gamma_N)$.

$W$ quadratic $\Rightarrow$ $\mathcal{I}(t,s) = t^2\mathcal{I}(1,s), \quad \mathcal{G}(t,s) = -\partial_s\mathcal{I}(t,s) = t^2\mathcal{G}(1,s)$.

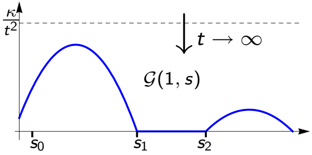
Question: Behavior of $s(t)$ for this particular loading?

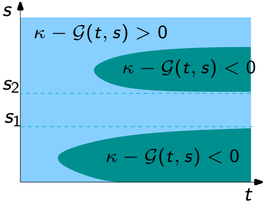


Assumption: monotone loading: $h(t) = th_1,\ h_1 \in L^2(\Gamma_N)$.

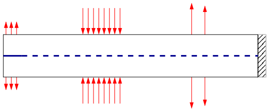
$W$ quadratic $\Rightarrow$ $\mathcal{I}(t,s) = t^2\mathcal{I}(1,s)$, $\mathcal{G}(t,s) = -\partial_s\mathcal{I}(t,s) = t^2\mathcal{G}(1,s)$.

Question: Behavior of $s(t)$ for this particular loading?



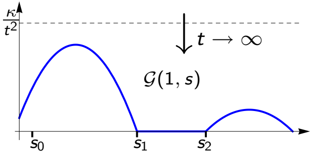

Local stability: $\frac{\kappa}{t^2} > \mathcal{G}(1,s) \iff \kappa - \mathcal{G}(t,s) > 0$

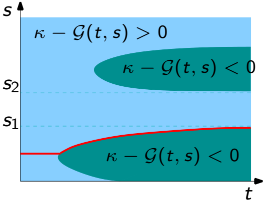


Assumption: monotone loading: $h(t) = th_1,\ h_1 \in L^2(\Gamma_N)$.

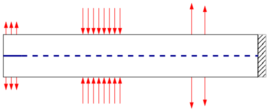
$W$ quadratic $\Rightarrow$ $\mathcal{I}(t,s) = t^2\mathcal{I}(1,s)$, $\mathcal{G}(t,s) = -\partial_s\mathcal{I}(t,s) = t^2\mathcal{G}(1,s)$.

Question: Behavior of $s(t)$ for this particular loading?



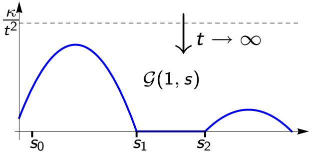

Local stability: $\frac{\kappa}{t^2} > \mathcal{G}(1,s)$ $\Longleftrightarrow$ $\kappa - \mathcal{G}(t,s) > 0$

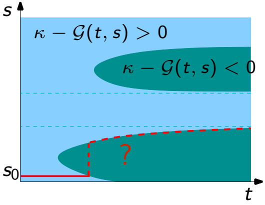


Assumption: monotone loading: $h(t) = th_1,\ h_1 \in L^2(\Gamma_N)$.

$W$ quadratic $\Rightarrow$ $\mathcal{I}(t,s) = t^2\mathcal{I}(1,s)$, $\mathcal{G}(t,s) = -\partial_s\mathcal{I}(t,s) = t^2\mathcal{G}(1,s)$.

Question: Behavior of $s(t)$ for this particular loading?




Local stability: $\frac{\kappa}{t^2} > \mathcal{G}(1,s)$ $\Longleftrightarrow$ $\kappa - \mathcal{G}(t,s) > 0$

$\Longrightarrow$ Extend the model such that discontinuous evolutions are possible.

Appropriate function space: $s \in BV([0,T];\mathbb{R})$

Strategies to improve the model: Derive suitable jump conditions via

- Global energetic approach
- Local approach based on viscous approximations

**Theorem**

*There exists a nondecreasing function $s \in BV([0,T],\mathbb{R})$ satisfying*

(S) **Global** *stability*

$$\mathcal{I}(t,s(t)) \le \mathcal{I}(t,\tilde{s}) + \mathcal{R}(\tilde{s}-s(t)) \qquad \forall \tilde{s} \in [s(t),s_1],$$

(E) *Energy balance*

$$\mathcal{I}(t,s(t)) + \int_{s(0)}^{s(t)} \kappa \mathrm{d}\sigma = \mathcal{I}(0,s(0)) + \int_0^t \partial_t \mathcal{I}(\tau,s(\tau))\mathrm{d}\tau.$$

Remark. For $t$, where $\dot{s}(t)$ exists, we have $\quad 0 \in \partial\mathcal{R}(\dot{s}(t)) - \mathcal{G}(t,s(t))$.

**Definition**

$s \in BV([0,T],\mathbb{R})$ satisfying (S)&(E) is a **global energetic solution**.

**Theorem**
*There exists a nondecreasing function $s \in BV([0,T],\mathbb{R})$ satisfying*

(S) **Global** *stability*

$$\mathcal{I}(t,s(t)) \le \mathcal{I}(t,\tilde{s}) + \mathcal{R}(\tilde{s}-s(t)) \qquad \forall \tilde{s} \in [s(t),s_1],$$

(E) *Energy balance*

$$\mathcal{I}(t,s(t)) + \int_{s(0)}^{s(t)} \kappa \mathrm{d}\sigma = \mathcal{I}(0,s(0)) + \int_0^t \partial_t \mathcal{I}(\tau,s(\tau))\mathrm{d}\tau.$$

Remark. For $t$, where $\dot{s}(t)$ exists, we have $\quad 0 \in \partial\mathcal{R}(\dot{s}(t)) - \mathcal{G}(t,s(t))$.

**Definition**
$s \in BV([0,T],\mathbb{R})$ satisfying (S)&(E) is a **global energetic solution**.

**Remark**
The global energetic formulation is a general concept for modeling rate indep. problems (e.g. elasto-plasticity, damage, shape-memory,...) (Mielke, Theil).

The Francfort/Marigo fracture model is based on a global energetic formulation.

$\tau$ time step size, $s_\tau^i$ crack length at time $t_i = i\tau$.

Global minimization

Find $s_\tau^i \geq s_\tau^{i-1}$ such that

$$s_\tau^i \in \operatorname{Argmin}\left\{\mathcal{I}(t_i,\sigma) + \tau\mathcal{R}\left(\frac{\sigma - s_\tau^{i-1}}{\tau}\right) ;\, \sigma \geq s_\tau^{i-1}\right\}.$$

$\tau$ time step size, $s_\tau^i$ crack length at time $t_i = i\tau$.

## Global minimization

Find $s_\tau^i \geq s_\tau^{i-1}$ such that

$$s_\tau^i \in \operatorname{Argmin}\left\{\mathcal{I}(t_i,\sigma) + \tau\mathcal{R}\Big(\frac{\sigma - s_\tau^{i-1}}{\tau}\Big)\,;\, \sigma \geq s_\tau^{i-1}\right\}.$$

Notation: $\hat{s}_\tau(t)$ piecewise affine interpolation.

- derive uniform estimates for the interpolants
- by Helly's selection principle: there exists a subsequence $\tau \searrow 0$ and $s \in BV([0,T,];\mathbb{R})$ such that

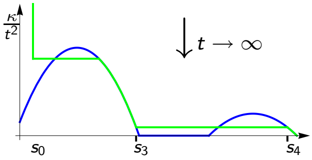
$$\hat{s}_\tau \overset{*}{\rightharpoonup} s \quad \text{in } BV([0,T],\mathbb{R}) \quad \text{and} \quad \hat{s}_\tau(t) \to s(t) \quad \text{for all } t \in [0,T].$$

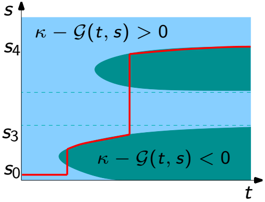
- prove that $s$ is a solution of the energetic model.

Global energetic solution

Global stability (S) $\Longrightarrow$ local stability $\forall t$: $\frac{\kappa}{t^2} - \mathcal{G}(1, s(t)) \geq 0$,

Energy balance (E) $\Longrightarrow$ jump condition: $\int_{s(t-)}^{s(t+)} \left(\frac{\kappa}{t^2} - \mathcal{G}(1,\sigma)\right) \mathrm{d}\sigma = 0$




- jump through the compressed region!

Penalize high velocities by adding a viscous term ($\nu > 0$):

$$0 \in \nu \dot{s}^{\nu}(t) + \partial \mathcal{R}(\dot{s}^{\nu}(t)) - \mathcal{G}(t, s^{\nu}(t)). \tag{1}$$

Penalize high velocities by adding a viscous term ($\nu > 0$):

$$0 \in \nu \dot{s}^\nu(t) + \partial \mathcal{R}(\dot{s}^\nu(t)) - \mathcal{G}(t, s^\nu(t)). \tag{1}$$

**Theorem**

*For every $\nu > 0$ exists a viscous solution $s^\nu \in H^1([0,T],\mathbb{R})$ with (1) for a.e. $t$. This is equivalent to*

($a^\nu$) $\dot{s}^\nu(t) \geq 0,$

($b^\nu$) $\kappa + \nu \dot{s}^\nu(t) - \mathcal{G}(t, s^\nu(t)) \geq 0,$

($c^\nu$) $\big(\kappa + \nu \dot{s}^\nu(t) - \mathcal{G}(t, s^\nu(t))\big)\dot{s}^\nu(t) = 0.$

Penalize high velocities by adding a viscous term ($\nu > 0$):

$$0 \in \nu \dot{s}^\nu(t) + \partial\mathcal{R}(\dot{s}^\nu(t)) - \mathcal{G}(t, s^\nu(t)). \tag{1}$$

**Theorem**

*For every $\nu > 0$ exists a viscous solution $s^\nu \in H^1([0,T],\mathbb{R})$ with (1) for a.e. $t$. This is equivalent to*

(a$^\nu$) $\dot{s}^\nu(t) \geq 0$,

(b$^\nu$) $\kappa + \nu\dot{s}^\nu(t) - \mathcal{G}(t, s^\nu(t)) \geq 0$,

(c$^\nu$) $\big(\kappa + \nu\dot{s}^\nu(t) - \mathcal{G}(t, s^\nu(t))\big)\dot{s}^\nu(t) = 0$.

*The energy balance is an immediate consequence of (c$^\nu$):*
$\forall\, 0 \leq t_1 < t_2 \leq T$

$$\mathcal{I}(t_2, s^\nu(t_2)) + \int_{t_1}^{t_2}\big(\kappa + \nu\dot{s}^\nu(t)\big)\dot{s}^\nu(t)\,\mathrm{d}t = \mathcal{I}(t_1, s^\nu(t_1)) + \int_{t_1}^{t_2}\partial_t\mathcal{I}(t, s^\nu(t))\,\mathrm{d}t.$$

**Proof**

Existence through time discretization or by considering the dual formulation.

- The continuity properties of $\mathcal{G}$ and $s^\nu$ imply (use $(\mathrm{c}^\nu)$)

$$\kappa - \mathcal{G}(t, s^\nu(t)) > 0 \quad \Rightarrow \quad \exists \delta > 0 : \ s^\nu\big|_{[t-\delta, t+\delta]} \text{ is constant.}$$

- A-priori estimates: $\exists c > 0 \ \forall \nu > 0$ and for every viscous solution $s^\nu$:

$$\|s^\nu\|_{L^\infty(0,T)} \le c, \quad \int_0^T \mathcal{R}(\dot{s}^\nu)\,\mathrm{d}t + \int_0^T \tfrac{\nu}{2}\,|\dot{s}^\nu|^2\,\mathrm{d}t \le c.$$

(from energy balance).

Theorem (KMZ08)

*There exists* $s \in BV([0,T],\mathbb{R})$ *and a subsequence* $\nu \searrow 0$ *with*

$$s^\nu \overset{*}{\rightharpoonup} s \quad \text{in } BV([0,T],\mathbb{R}) \quad \text{and} \quad s^\nu(t) \to s(t) \ \text{ for every } t \in [0,T].$$

*Moreover,*

(a) $s$ *is non-decreasing,*

(b) $\kappa - \mathcal{G}(t,s(t)) \geq 0$ *for every* $t \in [0,T]\backslash J(s)$,

(c) *if* $\kappa - \mathcal{G}(t,s(t)) > 0$, *then* $t \in D(s)$ *and* $\dot{s}(t) = 0$,

(d) $\forall\, t \in J(s)$, $\forall s_* \in [s(t_-), s(t_+)]$ *we have* $\kappa - \mathcal{G}(t,s_*) \leq 0$.

$J(s)$ jump set of $s$; $D(s)$ set of differentiable points of $s$.

Definition

$s \in BV([0,T],\mathbb{R})$ satisfying (a)–(d) is a **local energetic solution**.

Proof: A-priori estimates, Helley selection principle, continuity properties of $\mathcal{G}$, change of variables ($s \leftrightarrow t$) in (c$^\nu$) to obtain (d).

## Theorem (KMZ08)

*Every local energetic solution* $s : [0, T] \to [s_0, s_1]$, $s \in BV([0, T])$, *satisfies the energy balance:* $\forall\, 0 \leq t_1 < t_2 \leq T$, $t_i \notin J(s)$*:*

$$\mathcal{I}(t_2, s(t_2)) + \int_{s(t_1)}^{s(t_2)} \kappa(\sigma)\mathrm{d}\sigma + \mu([t_1, t_2]) = \mathcal{I}(t_1, s(t_1)) + \int_{t_1}^{t_2} \partial_t \mathcal{I}(t, s(t))\,\mathrm{d}t,$$

*where*

$$\mu([t_1, t_2]) = \sum_{t \in J(s) \cap ]t_1, t_2[} \int_{s(t-)}^{s(t+)} \big(\mathcal{G}(t, \sigma) - \kappa\big)\mathrm{d}\sigma,$$

*(d) implies that* $\mu([t_1, t_2]) \geq 0$.

Proof: Chain rule in $BV$ in combination with (b) and (c):

$$D\mathcal{I}(\cdot, s(\cdot)) = \partial_t \mathcal{I}(\cdot, s(\cdot))\,\mathrm{d}t + \underbrace{\partial_s \mathcal{I}(\cdot, s(\cdot))\widetilde{\mathrm{D}}s}_{=-\kappa(s(\cdot))\widetilde{\mathrm{D}}s} + \sum_{t \in J(s)} \underbrace{\big(\mathcal{I}(t, s(t+)) - \mathcal{I}(t, s(t-))\big)}_{=\int_{s(t-)}^{s(t+)} \partial_s \mathcal{I}(t,\sigma)\mathrm{d}\sigma} \delta_t.$$

$\widetilde{\mathrm{D}}s$ diffuse part of the distributional derivative of $s$.

$\tau$ time step size, $\quad s_\tau^i$ crack length at time $t_i = i\tau$.

Minimization with viscosity term

To find $\quad s_\tau^i \geq s_\tau^{i-1} \quad$ such that

$$s_\tau^i \in \operatorname{Argmin}\left\{ \mathcal{I}(t_i,\sigma) + \tau\mathcal{R}\left(\frac{\sigma - s_\tau^{i-1}}{\tau}\right) + \frac{\nu(\tau)}{2\tau}\left|\sigma - s_\tau^{i-1}\right|^2 ;\, \sigma \geq s_\tau^{i-1} \right\}$$

with viscosity $\nu(\tau)$ depending on time step size $\tau$.

Notation: $\quad \hat{s}_\tau^\nu \quad$ piecewise affine interpolation.

It holds:

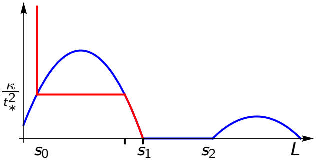
- Approximation of viscous solutions:
  $\nu > 0$ fixed, then for a subsequence we have
  $\hat{s}_\tau^\nu \overset{*}{\rightharpoonup} s^\nu$ in $BV([0,T],\mathbb{R})$ for $\tau \searrow 0$.
- Approximation of local energetic solutions:
  if $\nu(\tau) \to 0$ and $\frac{\nu(\tau)}{\tau} \to \infty$, then for a subsequence we have
  $\hat{s}_\tau^\nu \overset{*}{\rightharpoonup} s$ in $BV([0,T],\mathbb{R})$ for $\tau \searrow 0$.

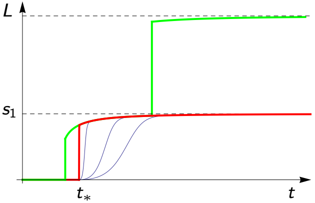
Local energetic solution

(b) $\frac{\kappa}{t^2} - \mathcal{G}(1, s(t)) \geq 0 \quad \forall t \in [0, T] \backslash J(s)$,

(c) $\frac{\kappa}{t^2} - \mathcal{G}(1, s(t)) > 0 \;\Rightarrow\; t \in D(s),\;\; \dot{s}(t) = 0$,

(d) $t \in J(s),\;\; s_* \in [s(t-), s(t+)] \qquad \Rightarrow \frac{\kappa}{t^2} - \mathcal{G}(1, s_*) \leq 0$.




$\longrightarrow$ crack does not pass through the compressed region!

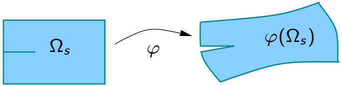
Linear elasticity:

Two possibilities to model discontinuous crack evolutions:

- Global energetic approach, solutions jump "early"
- Viscosity/local energetic approach, solutions jump later

Question:
What happens in the finite strain case with polyconvex energy densities?
Challenge: nonunique minimizers

$\varphi : \Omega_s \to \mathbb{R}^2$ deformation field,
$W : \mathbb{R}^{2\times 2} \to [0, \infty]$ elastic energy density.

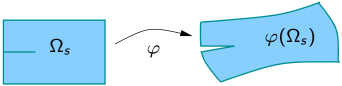



Assumptions: $F = \nabla\varphi \in \mathbb{R}^{2\times 2}$

- $W(F) = \infty$ if $\det F \leq 0$.
- Polyconvexity: $W(F) = g(F, \det F)$, $g$ convex and lower semi-continuous.
- Coercivity: $W(F) \geq c_1 |F|^p + c_2(\det F)^r - c_3$, $p \geq 2$, $r > 1$, $c_i > 0$.
- Multiplicative stress control $\forall F \in \mathbb{R}^{2\times 2}_+$: $\left|F^\top DW(F)\right| \leq c_1(W(F) + 1)$

Example: for $c_i > 0$ $\quad W(F) = c_1 |F|^2 + c_2(\det F)^2 - c_3 \log(\det F)$.

$\varphi : \Omega_s \to \mathbb{R}^2$ deformation field,
$W : \mathbb{R}^{2\times 2} \to [0,\infty]$ elastic energy density.




Assumptions: $F = \nabla\varphi \in \mathbb{R}^{2\times 2}$

- $W(F) = \infty$ if $\det F \le 0$.
- Polyconvexity: $W(F) = g(F, \det F)$, $g$ convex and lower semi-continuous.
- Coercivity: $W(F) \ge c_1 |F|^p + c_2(\det F)^r - c_3$, $p \ge 2$, $r > 1$, $c_i > 0$.
- Multiplicative stress control $\forall F \in \mathbb{R}^{2\times 2}_+$: $\left|F^\top DW(F)\right| \le c_1(W(F)+1)$

Example: for $c_i > 0$ $\quad W(F) = c_1 |F|^2 + c_2(\det F)^2 - c_3 \log(\det F)$.

Theorem (Ball '77)
*Let $g_D \in W^{1-\frac{1}{p},p}(\Gamma_D)$, $h(t) \in \left(W^{1-\frac{1}{p},p}(\Gamma_N)\right)'$. The functional*

$$\psi \mapsto \mathcal{E}(t,s,\psi) = \int_{\Omega_s} W(\nabla\psi)\,\mathrm{d}x - \langle h(t), \psi\rangle_{\Gamma_N}$$

*has a global minimizer $\varphi_s(t) \in V^p(\Omega_s) = \{\, \psi \in W^{1,p}(\Omega_s)\,;\, \psi\big|_{\Gamma_D} = g_D \,\}$.*

Right and left energy release rates:

$$\mathcal{G}^+(t,s) := -\lim_{\delta\searrow 0} \tfrac{1}{\delta}\left(\mathcal{I}(t,s+\delta) - \mathcal{I}(t,s)\right) = -\partial_s^+ \mathcal{I}(t,s),$$

$$\mathcal{G}^-(t,s) := -\lim_{\delta\searrow 0} \tfrac{1}{\delta}\left(\mathcal{I}(t,s) - \mathcal{I}(t,s-\delta)\right) = -\partial_s^- \mathcal{I}(t,s).$$

Griffith formula with Eshelby tensor $\mathbb{E}$: $\quad G(s,\varphi) := \int_{\Omega_s} \mathbb{E}(\nabla\varphi) : \nabla \varrho_s \,\mathrm{d}x$

**Theorem (KMZ09)**

*$\mathcal{I}(\cdot,\cdot) \in C^{lip}([0,T]\times(0,L))$ and*

$\mathcal{G}^+(t,s) = \max\{\, G(s,\varphi)\,;\, \varphi$ *minimizes* $\mathcal{E}(t,s,\cdot)\,\}$ *is usc. and right continuous,*

$\mathcal{G}^-(t,s) = \min\{\, G(s,\varphi)\,;\, \varphi$ *minimizes* $\mathcal{E}(t,s,\cdot)\,\}$ *is lsc. and left continuous.*

Right and left energy release rates:

$$\mathcal{G}^+(t,s) := -\lim_{\delta\searrow 0}\tfrac{1}{\delta}\left(\mathcal{I}(t,s+\delta)-\mathcal{I}(t,s)\right) = -\partial_s^+\mathcal{I}(t,s),$$
$$\mathcal{G}^-(t,s) := -\lim_{\delta\searrow 0}\tfrac{1}{\delta}\left(\mathcal{I}(t,s)-\mathcal{I}(t,s-\delta)\right) = -\partial_s^-\mathcal{I}(t,s).$$

Griffith formula with Eshelby tensor $\mathbb{E}$: $\quad G(s,\varphi) := \int_{\Omega_s}\mathbb{E}(\nabla\varphi):\nabla\varrho_s\,\mathrm{d}x$

**Theorem (KMZ09)**
$\mathcal{I}(\cdot,\cdot)\in C^{lip}([0,T]\times(0,L))$ *and*

$\mathcal{G}^+(t,s) = \max\{\,G(s,\varphi)\,;\,\varphi$ *minimizes* $\mathcal{E}(t,s,\cdot)\,\}$ *is usc. and right continuous,*
$\mathcal{G}^-(t,s) = \min\{\,G(s,\varphi)\,;\,\varphi$ *minimizes* $\mathcal{E}(t,s,\cdot)\,\}$ *is lsc. and left continuous.*

**Remark:**
$[\partial_s^+\mathcal{I}(t,s),\partial_s^-\mathcal{I}(t,s)] \;=\;$ Clarke subdifferential $\partial_s^{\mathrm{Cl}}\mathcal{I}$ of the map $s\mapsto\mathcal{I}(t,s)$.
Theorem $\Rightarrow$ $\partial_s^{\mathrm{Cl}}\mathcal{I}(t,\cdot):(0,L)\to\mathcal{P}(\mathbb{R})$ is upper semicont. as a setvalued map.

**Important:** $\mathcal{G}^-(t,s)<\mathcal{G}^+(t,s)$ is not excluded!
Which energy release rate should occur in the crack evolution model?

Like in the small strain case:

$\tau$ time step size, $s_\tau^i$ crack length at time $t_i = i\tau$.

## Minimization with viscosity term

Find $s_\tau^i \geq s_\tau^{i-1}$ such that

$$s_\tau^i \in \operatorname{Argmin}\left\{\mathcal{I}(t_i,\sigma) + \tau\mathcal{R}\Big(\frac{\sigma - s_\tau^{i-1}}{\tau}\Big) + \frac{\nu(\tau)}{2\tau}\big|\sigma - s_\tau^{k-1}\big|^2 ;\, \sigma \geq s_\tau^{i-1}\right\}$$

with viscosity $\nu(\tau) > 0$.

Like in the small strain case:

$\tau$ time step size, $s_\tau^i$ crack length at time $t_i = i\tau$.

### Minimization with viscosity term

Find $s_\tau^i \geq s_\tau^{i-1}$ such that

$$s_\tau^i \in \operatorname{Argmin}\left\{\mathcal{I}(t_i,\sigma) + \tau\mathcal{R}\Big(\frac{\sigma - s_\tau^{i-1}}{\tau}\Big) + \frac{\nu(\tau)}{2\tau}\big|\sigma - s_\tau^{k-1}\big|^2 ;\, \sigma \geq s_\tau^{i-1}\right\}$$

with viscosity $\nu(\tau) > 0$.

Minimizers satisfy

(a) local stability: $\kappa - \mathcal{G}^+(t_i, s_\tau^i) + \frac{\nu}{\tau}(s_\tau^i - s_\tau^{i-1}) \geq 0$

(b) complementarity condition: $\Big(\kappa - \mathcal{G}^+(t_i, s_\tau^i) + \nu(\tau)\frac{s_\tau^i - s_\tau^{i-1}}{\tau}\Big)\,\frac{s_\tau^i - s_\tau^{i-1}}{\tau} = 0.$

Problem: $\mathcal{G}^+$ is upper semicontinuous, only. (Recall: $\mathcal{G}^+ \geq \mathcal{G}^-$).

**Theorem (KMZ09)**

*There exists $s \in BV([0,T],\mathbb{R})$ and a subsequence $\tau \searrow 0$ with*

$$\hat{s}_\tau \overset{*}{\rightharpoonup} s \quad \text{in } BV([0,T],\mathbb{R}) \quad \text{and} \quad \hat{s}_\tau(t) \to s(t) \ \text{ for every } t \in [0,T].$$

*Moreover,*

(a) $s$ *is non-decreasing,*

(b$^-$) $\kappa - \mathcal{G}^-(t,s(t)) \geq 0$ *for every* $t \in [0,T] \backslash J(s)$,

(c$^+$) *if* $\kappa - \mathcal{G}^+(t,s(t)) > 0$, *then* $t \in D(s)$ *and* $\dot{s}(t) = 0$,

(d$^+$) $\forall\, t \in J(s)$, $\forall s_* \in [s(t_-), s(t_+)]$ *we have* $\kappa - \mathcal{G}^+(t,s_*) \leq 0$.

$J(s)$ jump set of $s$; $D(s)$ set of differentiable points of $s$.

Proof: A-priori estimates, Helley selection principle, continuity property of $\partial_s^{\mathrm{Cl}}\mathcal{I}$.

Complementarity condition:

For every $t \in D(s)$ exists $g(t) \in [\mathcal{G}^-(t,s(t)), \mathcal{G}^+(t,s(t))]$ with

$$\big(\kappa - g(t)\big)\dot{s}(t) = 0, \qquad 0 \in \partial\mathcal{R}(\dot{s}(t)) - g(t).$$

**Theorem**
*There exists* $s \in BV([0,T],\mathbb{R})$ *which satisfies*

(a) $s$ *is non-decreasing,*

(b$^+$) $\kappa - \mathcal{G}^+(t,s(t)) \geq 0$ *for every* $t \in [0,T]\backslash J(s)$,

(c$^+$) *if* $\kappa - \mathcal{G}^+(t,s(t)) > 0$, *then* $t \in D(s)$ *and* $\dot{s}(t) = 0$,

(d$^+$) $\forall\, t \in J(s)$, $\forall s_* \in [s(t_-), s(t_+)]$ *we have* $\kappa - \mathcal{G}^+(t,s_*) \leq 0$.

$J(s)$ jump set of $s$; $D(s)$ set of differentiable points of $s$.

**Theorem**

*There exists $s \in BV([0,T],\mathbb{R})$ which satisfies*

(a) *$s$ is non-decreasing,*

($b^+$) $\kappa - \mathcal{G}^+(t,s(t)) \geq 0$ *for every* $t \in [0,T]\backslash J(s)$,

($c^+$) *if* $\kappa - \mathcal{G}^+(t,s(t)) > 0$, *then* $t \in D(s)$ *and* $\dot{s}(t) = 0$,

($d^+$) $\forall\, t \in J(s)$, $\forall s_* \in [s(t_-), s(t_+)]$ *we have* $\kappa - \mathcal{G}^+(t,s_*) \leq 0$.

$J(s)$ jump set of $s$; $D(s)$ set of differentiable points of $s$.

Conclusion: For every $t \in D(s)$ the complementarity condition is satisfied:

$$\big(\kappa - \mathcal{G}^+(t,s(t))\big)\dot{s}(t) = 0 \quad \text{and} \quad 0 \in \partial\mathcal{R}(\dot{s}(t)) - \mathcal{G}^+(t,s(t)).$$

Proof: $s_{\max}(t) = \max\{\, s(t)\,;\, s \text{ local energetic sol.}\,\}$ satisfies (a),($b^+$)–($d^+$).

Open question:

Does the viscous approximation lead to local energetic solutions with $\mathcal{G}^+$?

- Small strain elasticity:
  - discontinuous evolutions are not excluded in the rate independent setting
  - Models which enable discontinuous solutions:
    - global energetic approach
    - local energetic approach
  - energy release rate is continuous

- Finite strain elasticity
  - nonunique minimizers $\Longrightarrow$ multivalued ERR with $\mathcal{G}^- < \mathcal{G}^+$ possible
  - Open question: viscous approximation $\Rightarrow$ special local energetic solutions?
  - example with $\mathcal{G}^- < \mathcal{G}^+$?

- Viscous time-incremental problem suitable for numerical approximations.
- Energy balance holds for local energetic solutions.
- Analysis available for smooth curved cracks + non-penetration condition.

- Efendiev/Mielke, *On the rate independent limit of systems with dry friction and small viscosity*, J. Convex Analysis, 2006.
- Knees/Mielke, *Energy release rate for cracks in finite–strain elasticity*, M2AS, 31, 501–528, 2008.
- Knees/Mielke/Zanini, *On the inviscid limit of a model for crack propagation*, M3AS, 2008.
- Knees/Mielke/Zanini, *Crack propagation in polyconvex materials*, Physica D, 2009.
- Negri/Ortner, Quasi–static crack propagation by Griffith's criterion, M3AS, 2009.
- Toader/Zanini, *An artificial viscosity approach to quasistatic crack growth*, SISSA-Preprint 43/M, 2006.