Some mathematical questions studied by MACADAM

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http://w3.bretagne.ens-cachan.fr/math/macadam/

Asymptotic methods, mechanics and other applications

Antenne de Bretagne de l'ENS Cachan

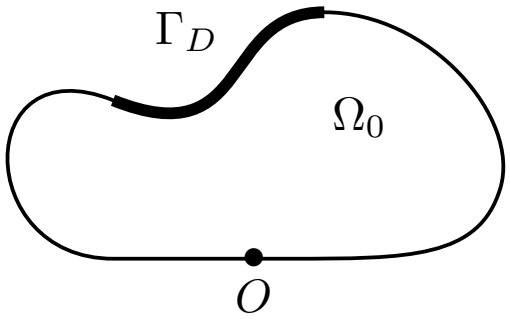
Motivations

Multiscale asymptotic and numerical approximation for surface defects and applications in mechanics.

- take into account surface and volume microdefects of materials
  $\rightsquigarrow$ multiscale asymptotic analysis
- propose a cheap numerical method
  $\rightsquigarrow$ superposition method
- application in mechanics
  $\rightsquigarrow$ cracks initiation and propagation

The leading ideas

Assumptions on the unperturbed problem



- $O \in \partial\Omega_0 \setminus \Gamma_D$
- $\partial\Omega_0$ is flat in the neighborhood of O
- $f \in C_0^\infty(\Omega)$

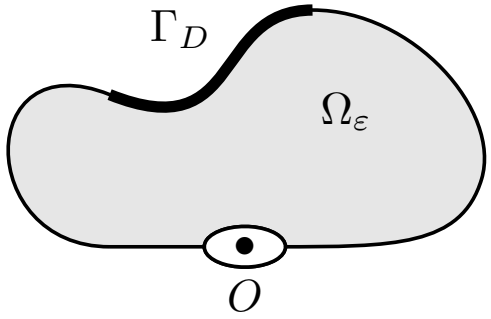
Solution of the unperturbed problem

$$\begin{cases} -\Delta u_0 &= f & \text{in } \Omega_0 \\ u_0 &= 0 & \text{on } \Gamma_D \\ \partial_{\mathbf{n}} u_0 &= 0 & \text{on } \partial\Omega_0 \setminus \Gamma_D \end{cases}$$

A single defect

[Mazja-Nazarov-Plamenevskii, Dambrine-Vial]

Assumption



- $\Omega_\varepsilon = \Omega_0 \setminus \varepsilon\omega$ with ω starlike w.r.t. O
- $f \in C_0^\infty(\Omega)$

Solution of the perturbed problem

$$\left\{ \begin{array}{rcll} -\Delta u_\varepsilon & = & f & \text{in } \Omega_\varepsilon \\ u_\varepsilon & = & 0 & \text{on } \Gamma_D \\ \partial_{\mathbf{n}} u_\varepsilon & = & 0 & \text{on } \partial\Omega_\varepsilon \setminus \Gamma_D \end{array} \right.$$

Compare u_ε and u_0

The main ideas of the asymptotic analysis

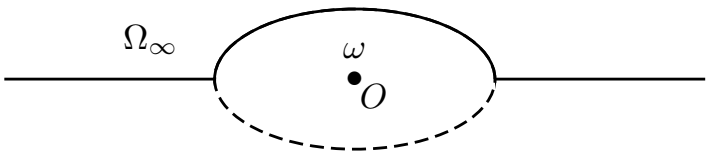
- use multiscale expansion
  - ♢ *slow variable* x at the scale of the domain,
  - ♢ *fast variable* x/ε at the scale of the perturbation

- compare u_ε to the limit u_0
  $\Rightarrow$ correctors to *compensate* the Taylor expansion of u_0 at 0

- to make the correctors stick on Ω_ε
  (through a cutoff function for boundary inclusions)
  $\Rightarrow$ generate correctors in slow variables

Profile

Solution of a boundary value problem in the infinite domain $\Omega_\infty = \lim_{\varepsilon \to 0} \Omega_\varepsilon / \varepsilon$:



$$\begin{cases} -\Delta V^1 &= 0 & \text{in } \Omega_\infty \\ \partial_n V^1 &= g & \text{on } \partial\Omega_\infty \\ V^1 &\to 0 & \text{at infinity} \end{cases}$$

$$g = -\nabla u_0(0) \cdot \mathbf{n}$$

Theorem

There is a unique weak solution V^1 in the variational space

$$\left\{ V \ ; \ \nabla V \in \mathrm{L}^2(\Omega_\infty) \quad \text{and} \quad \frac{V}{(1+|X|)\log(2+|X|)} \in \mathrm{L}^2(\Omega_\infty) \right\}$$

Behavior at infinity:

$$V^1(X) = \mathcal{O}(|X|^{-1}) \text{ and } \nabla V^1(X) = \mathcal{O}(|X|^{-2}) \quad \text{when } |X| \to \infty$$

A model result: expansion at order N

For all N, the solution u_ε of

$$\begin{cases} -\Delta u_\varepsilon &= f \text{ in } \Omega_\varepsilon \\ u_\varepsilon &= 0 \text{ on } \Gamma_D \\ \partial_{\mathbf{n}} u_\varepsilon &= 0 \text{ on } \partial\Omega_\varepsilon \backslash \Gamma_D \end{cases}$$

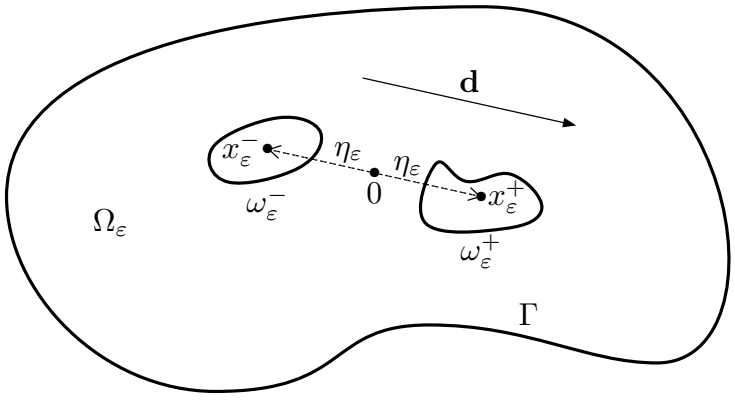
reads

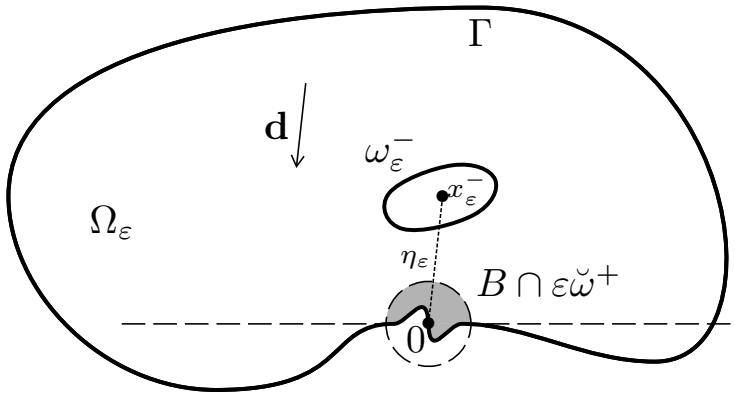
$$u_\varepsilon(x) = u_0(x) + \chi(x) \sum_{i=1}^{N} \varepsilon^i V^i\left(\tfrac{x}{\varepsilon}\right) + \sum_{i=2}^{N} \varepsilon^i w^i(x) + \mathcal{O}_{\mathrm{H}^1(\Omega_\varepsilon)}(\varepsilon^{N+1})$$

- the profiles V^i compensate the ith term u^i of Taylor expansion of u_0 and of w^j for $j < i$
- the correctors (in the slow variable) w^i compensate the error generated by the cutoff. $\|w^i\|_{\mathrm{H}^1(\Omega_\varepsilon)} = \mathcal{O}(1)$

Two inclusions

[Bonnaillie-Noël, Dambrine, Tordeux, Vial, 2008]





$$\left\{ \begin{array}{rcll} -\Delta u_\varepsilon & = & f & \text{in } \Omega_\varepsilon \\ u_\varepsilon & = & 0 & \text{on } \Gamma \\ \partial_{\mathbf{n}} u_\varepsilon & = & 0 & \text{on } \partial\Omega_\varepsilon^\pm \end{array} \right.$$

- $\eta_\varepsilon = \mathcal{O}(1)$: no interaction

$$u_\varepsilon(x) = u_0(x) + \varepsilon \left[V_0^+\left(\frac{x-x^+}{\varepsilon}\right) + V_0^-\left(\frac{x-x^-}{\varepsilon}\right) \right] + \mathcal{O}_{\mathrm{H}^1(\Omega_\varepsilon)}(\varepsilon^2)$$

- $\eta_\varepsilon = \mathcal{O}(\varepsilon)$: total interaction

$$u_\varepsilon(x) = u_0(x) + \varepsilon W_0\left(\frac{x}{\varepsilon}\right) + \mathcal{O}_{\mathrm{H}^1(\Omega_\varepsilon)}(\varepsilon^2)$$

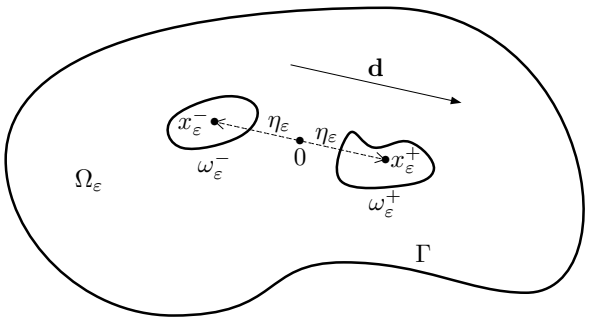
W_0 : profile associated with $\omega = \omega^+ \cup \omega^-$

- $\eta_\varepsilon = \mathcal{O}(\varepsilon^\alpha), 0 < \alpha < 1$: case in between

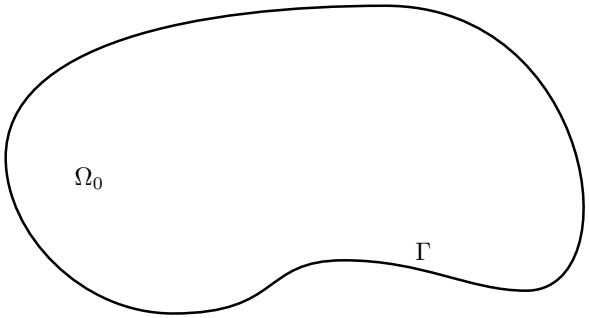
$$u_\varepsilon(x) = u_0(x) + \varepsilon \left[V_0^-\left(\frac{x-x_\varepsilon^-}{\varepsilon}\right) + V_0^+\left(\frac{x-x_\varepsilon^+}{\varepsilon}\right) \right] + \mathcal{O}(???)$$

Asymptotic expansion for two inclusions

$$\Omega_\varepsilon = \Omega_0 \setminus \overline{\left(\omega_\varepsilon^- \cup \omega_\varepsilon^+\right)}, \text{ with } \omega_\varepsilon^\pm = x_\varepsilon^\pm + \varepsilon\omega^\pm, \quad x_\varepsilon^\pm = \pm\varepsilon^\alpha \mathbf{d}$$



$$\begin{cases} -\Delta u_\varepsilon &= f & \text{in } \Omega_\varepsilon \\ u_\varepsilon &= 0 & \text{on } \Gamma = \partial\Omega_0 \\ \partial_{\mathbf{n}} u_\varepsilon &= 0 & \text{on } \partial\omega_\varepsilon^\pm \end{cases}$$



$$\begin{cases} -\Delta u_0 &= f & \text{in } \Omega_0 \\ u_0 &= 0 & \text{on } \Gamma = \partial\Omega_0 \end{cases}$$

Two scale expansion

Theorem

The solution u_ε has the order N asymptotic expansion

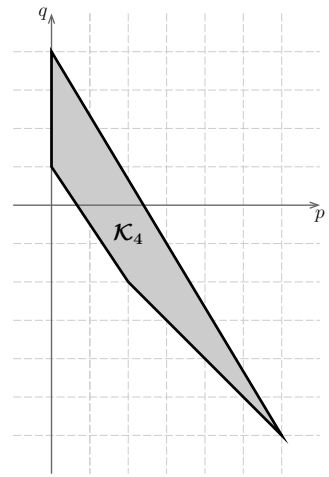
$$u_\varepsilon(x) = u_0(x) + \varepsilon\left[V_0^-\left(\tfrac{x-x_\varepsilon^-}{\varepsilon}\right) + V_0^+\left(\tfrac{x-x_\varepsilon^+}{\varepsilon}\right)\right]$$

$$+ \sum_{(p,q)\in\mathcal{K}_N} \varepsilon^{p+\alpha q}\left(v_{p+\alpha q}(x) + \varepsilon\left[V_{p+\alpha q}^-\left(\tfrac{x-x_\varepsilon^-}{\varepsilon}\right) + V_{p+\alpha q}^+\left(\tfrac{x-x_\varepsilon^+}{\varepsilon}\right)\right]\right) + r_\varepsilon^N(x)$$

with

$$\mathcal{K}_N = \{(p,q)\in\mathbb{Z}^2 \mid p\geq 0,$$
$$q\geq -\tfrac{3}{2}p+1,\ q\geq -p \text{ and } p+\alpha q\leq N\}$$

$$\|r_\varepsilon^N\|_{\mathrm{H}^1(\Omega_\varepsilon)} = o(\varepsilon^N)$$



$\mathcal{K}_4$ for $\alpha = \frac{3}{5}$

Interpretation of the leading terms

$$u_\varepsilon(x) = u_0(x) + \varepsilon \left[V_0^-\left(\frac{x-x_\varepsilon^-}{\varepsilon}\right) + V_0^+\left(\frac{x-x_\varepsilon^+}{\varepsilon}\right)\right] + r_\varepsilon^1(x)$$

with

$$\|r_\varepsilon^1\|_{\mathrm{H}^1(\Omega_\varepsilon)} = \mathcal{O}(\varepsilon^{\min(1+\alpha,3-2\alpha)})$$

The reminder r_ε^1 contains information about higher order terms:

- ◇ if $\alpha < 2/3$: the inclusions are rather distant
  the main error comes from the Taylor expansion of u_0 at 0
- ◇ if $2/3 < \alpha < 1$: the inclusions are rather close
  the main error comes from the *interaction* between profiles V_0^- and V_0^+
- ◇ when $\alpha = 2/3$: both contributions are of similar weight

Applications for numerical computations

Idea

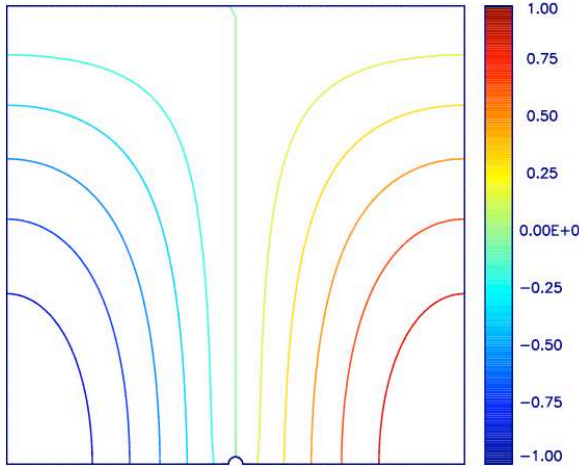
Approximate u_ε by the first order expansion

$$u_\varepsilon(x) \approx u_1(x) = u_0(x) + \varepsilon\chi(x)V(\tfrac{x}{\varepsilon})$$

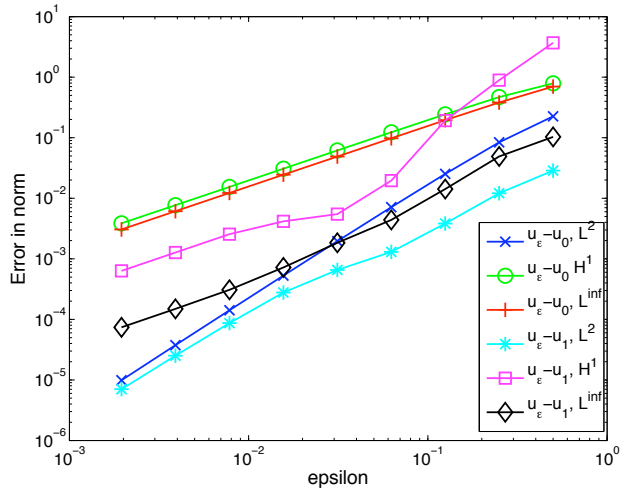
Compute

- u_0 : the solution of the problem set in the unperturbed domain
- V : the profile defined in the unbounded domain

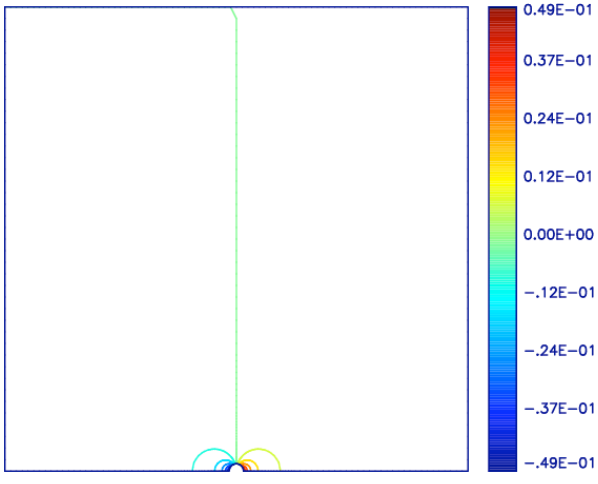
Computations for Neumann problem



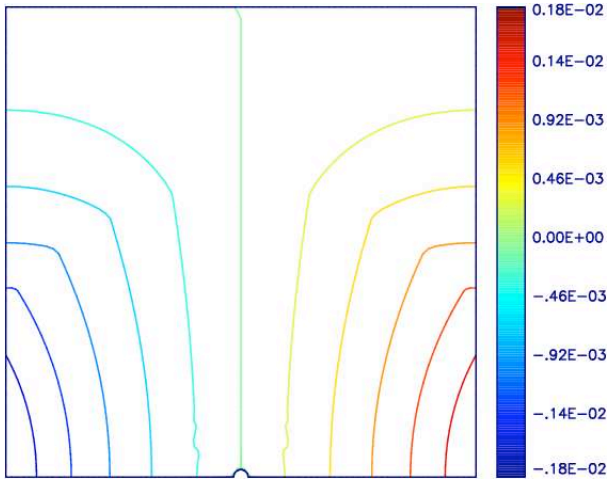
u_ε $(\varepsilon = \frac{1}{32})$



Errors norms

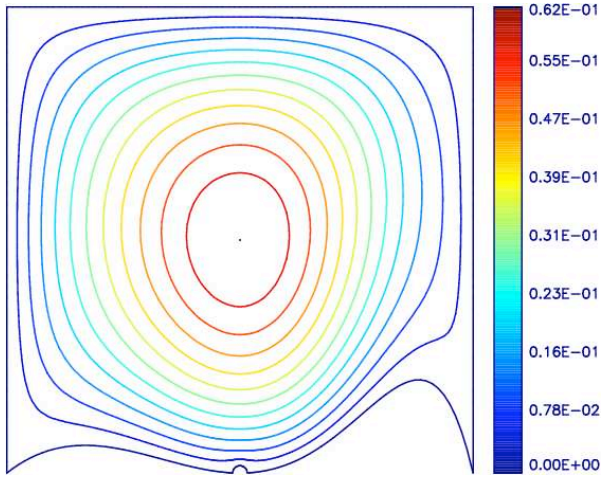


$u_\varepsilon - u_0$ $(\varepsilon = \frac{1}{32})$

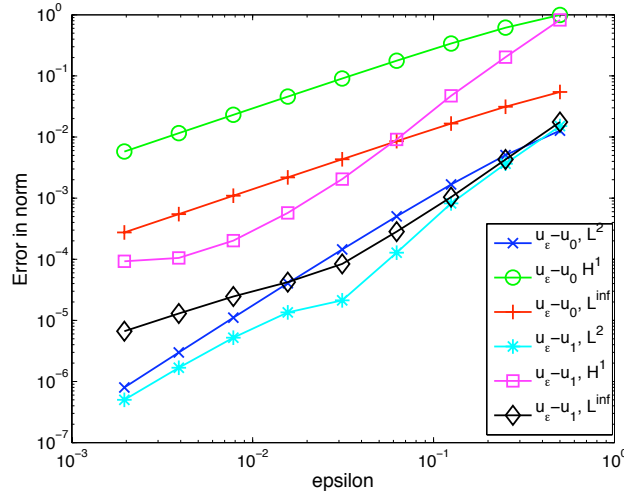


$u_\varepsilon - u_1$ $(\varepsilon = \frac{1}{32})$

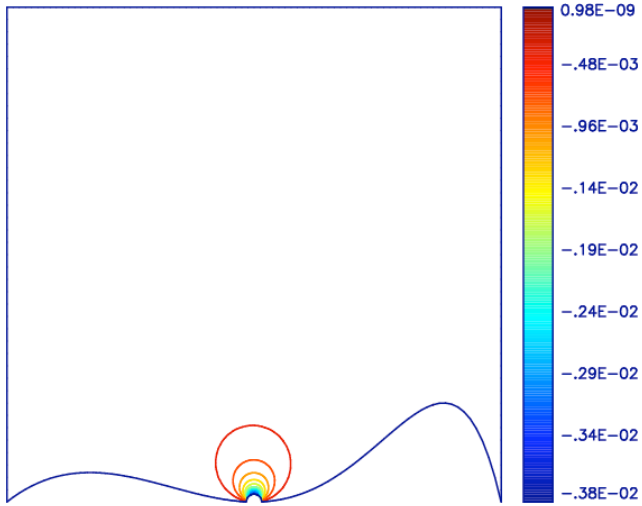
Computation for Dirichlet problem *(curved boundary)*



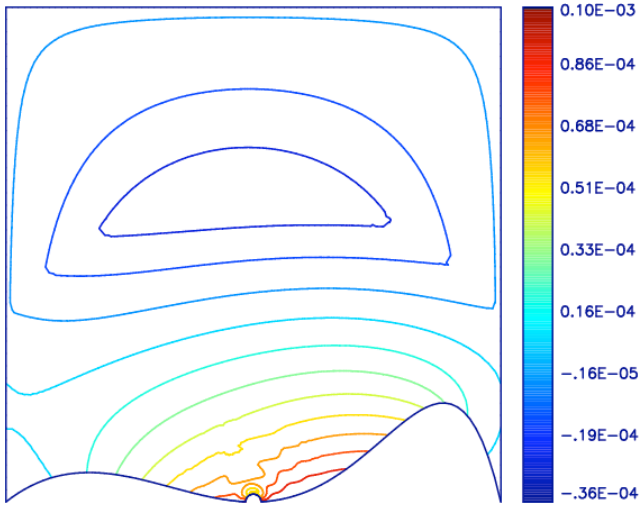
u_ε $(\varepsilon = \frac{1}{32})$



Errors norms



$u_\varepsilon - u_0$ $(\varepsilon = \frac{1}{32})$



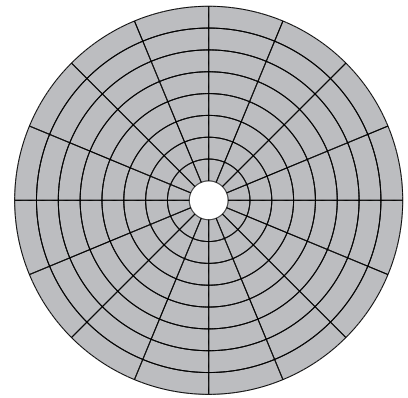
$u_\varepsilon - u_1$ $(\varepsilon = \frac{1}{32})$

Computation of the profile

$$\left\{\begin{array}{rcll} -\Delta V & = & 0 & \text{in } \mathbb{R}^2 \setminus \overline{\omega} \\ \partial_{\mathbf{n}} V & = & g & \text{on } \partial\omega \\ V & \to & 0 & \text{at infinity} \end{array}\right.$$

- Strategy 1 : absorbing conditions on $|x| = R$

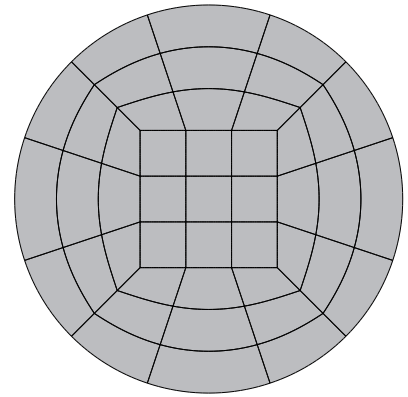
Dirichlet	$V = 0$
Robin	$V + R\partial_{\mathbf{n}} V = 0$
Wentzell	$V + \frac{3R}{2}\partial_{\mathbf{n}} V - \frac{R^2}{2}\partial_\tau^2 V = 0$



- Strategy 2 : inversion $\varphi : z \mapsto 1/z$

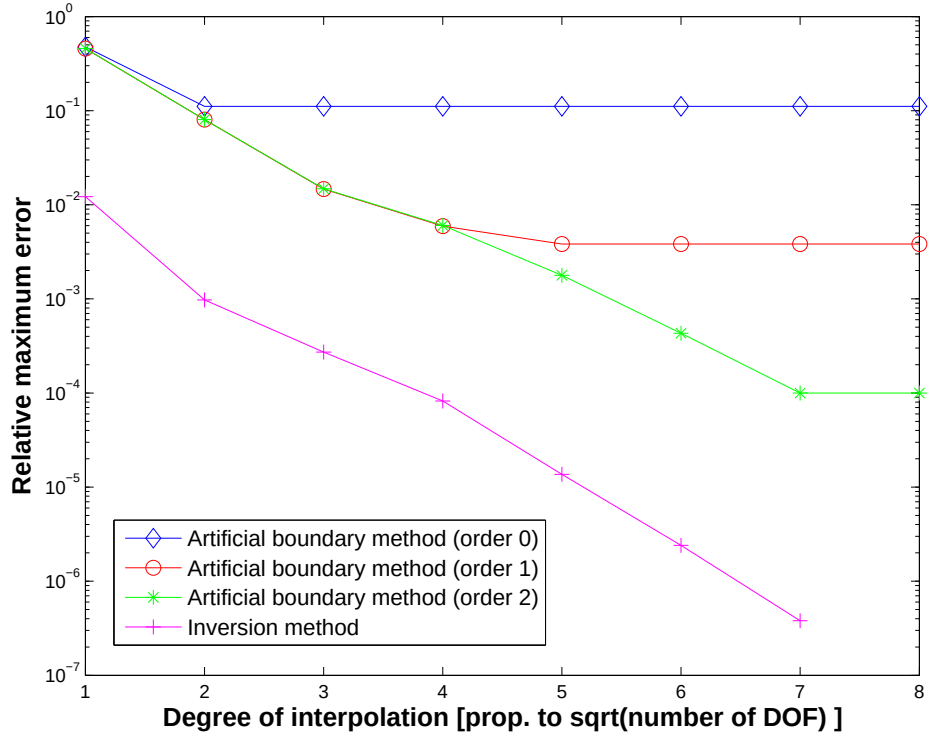
$W = V \circ \varphi$ solution of

$$\left\{\begin{array}{rcl} -\Delta W & = & 0 \text{ in } \varphi(\omega) \\ \partial_{\mathbf{n}} W & = & \partial_{\mathbf{s}}\varphi \; (g \circ \varphi) \text{ on } \partial\varphi(\omega) \\ W(0) & = & 0 \end{array}\right.$$



Comparison of strategies with a fixed number of d.o.f.

- ω is a disk
- $R = 10$
- known profile: $V(x) = \frac{\cos\theta}{r}$ inversion : $W(x) = x_1$
  boundary value condition: $g(x) = \cos\theta - 2\cos(2\theta) - 3\cos(3\theta)$



Fixed $\mathbb{Q}_8$ mesh and variable interpolation degree.

Computation of the profile for planar linear elasticity

[Bonnaillie-Noël, Brancherie, Dambrine, Vial], in preparation

$$\begin{cases} -\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\mathrm{div}\,(\mathbf{u}) &= 0 & \text{in } \mathbb{R}^2 \setminus \overline{\omega} \\ \sigma(\mathbf{u}).\mathbf{n} &= g & \text{on } \partial\omega \\ \mathbf{u} &= \mathbf{0} & \text{on } \partial(\mathbb{R}^2 \setminus \overline{\omega}) \setminus \partial\omega \\ \mathbf{u} &\to \mathbf{0} & \text{at infinity} \end{cases}$$

We compute the leading terms at infinity in the upper half plane and we seek an algebraic expression cancelling them. We find the absorbing boundary condition on $|x| = R$

$$\sigma(\mathbf{u})\mathbf{n} + \frac{1}{R}\,\frac{E}{1+\nu}\begin{bmatrix} \frac{1}{1-\nu} & 0 \\ 0 & 1 \end{bmatrix}\mathbf{u} + \frac{1}{R}\,\frac{E(1-\nu)}{2(1+\nu)(1-2\nu)}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\Delta_\tau\mathbf{u} = 0.$$

This is a degenerate Wentzell boundary value condition. The main difficulty is that

$$\frac{E(1-\nu)}{2(1+\nu)(1-2\nu)} > 0 \text{ since } E > 0 \text{ and } \nu \in (-1, 0.5)$$

How to solve a scalar Wentzell boundary value problem ?

The weak formulation of problem

$$\left\{\begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega, \\ \partial_n u + \alpha u + \beta \Delta_\tau u & = & 0 & \text{on } \partial\Omega, \end{array}\right.$$

then reads

$$\text{Find } u \in \mathcal{H}(\Omega) \text{ such that for all } v \in \mathcal{H}(\Omega),\ A(u,v) = B(v),$$

where

$$A(u,v) = \int_\Omega \nabla u \nabla v + \int_{\partial\Omega} \alpha\, uv - \beta\, \nabla_\tau u . \nabla_\tau v \quad \text{and} \quad B(v) = \int_\Omega fv,$$

on the Hilbert space

$$\mathcal{H}(\Omega) = \left\{ u \in \mathrm{H}^1(\Omega),\ u_{|\partial\Omega} \in \mathrm{H}^1(\partial\Omega) \right\}.$$

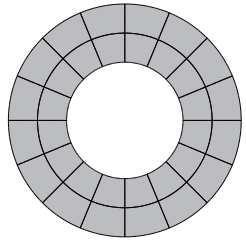
When $\alpha > 0$ and $\beta < 0$, the bilinear form A is coercive.

What happens when $\beta > 0$?

[Bonnaillie-Noël, Dambrine, Hérau, Vial] submitted

An analytic computation in a ring. Consider the boundary value problem:

$$\begin{cases} -\Delta u &= 0 & \text{in } \Omega = B_{R_e} \setminus \overline{B_{R_i}}, \\ \partial_n u &= g & \text{on } \partial B_{R_i}, \\ \partial_n u + \dfrac{\alpha}{R_e} u + \dfrac{\beta}{R_e} \partial^2_{\theta\theta} u &= 0 & \text{on } \partial B_{R_e}. \end{cases}$$



Seeking u in Ω under Laurent's series:

$$u(r,\theta) = d + c \ln r + \sum_{n=1}^{\infty} \left(a_n r^n + a_{-n} r^{-n}\right) \cos n\theta + \left(b_n r^n + b_{-n} r^{-n}\right) \sin n\theta,$$

leads to linear systems of the type

$$\begin{bmatrix} R_i^{n-1} n & -R_i^{-n-1} n \\ R_e^{n-1}\left(\alpha + n - \beta n^2\right) & R_e^{-n-1}\left(\alpha - n - \beta n^2\right) \end{bmatrix} \begin{bmatrix} a_n \\ a_{-n} \end{bmatrix} = \begin{bmatrix} -a_n(g) \\ 0 \end{bmatrix}.$$

What happens when $\beta > 0$?

The linear systems have a unique solution if and only if

$$nR_e^{-n-1}R_i^{n-1}\left(\alpha - n - \beta n^2\right) + nR_e^{n-1}R_i^{-n-1}\left(\alpha + n - \beta n^2\right) \neq 0.$$

This is possible except for the values of n and of the radii R_i and R_e characterized by

$$\left(\frac{R_e}{R_i}\right)^{2n} = -\frac{\alpha - n - \beta n^2}{\alpha + n - \beta n^2} = -1 + \frac{2n}{\alpha + n - \beta n^2}.$$

Since $R_e/R_i > 1$ by definition, a necessary condition on n is

$$\begin{cases} \alpha + n & > & \beta n^2, \\ \alpha & < & \beta n^2. \end{cases}$$

For a fixed value of R_i, Wentzell boundary value problem has a unique solution if R_e is large enough.

About existence and uniqueness of the solution

- Idea: reformulation as a nonlocal problem on the boundary

$$\left\{\begin{array}{rcll} -\Delta u & = & 0 & \text{in } \Omega, \\ \partial_n u + \alpha\, u + \beta\, \Delta_\tau u & = & \varphi & \text{on } \partial\Omega, \end{array}\right.$$

can be rewritten as the surface equation

$$\beta \Delta_\tau w + \Lambda w + \alpha w = \varphi \text{ on } \partial\Omega.$$

Here Λ is the Dirichlet-to-Neumann map associated to the Laplace operator on Ω:

$$\begin{array}{rcl} \mathrm{H}^{1/2}(\partial\Omega) & \to & \mathrm{H}^{-1/2}(\partial\Omega) \\ \psi & \mapsto & \Lambda(\psi) = \partial_n U \end{array}$$

where U is the solution of the boundary value problem

$$\left\{\begin{array}{rcll} -\Delta U & = & 0 & \text{in } \Omega, \\ U & = & \psi & \text{on } \partial\Omega. \end{array}\right.$$

- The result: generic existence and uniqueness

About existence and uniqueness of the solution

- Idea: reformulation as a nonlocal problem on the boundary

- The result: generic existence and uniqueness

  Operator $P_{\alpha,\beta} = -\beta\Delta_\tau - \Lambda - \alpha\mathrm{Id}$ is an elliptic selfadjoint semi-bounded from below pseudodifferential operator of order 2.

  Besides, for fixed $\beta > 0$, there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ growing to infinity such that for any $\phi \in \mathrm{H}^s(\partial\Omega)$ with $s \in \mathbb{R}$, we have

  1. If $\alpha \notin \{\alpha_n\}$, then equation $-P_{\alpha,\beta} w = \phi$ admits a unique solution in $\mathcal{S}'(\partial\Omega)$ which, in addition, belongs to $\mathrm{H}^{s+2}(\partial\Omega)$;

  2. If $\alpha \in \{\alpha_n\}$, then there is either no solution or a complete affine finite dimensional space of $\mathrm{H}^{s+2}(\partial\Omega)$ solutions.

Does the Wentzell boundary value problem have a unique solution if Ω is a ball of radius R chosen large enough ?

We make the change of variable $y = x/R$.

For a subdomain D of Ω, consider the differential operator L_D on $\mathrm{H}^1(\partial\Omega)$ with values in $\mathrm{H}^{-1}(\partial\Omega)$ defined by

$$L_D(u) = \alpha u + \Lambda_D(u) + \beta\Delta_\tau u,$$

where Λ_D is the Dirichlet-to-Neumann map defined as $\Lambda_D(\varphi) = \partial_n U$ and U denotes the solution of

$$\begin{cases} -\Delta U &= 0 & \text{in } \Omega \setminus \overline{D}, \\ U &= \varphi & \text{on } \partial\Omega, \\ U &= 0 & \text{on } \partial D. \end{cases}$$

Question: Is $L_{\omega_\varepsilon(x_0)}$ invertible ?

Does the Wentzell boundary value problem have a unique solution if Ω is a ball of radius R chosen large enough ?

- Idea: continuity of L_D around $\emptyset$
- If $L_\emptyset$ is invertible, then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the operator $L_{\omega_\varepsilon(x_0)}$ is invertible. The key estimate is

$$\|\Lambda_{\omega_\varepsilon(x_0)} - \Lambda_\emptyset\|_{\mathcal{L}(\mathrm{H}^{1/2}(\partial\Omega),\mathrm{H}^{-1/2}(\partial\Omega))} \leq \frac{C}{|\ln \varepsilon|}.$$

- Assume that
  Ω is the unit disk and that $\beta n^2 - n - \alpha \neq 0 \quad$ for all $n \in \mathbb{N}$,
  then $L_\emptyset$ is invertible.
  As a consequence, there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the operator $L_{\omega_\varepsilon(x_0)}$ is invertible.

  Conclusion: Wentzell boundary value problem has a unique solution if Ω is a ball of radius R chosen large enough.