Interactions between moderately close inclusions for the Laplace equation

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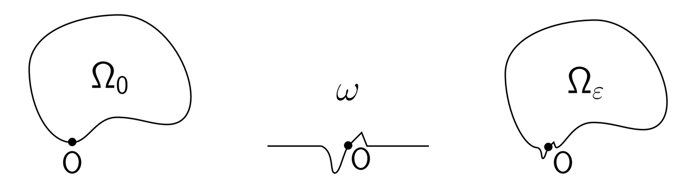
CONTEXT

The characteristic properties of a materiel (mechanical, acoustic, electromagnetic, etc) are altered by the presence of micro-defects. Taking them into account requires generally a strong mesh refinement.

We aim here at giving an alternative method based on an asymptotic analysis.

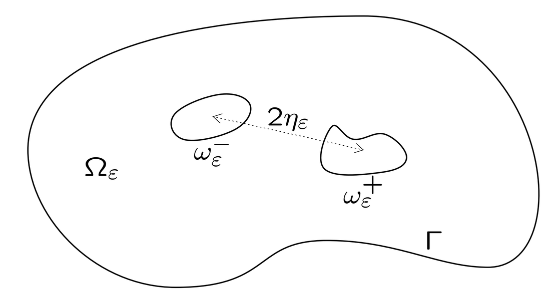
A SINGLE PERTURBATION



$$\begin{cases} -\Delta u_\varepsilon = f \text{ in } \Omega_\varepsilon, \\ + \text{ Boundary conditions on } \partial\Omega_\varepsilon. \end{cases}$$

Aim : Describe the behavior of u_ε, solution of a singular perturbation problem of the boundary.

MODERATELY CLOSE INCLUSIONS

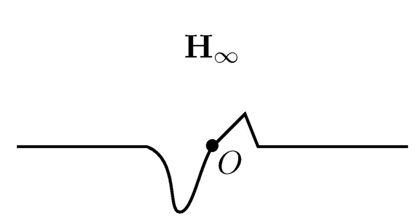


What about the interaction of two inclusions of size ε, at distance $\eta_\varepsilon \gg \varepsilon$?

MAIN THEORETICAL RESULTS

Basic ideas

- Use of a multiscale expansion involving:
  - ◇ the "slow variable" x at the scale of the domain,
  - ◇ the "fast variable" $\frac{x}{\varepsilon}$ to the scale of the perturbation.
- Comparing u_ε and the limit u_0, correctors appear in the fast variable: they are harmonic in an infinite domain obtained after blow-up of the perturbed domain, they arise from the Taylor expansion at the origin O of the limit solution u_0 in the unperturbed domain.



- The correctors (**singular profiles**) are plugged into the bounded domain via cut-off functions, which generate themselves correctors in slow variable.

Asymptotics for a single perturbation

In the simple case of Dirichlet conditions:

Theorem *We assume $f \in \mathcal{C}^\infty$ with compact support in Ω. Then the solution u_ε of*

$$-\Delta u_\varepsilon = f \text{ in } \Omega_\varepsilon,$$
$$u_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon,$$

admits the asymptotic expansion for $N < K$

$$u_\varepsilon(x) = \zeta(\tfrac{x}{\varepsilon})u_0(x) + \chi(x)\sum_{i=1}^{N}\varepsilon^i V_i(\tfrac{x}{\varepsilon}) + \zeta(\tfrac{x}{\varepsilon})\sum_{i=2}^{N}\varepsilon^i w_\varepsilon^i(x) + \mathcal{O}_{H^1(\Omega_\varepsilon)}(\varepsilon^{N+1}).$$

The profiles V_i correct the i^{th} term in the Taylor expansion of u_0 in O, and the w_ε^i are cut-off correctors satisfying $\|w_\varepsilon^i\|_{H^1(\Omega_\varepsilon)} = \mathcal{O}(1)$.

Similar results have been obtained for Neumann conditions, and for linear elasticity.

Case of two inclusions

We consider the simple case of two Neumann inclusions of size ε, centered in x_ε^- and x_ε^+, separated by a distance η_ε.

Theorem *With the same notations, the first terms of the asymptotic expansion of u_ε reads*

$$u_\varepsilon(x) = u_0(x) + \varepsilon\left[V_1^-\left(\tfrac{x-x_\varepsilon^-}{\varepsilon}\right) + V_1^+\left(\tfrac{x-x_\varepsilon^+}{\varepsilon}\right)\right] + r_\varepsilon^1(x),$$

with $\|r_\varepsilon^1\|_{H^1(\Omega_\varepsilon)} = o(\varepsilon)$.

The profiles V_1^- and V_+^1 are defined in the exterior domain of each inclusion rescaled to $\mathcal{O}(1)$.

Interaction: for $\eta_\varepsilon = \varepsilon^\alpha$ $(\alpha < 1)$,

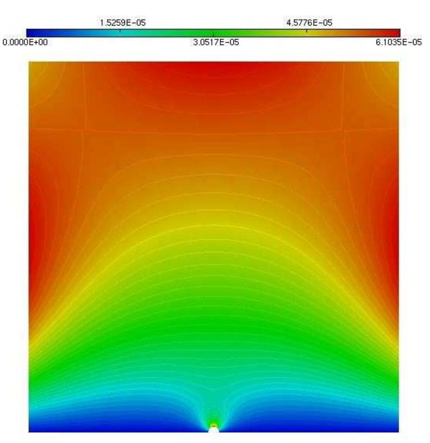
- if $\alpha < 2/3$, the main error in the remainer r_ε^1 comes from the Taylor expansion of u_0 in O (weak interaction∼superposition),
- if $\alpha > 2/3$, the main error arises from the interaction between the profiles (strong interaction).

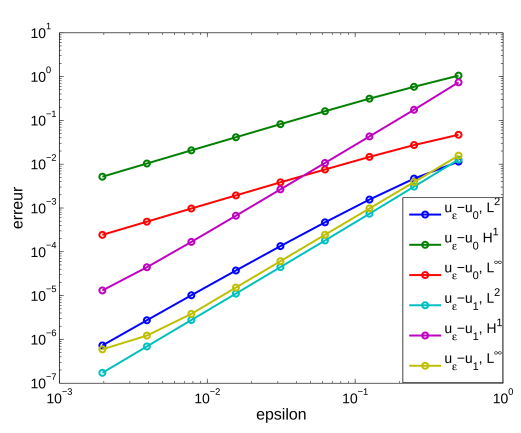
NUMERICS (SINGLE PERTURBATION)

Validation

We correct with the first profile, and plot the remainder

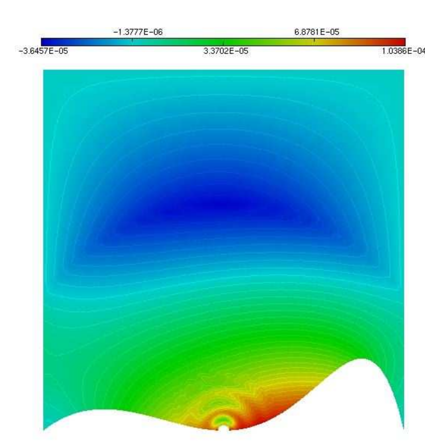
$$u_\varepsilon(x) - \underbrace{\left(u_0(x) + \varepsilon V_1(\tfrac{x}{\varepsilon})\right)}_{u_1(x)}$$

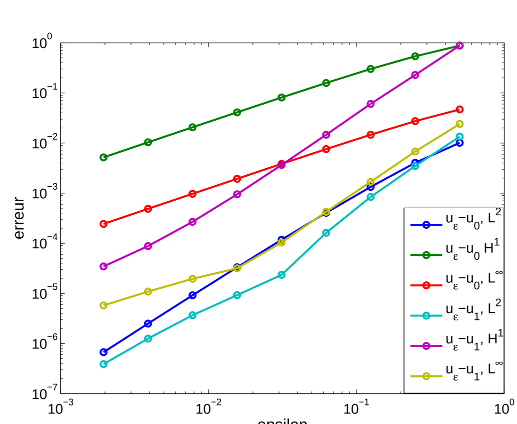




Case of a curved boundary

The proof is much more technical, but the same results still hold





CONCLUSIONS

- Extensions to other boundary conditions or elliptic problems (linear elasticity), or geometric situations (inclusions of size ε at a distance η_ε of the boundary,
- Accurate computation of the profiles (absorbing conditions, integral representations),
- Practical application to crack initiation,
- Interaction between *very close* inclusions (i.e. $\eta_\varepsilon \ll \varepsilon$).

References.

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