Enhanced resolution in structured media

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Outline:
1. An experiment of ‘super–resolution’
2. Small volume asymptotics
3. Periodicity defect in a composite
4. The main ingredients for the proofs
5. Back to ‘super–resolution’
1. Super (enhanced) resolution:

Experiment conducted at LOA by M. Fink and co-workers
Focusing beyond the diffraction limit with far-field time reversal (G. Lerosey et al, Science, 315, 1120, 2007)

Microwave sources surrounded by scatterers made of thin copper wires

Inter-scatterer distance = $\lambda/100$. Sources are $\lambda/30$ apart
The time reversal experiment:

- One of the sources emits a signal at time 0, at a frequency $\omega = 2\pi / \lambda$
- The signal is recorded far away from the source by a series of transducers, during some time interval $(0, T)$
- The signal is then time reversed $\phi(t) = \phi(T - t)$ and sent back into the medium by the transducers
- As the wave equation is symmetric, the signal back propagates towards the originating source

In the absence of scatterers, the signal gets time-reversed into a spot of diameter $\sim \lambda/2$. The originating source cannot be distinguished

In the presence of scatterers, the time-reversed signal sharply refocuses on the originating antenna
2. Small volume asymptotics

Reference medium conductivity $a(x)$

\[
\begin{align*}
\text{div}(a \nabla u) &= 0 \\
a \nabla u \cdot n &= g \\
\int_{\partial \Omega} u &= 0
\end{align*}
\]

Perturbed medium

\[
a_d(x) = a(x) + (k_j - a(x)) 1_{\omega_j}(x)
\]

\[
\begin{align*}
\text{div}(a_d \nabla u_d) &= 0 \\
a \nabla u_{d/\partial \Omega} \cdot n &= g \\
\int_{\partial \Omega} u_d &= 0
\end{align*}
\]
Asymptotics of the voltage difference

In the presence of $N$ inclusions centered at the points $z_j$ \hspace{1cm} (Fengya-Moskow-Vogelius)

$$ u_d(x) - u(x) = \varepsilon^n \sum_{j=1}^{N} M_j : \nabla u(z_j) \otimes \nabla N(x, z_j) + o(\varepsilon^n) \quad x \in \partial \Omega $$

where $M_j$ is a polarization tensor that contains some information on the geometry of the j-th inclusion + conductivity contrast

$N(x, z)$ is the Neumann’s function of the reference medium
• Such asymptotics have been extended to the Helmholtz equations, the Maxwell system, elasticity, to strip–like inclusions (Ammari–Kang, Ammari–Moskow, Beretta–Francini–Vogelius, Movchan–Serkov, Volkov–Vogelius,...)

• Compensated compactness approach and bounds on the polarization tensors (Capdeboscq–Vogelius)

• The singularity in the Neumann or the Green function in the RHS is very interesting for detection purposes: MUSIC type algorithms prove quite efficient to determine the location of inclusions (Brühl–Hanke–Vogelius, Ammari–Iakovleva, see also factorisation method)

• Echoscan: EIT measurements perturbed by localized ultrasound waves. Small volume asymptotics show that asymptotically the data determine the pointwise values of the electrostatic energy density \( \rightarrow \) solve the ‘0 Laplacian’

\[
\text{div} \left( \frac{j(x)}{|\nabla u(x)|^2} \nabla u(x) \right) = 0
\]

(Ammari–B–Capdeboscq–Tanter–Fink)
3. Misplaced inclusions in a periodic composite

A situation where the size of the ‘defect’ compares to the size of the variations of the background conductivity

\( \Omega_\varepsilon \subset \mathbb{R}^3 \) perfectly periodic medium, with conductivity \( a_\varepsilon(x) = a(x/\varepsilon) \)

\( Y \) periodicity cell, contains an inclusion \( D \)

\[
\begin{align*}
  a(y) &= k \quad \text{in } D \\
  a(y) &= 1 \quad \text{in } Y \setminus D
\end{align*}
\]

\( \Omega_{\varepsilon,d} \) perturbed medium: the p-th cell \( \varepsilon(p + Y) \) contains a misplaced inclusion.

The pth cell lies in \( \omega_{\varepsilon,2} = \varepsilon(p + d + D) \), \( |d| < 1 \) instead of \( \omega_{\varepsilon,1} = \varepsilon(p + D) \).

\[
\omega_\varepsilon = \omega_{\varepsilon,1} \cup \omega_{\varepsilon,2}
\]
The perturbed conductivity is

\[
\begin{align*}
    a_{\varepsilon,d}(x) &= a_\varepsilon(x) & x \in \Omega \setminus \omega_\varepsilon \\
    a_{\varepsilon,d}(x) &= 1 & x \in \omega_\varepsilon,1 \\
    a_{\varepsilon,d}(x) &= k & x \in \omega_\varepsilon,2
\end{align*}
\]

\[\omega_\varepsilon = (\omega_\varepsilon,1 \cup \omega_\varepsilon,2)\]

State equations: Given \( g \) on \( \partial\Omega \), smooth,

\[
\begin{align*}
    \mathcal{L}_\varepsilon u_\varepsilon &= -\text{div}(a_\varepsilon \nabla u_\varepsilon) = 0 \quad \text{in } \Omega \\
    a_\varepsilon \nabla u_\varepsilon \cdot \nu &= g \quad \text{on } \partial\Omega \\
    \int_{\partial\Omega} u_\varepsilon &= 0
\end{align*}
\]

\[
\begin{align*}
    \mathcal{L}_{\varepsilon,d} u_{\varepsilon,d} &= -\text{div}(a_{\varepsilon,d} \nabla u_{\varepsilon,d}) = 0 \quad \text{in } \Omega \\
    a_{\varepsilon,d} \nabla u_{\varepsilon,d} \cdot \nu &= g \quad \text{on } \partial\Omega \\
    \int_{\partial\Omega} u_{\varepsilon,d} &= 0
\end{align*}
\]

Let \( z \in \Omega \), \( \text{dist}(z, \omega_\varepsilon) \gg \varepsilon \). Assume the defect is centered at \( x = 0 \).

\[\rightarrow \text{Asymptotics of } u_{\varepsilon,d}(z) \text{ to } u_\varepsilon(z) \text{ as } \varepsilon \to 0\]
Representing both $u_\varepsilon$ and $u_{\varepsilon,d}$ with the Green function $G_\varepsilon$, obtain

$$
(u_{\varepsilon,d} - u_\varepsilon)(z) + \text{term on } \partial \Omega = \int_{\omega_\varepsilon} [a_\varepsilon - a_{\varepsilon,d}] \nabla u_{\varepsilon,d}(x) \cdot \nabla G_\varepsilon(x, z) \, d\sigma_x.
$$

As $\varepsilon \to 0$, $u_\varepsilon, u_{\varepsilon,d}$ converge to the homogenized potential $u_*$

$$
\begin{cases}
\text{div}(A \nabla u_*) = 0 \text{ in } \Omega \\
A \nabla u_* \cdot \nu = g \text{ on } \partial \Omega \\
\int_\Omega u_* = 0,
\end{cases}
$$

where $\chi$ is the vector the components of which solve the cell problems

$$
\begin{cases}
\text{div}(A \nabla (\chi_j(y) + y_j)) = 0 \text{ in } Y \\
\chi_j \in H^1_*(Y), \int_Y \chi_j = 0 \text{ on } \partial \Omega
\end{cases}
$$
Ansatz: \[ u_{\varepsilon,d}(x) \sim u_{\varepsilon}(x) + \varepsilon v_d(x/\varepsilon) + r_{\varepsilon,d}(x) \]

where \( v_d \) solves the rescaled problem \( y = x/\varepsilon \)

\[
\begin{align*}
\text{div} \left( a_d(y) \left[ (I + \nabla y \chi) \nabla u_*(0) + \nabla v_d(y) \right] \right) &= 0 \quad \text{in } \mathbb{R}^3 \\
v_d(y) &\to 0 \quad \text{as } |y| \to \infty
\end{align*}
\]

\( v_d \) is the correction to the oscillatory potential with linear growth that would be solution if there were no perturbation.

Inject in the representation formula

\[
\begin{align*}
&u_{\varepsilon,d}(z) - u_{\varepsilon}(z) + \text{term on } \partial \Omega \\
&= \int_{\omega_{\varepsilon}} \left( a_{\varepsilon} - a_{\varepsilon,d} \right) \left[ \nabla_x u_{\varepsilon}(x) + \nabla_y v_d(x/\varepsilon) \right] \cdot \nabla_x G_\varepsilon(x, z) \\
&\quad + \int_{\omega_{\varepsilon}} \left( a_{\varepsilon} - a_{\varepsilon,d} \right) \nabla r_{\varepsilon,d}(x) \cdot \nabla_x G_\varepsilon(x, z) = I_1 + I_2.
\end{align*}
\]
Approximate
\[ I_1 = \int_{\omega_\varepsilon} (a_\varepsilon - a_{\varepsilon,d}) \left[ \nabla_x u_\varepsilon(x) + \nabla_y v_d(x/\varepsilon) \right] \cdot \nabla_x G_\varepsilon(x, z) \]

\[ u_\varepsilon(x) \sim u_\ast(x) + \varepsilon \chi(x/\varepsilon) \cdot \nabla u_\ast(x) \]
\[ \nabla u_\varepsilon(x) \sim [I + \nabla_y \chi(x/\varepsilon)] \nabla u_\ast(0) \]
\[ \nabla G_\varepsilon(x, z) \sim [I + \nabla_y \chi(x/\varepsilon)] \nabla G_\ast(0, z) \]

to obtain
\[ I_1 \sim \varepsilon^n \int_{\omega_1} (a - a_d) \left[ (I + \nabla_y \chi(y)) \nabla u_\ast(0) + \nabla_y v_d(y) \right] \cdot (I + \nabla_y \chi(y)) \nabla_x G_\ast(0, z) \]

while \[ I_2 = o(\varepsilon^n) \]
Rewritting the auxiliary function $v_d(y) = \varphi(y) \nabla u_*(0)$, with $\varphi$ defined by

$$
\begin{cases}
\text{div} [a_d(y) \nabla \varphi_j] = \text{div} [(a - a_d)(y) \nabla (y_j + \chi_j)] & \text{in } \mathbb{R}^n, \\
\varphi_j(y) \to 0 & \text{as } |y| \to 0
\end{cases}
$$

to get the **first order asymptotics**:

$$
(u_{\varepsilon,d} - u_{\varepsilon})(z) + \int_{\partial \Omega} a_{\varepsilon} \partial_{\nu_x} G_{\varepsilon}(x, z) (u_{\varepsilon,d} - u_{\varepsilon})(x) \\
= \varepsilon^n M : \nabla_x u_*(0) \otimes \nabla_x G_*(0, z) + o(\varepsilon^n)
$$

where the polarization tensor is given by

$$
M_{ij} = \int_{\partial \bar{\omega}} \left( \frac{a^-_d}{a^-} - 1 \right) (y_i + \chi_i) \left[ a^+ \partial_{\nu} \varphi_j^+ + a^- (y_j + \chi_j)^- \right]
$$

**Same structure** as in the case of a homogeneous medium, with homogenized potential and Green's function
4. Ingredients for the proofs

The expansions are based on $W^{1,\infty}$ estimates on $u_\varepsilon$ and on $G_\varepsilon$, which are uniform with respect to $\varepsilon$. In the case $n = 3$:

\[
\|\nabla_x u_\varepsilon\|_{L^\infty(\omega_\varepsilon)} \leq C \\
\|\nabla_x u_\varepsilon(\cdot) - (I + \nabla_y \chi(\cdot/\varepsilon)) \nabla_x u_\varepsilon(\cdot)\|_{L^\infty(\omega_\varepsilon)} \leq C\varepsilon \\
\|\nabla_x G_\varepsilon(\cdot, z) - (I + \nabla_y \chi(\cdot/\varepsilon)) \nabla_x G_\varepsilon(\cdot, z)\|_{L^\infty(\omega_\varepsilon)} \leq C\varepsilon^{1/4} \\
\|\nabla_y r_\varepsilon,d\|_{L^2(\varepsilon^{-1}\Omega)} \leq C\varepsilon^{3/2} \\
\nabla_y v_d(y) = O(|y|^{-2}),
\]

where the constant $C$ is uniform w.r.t. $\varepsilon$ (and w.r.t. $z$ away from the position of the defect).
Uniform estimates on the gradients $\nabla u_\varepsilon$ of the potential for a perfectly periodic medium, were first established by M. Avellaneda and Fang Hua Lin under the assumption that the coefficients are regular:

**Theorem 1.** Let $f \in L^{n+\delta}(\Omega)$, $g \in C^{1,\mu}(\partial\Omega)$, $\|a\|_{C^{0,\gamma}(Y)} \leq M$, with uniform ellipticity (with constants $\lambda, \Lambda$)

Let $u_\varepsilon$ solve

\[
\begin{cases}
-\div(a(x/\varepsilon)\nabla u_\varepsilon) = f & \text{in } \Omega \\
u_\varepsilon = g & \text{on } \partial\Omega,
\end{cases}
\]

Then for $C = C(n, \Omega, \lambda, \Lambda, \delta, \mu, \gamma, M)$, one has

\[
\|u_\varepsilon\|_{C^{0,1}(\Omega)} \leq C \left(\|g\|_{C^{1,\mu}(\Omega)} + \|f\|_{L^{n+\delta}(\Omega)}\right).
\]
Extension to the case of piecewise Hölder coefficients:

- \( Y = \bigcup_{l=1}^{L} \overline{D_l} \), matrix phase is \( D_L = Y \setminus \bigcup_{l=1}^{L-1} \overline{D_l} \)
- Each \( D_l \) has regularity \( C^{1,\alpha} \)
- Each point of \( \overline{Y} \) belongs to at most 2 \( D_l \)'s

**Theorem 2.** (YanYan Li and M. Vogelius, YanYan Li and L. Nirenberg)

Assume that \( \alpha \in C^\mu(\overline{D_l})_{1 \leq l \leq L} \) + uniform ellipticity.

Let \( f \in L^\infty(\Omega), \ g \in C^\mu(\overline{D_l})_{1 \leq l \leq L}, \)

and let \( u \) solve \(-\text{div}(a \nabla u) = f + \text{div}(g) \) in \( \Omega \).

Then, \( \forall \rho > 0 \), there is a constant \( C' \) such that

\[
\sum_{l=1}^{L} \|u\|_{C^{1,\alpha'}(\overline{D_l} \cap \Omega_\rho)} \leq C' \left[ \|u\|_{L^2(\Omega)} + \|f\|_{L^\infty(\Omega)} + \sum_{l} \|g\|_{C^{0,\alpha'}(\overline{D_l})} \right]
\]

for all \( 0 < \alpha' \leq \min(\mu, \frac{\alpha}{2(\alpha + 1)}) \).
One can combine both previous theorems to get uniform Lipschitz estimates on $u_\varepsilon$

**Theorem 3.**

Let $B = B(0, 1)$ and $a$, elliptic, $Y$—periodic, let $b$ $Y$—periodic, such that

$$a, b \in C^{0, \mu}(Y \setminus D) \quad \text{and} \quad a, b \in C^{0, \mu}(\overline{D})$$

Let $f \in L^\infty(B)$, $h \in C^{0, \mu}(B)$ and $u_\varepsilon$ solves

$$-\text{div}(a_\varepsilon \nabla u_\varepsilon) = f + \varepsilon \text{div}(b_\varepsilon h) \quad \text{in } B$$

then, for some $C'$ independent on $\varepsilon$,

$$\|u_\varepsilon\|_{C^{0, \mu}(B_{1/2})} + \|\nabla u_\varepsilon\|_{L^\infty(B_{1/2})} \leq C \left[\|u_\varepsilon\|_{L^2(B)} + \|f\|_{L^\infty(B)} + \|h\|_{C^{0, \mu}(B)}\right]$$
Consequences:

Corrector result:
Assume that \( \| u_\varepsilon - u_0 \|_{L^2(\Omega)} \leq C \varepsilon^{\sigma} \), with \( 0 < \sigma < 1 \), then, for some \( C \) independent on \( \varepsilon \),

\[
\| u_\varepsilon - u_0 \|_{L^\infty(\omega)} \leq C \varepsilon^{\sigma},
\]

\[
\| \nabla u_\varepsilon(.) - [I + \nabla_y x(./\varepsilon)] \nabla u_0(.) \|_{L^\infty(\omega)} \leq C \varepsilon^{\sigma}.
\]

Estimation of \( G_\varepsilon(., z) \):
Let \( \omega \subset \subset \Omega \), then for some \( C \) independent on \( \varepsilon \)

\[
\| G_\varepsilon(., z) - G_0(., z) \|_{L^\infty(\omega)} \leq C \varepsilon^{1/4},
\]

\[
\| \nabla G_\varepsilon(., z) - [I + \nabla_y x(./\varepsilon)] \nabla G_0(., z) \|_{L^\infty(\omega)} \leq C \varepsilon^{1/4}.
\]
5. Back to super resolution:

Time reversal operator in the case of a small inclusion centered at $z$, embedded in a homogeneous medium

$$(u_\delta - u)(x, t) \sim -|\omega_\delta| \mathbf{M} \nabla U_y(z, T) \cdot \nabla \left[ U_z(x, t_0 - t) - U_z(x, t - t_0) \right]$$

where $T = |y - z|$, $U_y(x, t) = \frac{\delta_{t = |x - y|}}{4\pi |x - y|}$

In Fourier space

$$\mathcal{F}(u_\delta - u)(x, k) \sim |\omega_\delta| p \cdot \nabla \left( \frac{\sin(k|x - z|)}{|x - z|} \right)$$
Harmonic regime \( \hat{U}_z(x, k) = \frac{e^{ik|x-z|}}{4\pi|x-z|} \)

\[
\delta \hat{u}(x) = (\hat{u}_\delta - \hat{u})(x, k) \sim |\omega_\delta| \nabla \hat{U}_y(z, k) M \nabla \hat{U}_x(z, k)
\]

The response function corresponds to

\[
\hat{w}(x) = \int_S \hat{U}_x(x', k) \partial_\nu \delta \hat{u}(x') - \delta \hat{u}(x') \partial_\nu \hat{U}_x(x', k) \, d\sigma(x')
\]

is proportionnal to \( \nabla \text{Im}(\hat{U}_z(x, k)) \sim \nabla \left( \frac{\sin(k|x-z|)}{|x-z|} \right) \)

In the case of a homogeneous background medium, the width of the first lobe of the Green’s function is roughly equal to \( k^{-1} = (\omega \sqrt{\varepsilon_0 \mu})^{-1} \)
In $O \subset \mathbb{R}^2$, consider inclusions distributed periodically in a region $\Omega$, which also contains a defect

\[
a_{\delta,d} = \begin{cases} 
(\varepsilon_s + i\sigma_s/\omega)^{-1} & \text{in the inclusions} \\
\varepsilon_d^{-1} & \text{in the defect} \\
\varepsilon_0^{-1} & \text{elsewhere}
\end{cases}
\]

and the Helmholtz equation

\[
\begin{align*}
\text{div}(a_{\delta,d}(x) \nabla u_{\delta,d}) + \omega^2 \mu u_{\delta,d} &= 0 \\
u_{\delta,d}/\partial O &= \phi
\end{align*}
\]

We assume that

\[
\begin{cases} 
\varepsilon_0, \varepsilon_d, \varepsilon_s > 0, & \sigma_s > 0 \\
\mu_0 = \mu_d = \mu_s, & k = \omega^2 \mu_0
\end{cases}
\]

As above, consider corresponding defect-free field $u_\delta$ and homogenized field $u_*$

Assume that $\omega$ is not an eigenvalue of $\text{div}(A_* \nabla v) + \omega^2 \mu v$
**Ingredients:**

- The smoothness of the homogenized limit $u_*$
- Pointwise interior estimates on $u_\delta - u_*$
- A convergence estimate
  \[
  \|u_{\delta,d} - u_\delta\|_{1,2,O} = O(|D_\delta|^{1/2}), \quad \|u_{\delta,d} - u_\delta\|_{0,2,O} = O(|D_\delta|^{1/2+\eta})
  \]

  to prove the asymptotic expansion:

  \[
  (u_{\delta,d} - u_\delta)(x) = |\omega_\delta|(a_{\delta,d} - a_\delta)/\partial \Omega M_* : \nabla u_*(0) \otimes \nabla G_*(0,x) + o(|\omega_\delta|)
  \]

The response function involves the homogenized Green function.

When the homogenized matrix is isotropic (and real), the resolution is proportional to the effective wavelength

\[
k_*^{-1} = (\omega \sqrt{\varepsilon_*, \mu})^{-1}
\]
Conclusions/perspectives

- Small volume asymptotic in a periodic background medium: The asymptotics of the fields have a structure similar to the asymptotics obtained in the case of a smooth (slowly varying) background.

- Super-resolution experiment: Choosing the proper dielectric parameters for the scatterers may indeed improve resolution: In the case when $\sigma_s = 0$, $\varepsilon_s > \varepsilon_0$ then $k_*^{-1} < k_0^{-1}$.

- What are the distributions/geometries of the scatterers that achieve optimal resolution (generically, homogenized media are anisotropic)?

- When $\sigma_s > 0$ energy is absorbed in the scatterers and the diameter of the scatterers, the size of $\Omega$ might play a role.
The 3–step compactness method to obtain uniform bounds on $u_\varepsilon$

**Theorem 4.** Assume $f_h \in L^\infty(B_1)$ and $u_\varepsilon$ solves

$$-\text{div}(a_\varepsilon \nabla u_\varepsilon) = f, \quad \text{in } B_1$$

Then there exists $C = C(B_1, A)$ independent of $h$ such that

$$\|u_\varepsilon\|_{W^{1,\infty}(B_1/2)} \leq C \left(\|u_\varepsilon\|_{L^2(B_1)} + \|f\|_{L^\infty(B_1)}\right)$$
Sketch of proof: for a uniform Hölder estimate

step 1: Let $0 < \mu = \alpha/2(\alpha + 1)$.

Show that $\exists 0 < \theta < 1, \exists 0 < \varepsilon_0 < 1$ such that, if

\[
\begin{aligned}
-\text{div}(a_\varepsilon \nabla u_\varepsilon) &= f \quad \text{in } B_1 \\
\int_{B_1} |u_\varepsilon|^2 &\leq 1 \\
\|f\|_{L^\infty(B_1)} &\leq \varepsilon_0
\end{aligned}
\]

then for $0 < \varepsilon < \varepsilon_0$,

\[
\int_{B_\theta} |u_\varepsilon - (\underline{u}_\varepsilon)_{0,\theta}|^2 \leq \theta^{2\mu}
\]

proof: Let $\mu < \mu' < 1$. By elliptic regularity, the solutions to the limiting problem

\[
\text{div}(A \nabla u_*) = 0 \quad \text{in } B_1
\]

are smooth:

$\exists 0 < \theta < 1$ such that

\[
\int_{B_\theta} |u_* - (\underline{u}_*)_{0,\theta}|^2 \leq \theta^{2\mu'} \int_{B_1} |u_*|^2
\]
Assume that a sequence $\varepsilon_n \to 0$, $u_{\varepsilon_n}$, $f_{\varepsilon_n}$ does not satisfy the estimate

We can extract a subsequence such that $u_{\varepsilon_n} \rightharpoonup u^*$ in $H^1_{loc}(B_1)$, a solution to the limiting equation.

Passing to the limit one gets

$$\theta^{2\mu} \leq \liminf \int_{B_1} |u_{\varepsilon_n} - (\overline{u_{\varepsilon_n}})_{0,\theta}|^2$$

$$= \int_{B_1} |u^* - (\overline{u^*})\theta|^2 = \theta^{2\mu'}$$

a contradiction.
**step 2 : Iteration**

Show that for all $k \geq 1$, such that $\varepsilon / \theta^k \leq \varepsilon_0$

$$
\int_{B_{\theta^k}} |u_\varepsilon - (u_\varepsilon)_{0, \theta}|^2 \leq \theta^{2k\mu} \left( \|u_\varepsilon\|_{L^2(B_1)} + \varepsilon_0^{-1} \|f\|_{L^\infty(B_1)} \right)^2
$$

**proof :** by induction, applying step 1 to the rescaled functions

$$
\omega_\varepsilon(x) = J^{-1} \left( u_\varepsilon(\theta^k x) - (u_\varepsilon)_{0, \theta^k} \right)
$$

$$
J = \left( \|u_\varepsilon\|_{L^2(B_1)} + \varepsilon_0^{-1} ||f||_{L^\infty(B_1)} \right)^2
$$

where we use the fact that the discretization is uniform
step 3 : Blow up

Show the Hölder estimate

$$\sup_{0 < r < 1/2} \sup_{|x| < 1/2} \int_{Bx,r} |u_\varepsilon - (\overline{u_\varepsilon})_{x,r}|^2 \leq C r^{2\mu} \left( \|u_\varepsilon\|_{L^2(B_1)} + \|f\|_{L^\infty(B_1)} \right)$$

When $r \geq \varepsilon/\varepsilon_0$, the estimate holds by step 2

Let $w_\varepsilon(y) = \varepsilon^{-\mu}(u_\varepsilon(\varepsilon y) - (\overline{u_\varepsilon})_{0,2\varepsilon/\varepsilon_0})$

Then $L_1 w_\varepsilon = \tilde{f}_\varepsilon$ in $B_{2/\varepsilon_0}$

i.e., $w_\varepsilon$ solves an equation the coefficients of which are independent of $\varepsilon$, and on a domain independent of $\varepsilon$.

Regularity results for piecewise Hölder coefficients $\rightarrow$ Hölder estimate on $w_\varepsilon$, then on $u_\varepsilon$